Asian Journal of Research and Reviews in Physics

4(1): 14-26, 2021; Article no.AJR2P.64107

ISSN: 2582-5992

Analytic Solutions of the Rotating and Stratified Hydrodynamical Equations

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Authors' contributions

This work was carried out in collaboration between both authors. Author IFB had the original idea, the design of the study, performed the main part of all analytical calculations, wrote the protocol, and wrote the first draft of the manuscript. Author LM performed the main part of the literature searches, and formulated the summary section. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/AJR2P/2021/v4i130133
Editor(s):
(1) Dr. Jelena Purenovic, Kragujevac University, Serbia.
Reviewers:
(1) Khaled Hamed El-Shorbagy, Atomic Energy Authority, Egypt.
(2) Vahid Shobeiri, The University of Adelaide, Australia.
Complete Peer review History: http://www.sdiarticle4.com/review-history/64107

Received 02 November 2021
Accepted 07 January 2021
Published 22 January 2021

Original Research Article

ABSTRACT

In this article we investigate the two-dimensional incompressible rotating and stratified, just rotating, just stratified Euler equations, comparing with each other and with the normal Euler equations with the self-similar Ansatz. The motivation of our study is the following the presented rotating stratified fluid equations can be interpreted as a well-established starting point of various more complex and more realistic meteorologic, oceanographic or geographic models. We present analytic solutions for all four models for density, pressure and velocity fields, most of them are some kind of power-law type of functions. In general the presented solutions have a rich mathematical structure. Some solutions show nonphysical explosive properties others, however are physically acceptable and have finite numerical values with power law decays. For a better transparency we present some figs for the most complicated velocity and pressure fields. To our knowledge there are no such analytic results available in the literature till today. Our results may attract attention in various scientific fields.

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Keywords: Mathematical formulation; conservation laws and constitutive relations; stratified flows.

PACS : 24.60.-k, 52.27.Ep, 95.30.Qd.

1 INTRODUCTION

There is no need to prove the evidence that geophysics, oceanography and meteorology have crucial importance for human society and civilization. Part of it is the special interest in science. Overwhelm problems in meteorology and oceanography are hydrodynamic in origin. This statement is partially true for geophysics as well. On the surface of Earth due to axial rotation and the additional gravity the question of stratified flows play an important role. Various hydrodynamic models of such kind for meteorology, oceanography and geophysics can be found in numerous monographs we just mention three of them [1, 2, 3] which are quite general. Later in this study we gave additional literature which are more specific for stratified of for rotating fluids.

Unlike the large number of highly-technical and numerical studies we investigate the time-dependent disperse self-similar solutions [4, 5] (not the blow-up type) of these kind of multidimensional Euler-type equations. The form of the original one-dimensional Ansatz reads as follows

$$V(x,t) = t^{-\alpha} f(x/t^{\beta}) = t^{-\alpha} f(\eta), \qquad (1)$$

where $V(x,t)$ is the dynamical variable, $f(\eta)$ is the shape function with the reduced variable η and α, β are the self-similar exponents. Usually $\alpha, \beta > 0$ present physically relevant power-law decaying physical solutions of the problem. This transformation is based on the assumption that a self-similar solution exists, i.e., every physical parameter preserves its shape during the expansion. Self-similar solutions usually describe the asymptotic behavior of an unbounded or a far-field problem; the time t and the space coordinate x appear only in the combination of $\eta = x/t^{\beta}$. It means that the existence of self-similar variables implies the lack of characteristic lengths and times. The geometrical and physical interpretations of this Ansatz were exhaustively explained in all our former studies [6, 7, 8], therefore we skip it here.

The aim of our analytic solutions is twofold. Firstly, it helps to analyze and understand the asymptotic properties of the solutions for asymptotic times and distances. This is a pure academic interest. Secondly, our clear-cut solutions may help to test highly complex numerical program packages which are used to forecast in meteorology, oceanography or geophysics. Our solutions can be taken as boundary conditions for a given time point then the propagation process of such complex models can be compared to our solutions at later times.

This study is part of our long-time program which systematically goes over fundamental hydrodynamic systems. Till now we published about half a dozen papers [6, 7] and a book chapter [8] in this field. To the best of our knowledge, there are no such time-dependent self-similar solutions known, presented and analyzed in the scientific literature for these systems. The structure of this paper is the following: to give a broader overview we investigate and compare the solutions of two dimensional rotating and stratified Euler equations with just stratified, just rotating and pure Euler equations. So four different flow systems will be discussed, We already published studies with the similar logic where several cases were investigated like the surface growth KPZ equation with numerous different noise terms [9] or the compressible one dimensional Euler equations where various equation-of-states were applied [10]. In the next chapter we present the investigated four systems one after another with the analytic solutions and with the corresponding parameter study. The paper ends with a summary where we overview our results and give an outlook to the reader of future generalizations or open problems.

2 THEORY

To have a complex analysis for all four cases we present the corresponding original partial differential equation (PDE) systems, the applied Ansatz with the obtained self-similar exponents, the obtained coupled ordinary differential equation (ODE) system and the solutions for the dynamical variables, the velocity and pressure fields (in two cases even for the densities). For a better transparency and for a clearer understanding we present figs for the most complicated solutions. These non-trivial shape functions and the corresponding final dynamical variables (velocity and pressure) are plotted and analyzed. We think that it is unnecessary to plot all shape functions and all dynamical variables for all four models for trivial solutions.

2.1 The Rotating and Stratified System

We start our study with the most complex flow where both rotation and stratification are present. The stability and turbulence of such systems were extensively studied by Koba [11] and Davidson [12]. The dynamics of the vortex structures in rotating stratified flows were investigated by Sokolovskiy [13]. Regarding the governing equations we consider the mass conservation law for an incompressible fluid. The dynamics of the flow is determined by the Euler's momentum equation, where a possible rotation of the system is also taken into account. According to the book of Dolzhansky [2] the rotating stratified fluid equations in two Cartesian dimensions in vectorial notation read as follows:

$$\begin{aligned} \nabla \mathbf{v} &= 0, \\ \rho_t + (\mathbf{v}\nabla)\rho &= 0, \\ \mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} + 2\mathbf{\Omega}_0 \times \mathbf{v} &= -\frac{\nabla p}{\rho_0} + \frac{G}{\rho_0}\rho, \end{aligned} \tag{2}$$

where $\mathbf{v}, \rho, p, \mathbf{\Omega_0}, G$ denote respectively the two-dimensional velocity field, density, pressure, angular velocity and an external force (now gravitation) of the investigated fluid. In the following ρ_0, is one physical parameter of the flow. For a better overview we use the coordinate notation $\mathbf{v}(x,y,t) = u(x,y,t), v(x,y,t)$ for the velocity and $p(x,y,t)$ for the scalar pressure field. To have a trivial rotation contribution we consider the $\mathbf{\Omega_0} = (0,0,\Omega_0^z(x,y,t))$ angular velocity vector. The direct form, (coordinate form) of the equations are:

$$\begin{aligned} u_x + v_y &= 0, \\ \rho_t + u\rho_x + v\rho_y &= 0, \\ u_t + uu_x + vu_y - 2v\Omega_0 &= -\frac{p_x}{\rho_0}, \\ v_t + uv_x + vv_y + 2u\Omega_0 &= -\frac{p_y}{\rho_0} + \frac{G}{\rho_0}\rho, \end{aligned} \tag{3}$$

where the subscripts mean partial derivations with respect to time and spatial coordinates. (For the following three models we skip the vectorial form, and just write out all the coordinates. We think that to perform direct calculations this is the proper form of the equations and the reader can see how the derivation act on various functions.) Therefore, this is our starting point. In this study – in all four models – we consider the Euler equation only and skip additional viscous terms. Such an investigation could be the topic of our next investigation. We think that if all the usual Newtonian viscosity term are added, the same analysis can be done and analytic solutions (probably containing Kummer special functions) can be derived as well. We also think that exploding type of solutions (which are not finite at infinite time or space coordinate) will be partially missing, of course the effect of the rotation is now unknown for us. Therefore the application of the present results to reality is strongly limited to cases where viscosity is negligible, one example could be the high speed flow of air.

We have to define our self-similar Ansatz for all four dynamical variables. We consider the form of:

$$\begin{aligned} \rho(x,y,t) = t^{-\alpha} f(\eta), \quad u(x,y,t) = t^{-\delta} g(\eta), \\ v(x,y,t) = t^{-\epsilon} h(\eta), \quad p(x,y,t) = t^{-\gamma} i(\eta), \end{aligned} \tag{4}$$

with the new variable of $\eta = \frac{x+y}{t^\beta}$. All the exponents $\alpha, \beta, \gamma, \delta, \epsilon,$ are real numbers. (Solutions with integer exponents are called self-similar solutions of the first kind, non-integer exponents generate self-similar solutions which are of called second kind.) The shape functions f, g, h, i could be any continuous functions with existing first derivatives and will be evaluated later on. The logic, the physical and the geometrical interpretation of the Ansatz were exhaustively analyzed in all our former publications [6, 7, 8, 9, 10] therefore we neglect it here.

To derive a consistent coupled ODE system for the shape functions the exponents have to fulfill the next constraints

$$\alpha = 3/2, \quad \beta = \delta = \epsilon = 1/2, \quad \gamma = 1. \tag{5}$$

Note, that all exponents have a fixed numerical value, which clearly defines the solutions. Each exponent is positive so the solutions are expected to be physical (which mean that all will have power law decays at large times). It is important to emphasize, that only the $\mathbf{\Omega}_0^z = \omega_0/t$ angular velocity function (which is trivial from dimensional consideration) leads to the following clean-cut ordinary differential equation (ODE) system

$$\begin{aligned} f' + g' &= 0, \\ -\frac{3}{2}f - \frac{1}{2}\eta f' + gf' + hf' &= 0, \\ -\frac{1}{2}g - \frac{1}{2}\eta g' + gg' + hg' - 2h\omega_0 &= -\frac{i'}{\rho_0}, \\ -\frac{1}{2}h - \frac{1}{2}\eta h' + gh' + hh' + 2g\omega_0 &= -\frac{i'}{\rho_0} + \frac{G}{\rho_0}f. \end{aligned} \tag{6}$$

At first sight we may be scared from the coupled non-linearity of the ODE system, these kind of ODE systems are not usual. From the first (continuity) equation we automatically get $f + g = c_0$, where c_0 is proportional with the constant mass flow rate. Implicitly, larger c_0 means larger velocities. From the first and second Eq. of (6) the ODE for the density shape function can be easily derived

$$f'\left(c_0 - \frac{\eta}{2}\right) - \frac{3f}{2} = 0. \tag{7}$$

The solution is almost trivial

$$f = \frac{c_1}{(2c_0 - \eta)^3}, \tag{8}$$

where c_1 stands for the usual integration constant. The function is a shifted third order hyperbola with a singularity at $\eta = 2c_0$ for $\eta > 0$ it is positive and strictly monotone decreases. The density has a power-law decay for large times which is physically desirable

$$\rho(x, y, t) = \frac{1}{t^{\frac{3}{2}}} \cdot \frac{c_1}{\left(2c_0 - \frac{x+y}{t^{\frac{1}{2}}}\right)^3} \simeq \frac{1}{(x+y)^3}. \tag{9}$$

Extracting the fourth equation from the third one in 6 the ODE for the shape function of the velocity component v can be easily given:

$$h'(\eta - 2c_0) + h - c_0\left(2\omega_0 + \frac{1}{2}\right) + \frac{Gf}{\rho_0} = 0, \tag{10}$$

with the solution of

$$h = \frac{\eta(-c_0 - 4c_0\omega_0)}{2c_0 - \eta} - \frac{Gc_1}{\rho_0(\eta - 2c_0)^3} + \frac{c_2}{2c_0 - \eta}. \tag{11}$$

The function has a singularity at $\eta = 2c$ and it it strictly monotone growing for all positive ηs where $\eta > 2c_0$. The solution is the sum of a shifted first and third order hyperbola. All the parameters are responsible for the scaling and the shift of the singularity. It is straightforward to show that the asymptotic behavior of the velocity field is

$$v(x, y, t) = t^{-\epsilon}h(\eta) \simeq t^{-1/2}\left(\frac{Gc_1}{\rho_0\left[\left(\frac{(x+y)}{t^{1/2}} - 2c_0\right]^3}\right) \simeq \frac{t}{(x+y)^3}, \tag{12}$$

which makes it a physically acceptable solution.

Adding the last two equations of Eq. (6) the ODE of the pressure shape function can be derived

$$-\frac{2i(\eta)'}{\rho_0} + \frac{Gf(\eta)}{\rho_0} + 4\omega_0 h(\eta) - c_0\left(\omega_0 + \frac{1}{2}\right) = 0. \quad (13)$$

The solution can be easily evaluated with quadrature

$$i = 2\omega_0 ln(\eta - 2c_0)(c_2 - 4\omega_0 c_0 - \rho_0 c_0^2) + \frac{Gc_1}{2(\eta-2c_0)^2}\left(\frac{1}{2} - \omega_0\right) + \eta\left(\frac{1}{2}\omega_0\rho_0 c_0 - 4\rho_0\omega_0^2 c_0 - \frac{c_0\rho_0}{4}\right) + c_3. \quad (14)$$

The asymptotic of the pressure as field variable is:

$$p(x,y,t) = t^{-\gamma} i(\eta) \simeq t^{-1}\frac{Gc_1}{2\left[\frac{x+y}{t^{1/2}} - 2c_0\right]^2}\left(\frac{1}{2} - \omega_0\right) \simeq \frac{1}{(x+y)^2}. \quad (15)$$

Fig. 1. shows the pressure shape function for two different angular velocities giving qualitatively different curves. The integration constants c_0, c_1, c_2, c_3 play no relevant role just shift and scale the results. The key parameter is the angular velocity with the turning point of $\omega = 0.5$. In the case of $\omega_0 > 0.5$ there is a global maximum of the pressure. Larger ω means quicker decay. Larger densities makes quicker pressure decays as well.

To have a feeling about the general properties of the pressure, Fig. (2-3) present the ten-based logarithm of the solution for two different angular velocities. In both cases the pressure functions have clear asymptotic values.

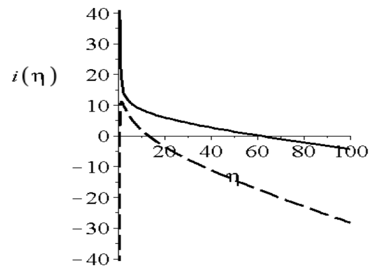

Fig. 1. Graphs of two different pressure shape functions Eq. (14) where the common parameters are $G = 10, \rho_0 = 1, c_0 = 1, c_1 = 3.25, c_2 = -3.1, c_3 = 15$. The solid and dashed curves are for $\omega_0 = 0.135$ and $\omega_0 = 75$, respectively

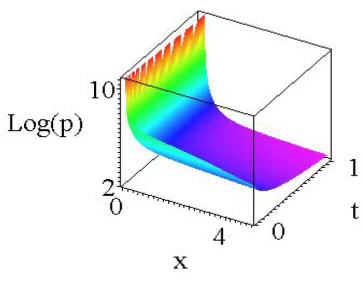

Fig. 2. The ten-based logarithm of the pressure $Log(p(x, y = 0, t))$ for $\omega_0 = 0.135$ angular velocity, all other parameters are given above

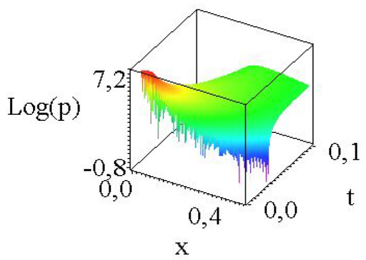

Fig. 3. The ten-based logarithm of the pressure $Log(p(x, y = 0, t))$ for $\omega_0 = 0.75$ angular velocity, all other parameters are given above

2.2 The stratified system without rotation

Now we consider the $\mathbf{\Omega} = 0$ special case. The stratification and the change of density with the altitude has its importance in the Earth science. Turbulence issues in stratified planetary boundary layers is studied by Ansorge [14]. Numerous general hydrodynamic questions in environmental stratified flows can be found in various textbooks [15, 16, 17, 18, 19]. The vortex structures in stratified fluids was investigated by Voropayev and Afanasyev [20]. Additional wave propagation issues were extensively described in [21, 22]. In 1975 Ono [23] presented algebraic solitary wave solutions for stratified fluids. An enhanced decrease of density of air with the altitude may lead to static stability which usually yields an increase of concentration of certain pollutants [24, 25, 26]. The effect of temperature in the hydrodynamics of stratified flows may lead to specific convection phenomena even on small scales [27]. Interesting aspects were investigated and discussed related to sedimentation in stratified flows by [28].

Our reduced PDE system is now:

$$
\begin{aligned}
u_x + v_y &= 0, \\
\rho_t + u\rho_x + v\rho_y &= 0, \\
u_t + uu_x + vu_y &= -\frac{p_x}{\rho_0}, \\
v_t + uv_x + vv_y &= -\frac{p_y}{\rho_0} + \frac{G}{\rho_0}\rho, \qquad (16)
\end{aligned}
$$

where subscripts means partial derivatives with respect to time and both coordinates x and y. The applied Ansatz is the following:

$$
\begin{aligned}
\rho(x, y, t) = t^{-\alpha} f(\eta), \quad u(x, y, t) = t^{-\delta} g(\eta), \\
v(x, y, t) = t^{-\epsilon} h(\eta), \quad p(x, y, t) = t^{-\gamma} i(\eta), \qquad (17)
\end{aligned}
$$

The corresponding ODE system reads as:

$$
\begin{aligned}
f' + g' &= 0, \\
-(2-\beta)f - \beta\eta f' + gf' + hf' &= 0, \\
-(1-\beta)g - \beta\eta g' + gg' + hg' &= -\frac{i'}{\rho_0}, \\
-(1-\beta)h - \beta\eta h' + gh' + hh' &= -\frac{i'}{\rho_0} + \frac{G}{\rho_0} f. \qquad (18)
\end{aligned}
$$

The slightly modified corresponding ODE system due to an undefined free self-similar exponent has a much larger degree of freedom. So, in this sense all the exponents can be expressed with a fixed one (we may say with β)

$$\alpha = 2-\beta, \quad \delta = \epsilon = 1-\beta, \quad \gamma = 2(1-\beta). \tag{19}$$

(We use β as free parameter because it describes the common "spreading" property of all the dynamical variables, and now all "decay" parameters are free from each other. So the "decays" of all variables can be studied independently.) The three ODEs for the shape functions can be determined with the logic mentioned above,

$$f'(c_0-\beta\eta) - f(2-\beta) = 0, \tag{20}$$

$$2h'(\beta\eta - c_0) - (1-\beta)c_0 + \frac{Gf}{\rho_0} = 0, \tag{21}$$

$$-\frac{2i'}{\rho_0} + \frac{Gf}{\rho_0} + (1-\beta)c_0 = 0. \tag{22}$$

All the solutions can be derived with quadrature

$$f = c_1(c_0-\beta\eta)^{\frac{\beta-2}{\beta}}, \tag{23}$$

$$h = -\frac{Gc_1(c_0-\beta\eta)^{\frac{\beta-2}{\beta}}}{2\rho_0(\beta-2)} + \frac{c_0 ln\left([c_0-\beta\eta]^{\frac{\beta-2}{\beta}}\right)}{2(\beta-2)}(1-\beta) + c_2, \tag{24}$$

$$i = -\frac{c_1G(c_0-\beta\eta)^{\frac{2(\beta-1)}{2}}}{4(\beta-1)} + \frac{(1-\beta)c_0\rho_0}{2}\eta + c_3. \tag{25}$$

Note, that due to the free running self-similar exponent (now β) we got different kind of power-low dependent solutions, therefore this model has the richest mathematical structure. To show the features of (23 - 25) we present and discuss some solutions with various βs. Fig. 4. shows (23) for six different exponents. Note, that we can get back all the usual power law functions, constant hyperbola and parabola as well. Fig. 5. presents the $h(\eta)$ shape functions. We present 5 different kind of solutions, for reasonable βs. Fig. 6. shows the shape functions for $i(\eta)$, there are 7 qualitative different functions exist as solutions. (We say that the typical exponent lies in the $[-4..4]$ range, for lot of physical systems this is restricted to the $[-2..2]$ interval.)

Our decade long experience shows that mainly the solutions with all positive exponents are physically relevant describing power-law dependent solutions. (Solutions with negative exponents usually have exploding properties at large times and space coordinates which violates mass, momenta or energy conservation.) Fig. (7 - 9) present the tenth-based logarithm of the density, velocity and pressure for the common $\beta = 1/2$ value. Note, that all dynamical variables have a physically reasonable power-law decay for infinite times.

2.3 The Rotating System without Stratification

Rotating fluids are also relevant for science and engineering therefore the corresponding literature again enormous, without completeness we mention some general work of them [29, 30, 31, 32, 33, 34, 35]. Additional convection problems in rotating fluids were directly analyzed by Boubnov and Golitsyn [36]. Vadász [37] investigated the heat transfer of rotating porous fluids. A detailed mathematical analysis of fluids on rotating spheres is given by Skiba [38]. To investigate this case the complete second equation of (2) has to be neglected having the PDE system in the form of:

$$\begin{aligned} u_x + v_y &= 0, \\ u_t + uu_x + vu_y - 2v\omega_0 &= -\frac{p_x}{\rho_0}, \\ v_t + uv_x + vv_y + 2u\omega_0 &= -\frac{p_y}{\rho_0} + G. \end{aligned} \tag{26}$$

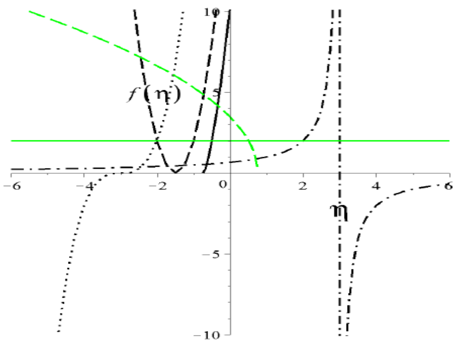

Fig. 4. The graphs of Eq. (23) the common parameters are $c_0 = 4, c_1 = 1.2$. The black solid, dashed, dotted and dash-dotted, the green solid and green dashed curves are for $\beta = -4, -2, -1, 1, 2, 4$, respectively

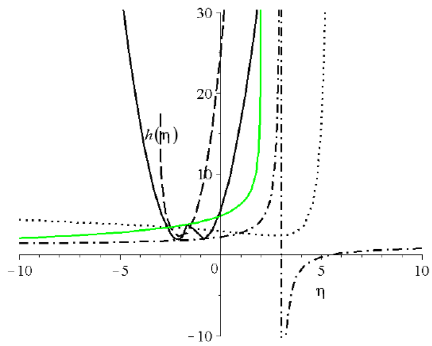

Fig. 5. The graphs of Eq. (24) the common parameters are $G = 10, \rho_0 = 1, c_0 = 3, c_1 = 1.2, c_2 = 1.2$. The black solid, dashed, dotted, dash-dotted and green solid curves are for $\beta = -2, -1, 1, 1.5, 2.5$, respectively

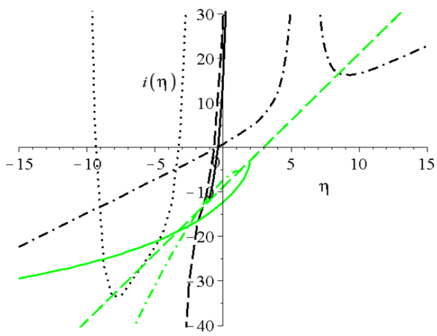

Fig. 6. The graphs of Eq. (25) the common parameters are the same as above with $c_3 = 0$. The black solid, dashed, dotted, dash-dotted and the green solid and green dashed and green dash-dotted curves are for $\beta = -3, -2, -0.5, 0.5, 1.5, 2, 2.5$, respectively

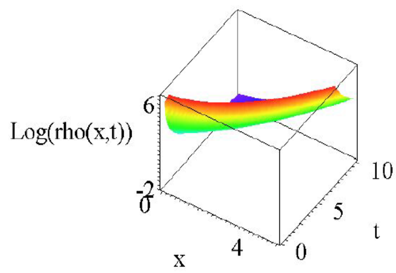

Fig. 7. The graph of the ten-based logarithm of the density function $\rho(x, y = 0, t)$ **for** $\beta = 1/2$

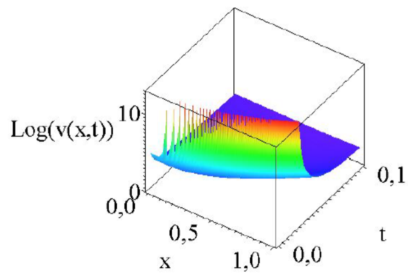

Fig. 8. The graph of the ten-based logarithm of the velocity function $v(x, y = 0, t)$ **for** $\beta = 1/2$

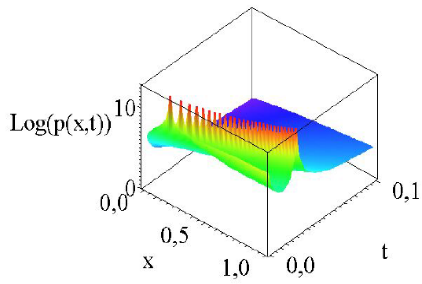

Fig. 9. The graph of the ten-based logarithm of the pressure function $p(x, y = 0, t)$ **for** $\beta = 1/2$

So the number of the four unknowns is now reduced to three, namely to the velocity components u, v and to the pressure p. To avoid contradiction among the exponents it is important to emphasize, that again only for $\mathbf{\Omega}_0^z = \omega_0/t$ angular velocity function we get a clean-cut ODE system. The trial functions for the solutions now read

$$u(x, y, t) = t^{-\delta} g(\eta), \quad v(x, y, t) = t^{-\epsilon} h(\eta), \quad p(x, y, t) = t^{-\gamma} i(\eta). \tag{27}$$

The relations among the exponents are the following:

$$\beta = 2, \quad \delta = \epsilon = -1 \quad \gamma = -2. \tag{28}$$

Note, all the fixed exponents. The coupled ODE system is

$$\begin{aligned} g' + h' &= 0, \\ g - 2\eta g' + gg' + hg' - 2h\omega_0 &= -\frac{i'}{\rho_0}, \\ h - 2\eta h' + gh' + hh' + 2g\omega_0 &= -\frac{i'}{\rho_0} + G. \end{aligned} \tag{29}$$

The ODEs for one velocity field component and for the pressure field are:

$$2h'(2\eta - c_0) - 2h - c_0(2\omega_0 - 1) + G = 0, \tag{30}$$

$$-\frac{2i'}{\rho_0} + 4\omega_0 hG - c_0(2\omega_0 + 1) + G = 0. \tag{31}$$

There is no coupling between the variables. The corresponding solutions are

$$h = c_1\sqrt{c_0 - 2\eta} + \frac{G}{2} - \frac{c_0(2\omega_0 - 1)}{2}, \tag{32}$$

$$i = \frac{1}{2}\rho_0\left(-\frac{4}{3}\omega_0 c_1[c_0 - 2\eta]^{\frac{3}{2}} + \eta[-2\omega_0 c_0\{2\omega_0 - 1\} + 2\omega_0 G - c_0\{2\omega_0 + 1\} + G]\right) + c_2. \tag{33}$$

The shape function of the velocity is a shifted square root function with negative argument. Note, the extra last positive shift term compared to the simple Euler case Eq. (42) which is proportional to the angular velocity of the rotation ω_0. The pressure shape function is a sum of a linear and an $\eta^{3/2}$ power law function with some shifts. Note, that the rotation is responsible to the first power law term,

$$\begin{aligned} v(x,y,t) &= t^{-\epsilon}h(\eta) = t\left(c_1\sqrt{c_0 - \frac{2(x+y)}{t^2}} + \frac{c_0(1-2\omega_0)+G}{2}\right) \simeq \sqrt{x+y} + t \\ p(x,y,t) &= t^{-\gamma}i(\eta) = \\ &= \frac{t^2\rho_0}{2}\left(-\frac{4}{3}\omega_0 c_1\left[c_0 - 2\frac{(x+y)}{t^2}\right]^{\frac{3}{2}} + \frac{(x+y)}{t^2}\left[\tilde{C}\right] + c_2,\right) \\ &\simeq (x+y)^{3/2}/t + x + y \end{aligned} \tag{34}$$

where

$$\tilde{C} = -2\omega_0 c_0\{2\omega_0 - 1\} + 2\omega_0 G - c_0\{2\omega_0 + 1\} + G. \tag{35}$$

The derived solutions are quite simple therefore we skip to present additional figs. At this point we have to make a comment. Parallel to this study we investigate an astrophysical relevant fictive media, the spherical symmetric self-gravitating compressible dark fluid with the same method [39]. We plan to complete that model with additional rotation therefore the obtained results presented in Eq. (34) are important for comparison.

2.4 No Rotation and No Stratification

This is the simplest system among the investigated four cases and this is the equation for the two dimensional incompressible ideal fluid as well. For completeness the starting PDE system reads

$$\begin{aligned} u_x + v_y &= 0, \\ u_t + uu_x + vu_y &= -\frac{p_x}{\rho_0}, \\ v_t + uv_x + vv_y &= -\frac{p_y}{\rho_0} + \frac{\rho}{\rho_0}G. \end{aligned} \tag{36}$$

We note, that with the usual Newtonian additional viscous term and with the additional third Cartesian coordinate z, the same analysis is possible (and was performed in our first paper) resulting Kummer functions with quadratic arguments [40].

The trial functions for the solutions are not changed from the previous case

$$u(x,y,t) = t^{-\delta}g(\eta), \quad v(x,y,t) = t^{-\epsilon}h(\eta), \quad p(x,y,t) = t^{-\gamma}i(\eta). \tag{37}$$

The self-similar exponents remained the same too:

$$\beta = 2, \quad \delta = \epsilon = -1 \quad \gamma = -2. \tag{38}$$

The ODE system is however a bit simpler:

$$\begin{aligned} g' + h' &= 0, \\ g - 2\eta g' + gg' + hg' &= -\frac{i'}{\rho_0}, \\ h - 2\eta h' + gh' + hh' &= -\frac{i'}{\rho_0} + G. \end{aligned} \tag{39}$$

The decoupled ODEs for the velocity and for the pressure are also simpler, (note the missing terms with ω_0)

$$2h'(2\eta - c_0) - 2h + c_0 + G = 0, \tag{40}$$

$$-\frac{2i'}{\rho_0} - c_0 + G = 0. \tag{41}$$

The analytic solutions, after all, are almost trivial and read

$$h = c_1\sqrt{c_0 - 2\eta} + \frac{c_0 + G}{2},, \tag{42}$$

$$i = \frac{(G - c_0)\rho_0\eta}{2} + c_2.. \tag{43}$$

The velocity shape function is a square root function with shifted negative arguments which means that the function domain becomes negative. The shape function of the pressure is a simple linear function. Note the difference to Eq. (33) is due to the rotation ω_0. For completeness the final field variables are

$$v(x,y,t) = t^{-\epsilon}h(\eta) = t\left(c_1\sqrt{c_0 - \frac{2(x+y)}{t^2}} + \frac{c_0 + G}{2}\right) \simeq \sqrt{x+y} + t, \tag{44}$$

$$p(x,y,t) = t^{-\gamma}i(\eta) = t^2\left(\frac{(G - c_0)\rho_0}{2}\frac{(x+y)}{t^2} + c_2\right) \simeq x + x + t^2. \tag{45}$$

Notice, that both dynamical variable have no decay property to large times, therefore we consider them nonphysical and skip to present additional figs. From physical considerations we may calculate the total kinetic energy term which is proportional to the volume integral of $\int_V \frac{\rho_0}{2}[u(x,y,t)^2 + v(x,y,t)^2]dxdy$ this quantity should however has a time decay at infinite times (where V means the volume of the dynamics).

3 SUMMARY

We investigated the two-dimensional incompressible rotating and stratified, just rotating, just stratified Euler equations by comparing them to each other and with the normal Euler equations applying the self-similar Ansatz. To emphasize the scientific relevance of the first three of these equations we mentioned numerous textbooks and monographs which were written in the recent decade. There are at least three scientific disciplines exist – meteorology, oceanography and geophysics – where these kind of equation are the very basic starting points of more sophisticated and complex numerical models and program packages. We found analytic solutions for all dynamical variables of all four models. Every solution can be expressed with various power-law type functions.

The solutions of the rotating stratified and the stratified flows are much more complex than the last two one, therefore we presented additional figs to enlighten the details. Overall the physically relevant, power-law time decaying solutions were emphasized. We think that due to the lack of higher order viscous terms in the Euler equations all solutions are quite simple contains no additional internal finer structure e.g. some waves or oscillations. We mentioned that additional viscous terms will effect second order ODE system with solutions of special functions (like Kummer or Whittaker functions) which could have additional oscillations and strong decay properties. The present study is an organic part of our decade long scientific program in which we investigate numerous basic hydrodynamic systems one after another giving closed solutions with in-depth parameter studies and analysis. We would like to publish this manuscript just as a precursor of planned later studies with more complex materials of viscous fluids like [41].

4 CONCLUSION

We investigated four different rotating and/or stratified ideal fluid equations with the self-similar Ansatz and proved that all models have power-law type of solutions for the velocity, density and pressure fields.

ACKNOWLEDGMENT

One of us (I.F. Barna) was supported by the NKFIH, the Hungarian National Research Development and Innovation Office. Finally, the authors dedicate this paper to their parents.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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Peer-review history:
The peer review history for this paper can be accessed here:
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