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# Multi-parametric Rational Solutions to the KdV Equation

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***Author's contribution***

*The sole author designed, analyzed, interpreted and prepared the manuscript.*

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## ABSTRACT

We construct multi-parametric rational solutions to the KdV equation. For this, we use solutions in terms of exponentials depending on several parameters and take a limit when one of these parameters goes to $0$. Here we present degenerate rational solutions and give a result without the presence of a limit as a quotient of polynomials depending on $3N$ parameters. We give the explicit expressions of some of these rational solutions.

*Keywords: KdV equation; rational solutions.*

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## 1 INTRODUCTION

We consider the KdV equation

$$4u_t = 6uu_x - u_{xxx}, \tag{1.1}$$

where as usual, the subscripts $x$ and $t$ denote partial derivatives and $u$ is a function of $x$ and $t$.

Korteweg and de Vries [1] introduced this equation $(1.1)$ for the first time in $1895$.

The KdV equation $(1.1)$ is the basis of the most common tool for the (1+1)-dimensional modelling of shallow water waves. It is now known that this equation describes the propagation of waves with weak dispersion in various nonlinear media.

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In $1967$, Gardner et al. [2] gave a method of resolution of this equation. Zakharov and Faddeev [3] proved that this is a complete integrable equation in $1971$.

Using the bilinear method, Hirota [4] constructed solutions in $1971$. Among the works which have been realized, we can mention some of them. Solutions in terms of Riemann theta functions [5] were given by Its and Matveev in $1975$. Lax gave in $1975$ the expressions of periodic and almost periodic solutions [6]; we can also quote, Matveev in $1992$ [7], Ma in $2004$ [8], Kovalyov in $2005$ [9].

We consider solutions to the KdV equation in this paper. From elementary exponential functions depending on several parameters, we construct rational solutions in performing a passage to the limit when one of these parameters goes to $0$. We obtain rational solutions as depending on $3N$ parameters at order $N$. To get a result which does not depend on a limit, we have to consider another type of functions. With this new functions, we can give explicit solutions and we get a hierarchy of rational solution of order $N$ to the KdV equation depending on $3N$ parameters.

# 2 SOLUTIONS TO THE KdV EQUATION

## 2.1 Solutions to the KdV Equation in Terms of Elementary Exponentials

We consider the KdV equation $(1.1)$

$$4u_t = 6uu_x - u_{xxx}.$$

We use the following notations. We consider the real numbers $\alpha_j$, $\beta_j$ and $\gamma_j$ defined by :

$$\alpha_j = \sum_{k=1}^{N} a_k (je)^{2k-1}, \quad \beta_j = \sum_{k=1}^{N} c_k (je)^{2k-1}, \quad \gamma_j = \sum_{k=1}^{N} d_k (je)^{2k-1}, \text{ for } 1 \leqslant i \leqslant N \tag{2.1}$$

We consider the elementary functions :

$$f_{ij}(x,t) = \alpha_j^{i-1} \exp(\alpha_j x - \alpha_j^3 t + \beta_j) - \alpha_j^{i-1} \exp(\alpha_j x - \alpha_j^3 t + \gamma_j), \text{ for } 1 \leqslant i \leqslant N \tag{2.2}$$

Then, we have the following statement:

**Theorem 2.1.** *The function $v$ defined by*

$$v(x,t) = -2\partial_x^2 \ln(\det(f_{ij})_{(i,j)\in[1,N]}) \tag{2.3}$$

*is a solution to the KdV equation (1.1) with $e$, $a_j$, $c_j$ and $d_j$, $1 \leqslant j \leqslant N$ arbitrarily real parameters.*

**Proof:** The corresponding Lax pair to the KdV equation $(1.1)$ is

$$\begin{cases} -\phi_{xx} + u\phi = \lambda\phi, \\ \phi_t = -\phi_{xxx} + \dfrac{3}{2}u\phi_x + \dfrac{3}{4}u_x\phi. \end{cases} \tag{2.4}$$

This system is covariant by the Darboux transformation. If $\phi_1, \ldots, \phi_N$ are solutions of the system $(2.4)$, then $\phi[N]$ defined by $\phi[N] = \dfrac{W(\phi_1, \ldots, \phi_N, \phi)}{W(\phi_1, \ldots, \phi_N)}$ is another solution of this system $(2.4)$ where $u$ is replaced by $u[N] = u - 2(\ln W(\phi_1, \ldots, \phi_N)_{xx}$ [10]. In the expression of $\phi[N]$, $W(\phi_1, \ldots, \phi_k)$ is the wronskian of the functions $\phi_1, \ldots, \phi_k$ defined by $(\det \partial_x^{i-1}\phi_j)_{(1\leqslant i \leqslant k,\, 1\leqslant j\leqslant k)}$.

We choose $u = 0$. Then the functions $\phi_j = f_{1j}$ verify the following system

$$\begin{cases} \phi_{xxx} - \dfrac{3}{4}\phi_x = \lambda\phi, \\ \phi_t = -\phi_{xx}. \end{cases} \tag{2.5}$$

Then the solution of $(1.1)$ can be written as $v(x,t) = -2(\ln W(\phi_1,\dots,\phi_N)_{xx}$which is nothing else that $(2.3)$ $v(x,t) = -2\partial_x^2 \ln(\det(f_{ij})_{(i,j)\in[1,N]})$. □

To the best of my knowledge, such solutions have not been built.

## 2.2 Rational Solutions to the KdV Equation

We are going to perform a limit when the parameter $e$ tends to $0$. With this idea, we can obtain rational solutions to the KdV equation.

### 2.2.1 Rational solutions as a limit case

We get the following result:

**Theorem 2.2.** *The function $v$ defined by*

$$v(x,t) = \lim_{e\to 0} -2\partial_x^2 \ln(\det(f_{ij})_{(i,j)\in[1,N]}) \tag{2.6}$$

*is a rational solution to the KdV equation (1.1) depending on $3N$ parameters $a_j$, $c_j$ and $d_j$, $1 \leqslant j \leqslant N$.*

**Proof:** It is a consequence of the previous result. □

### 2.2.2 Degenerate rational solutions

We can give give a more precise result avoiding the presence of a limit.

For this, we consider another type of functions $g_{ij}$ defined by:

$g_{ij} = \frac{\partial^{2j-1} f_{ij}}{\partial e^{2j-1}}_{|e=0}$, for $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant N$.

Then get the following result:

**Theorem 2.3.** *The function $v$ defined by*

$$v(x,t) = -2\partial_x^2 \ln(\det(g_{ij})_{(i,j)\in[1,N]} \tag{2.7}$$

*is a rational solution to the KdV equation (1.1) depending on $3N$ parameters $a_j$, $c_j$ and $d_j$, $1 \leqslant j \leqslant N$.*

**Proof:** We combine the columns of the determinant in oder to eliminate successively the terms in $e^{2k-1}$. Then when we take a passage to the limit when $e$ tends to $0$ for each column we get the result. □

So for each positive integer $N$, we get rational solutions to the KdV equation depending on $3N$ real parameters.

In the following we give some examples of rational solutions.

These results are consequences of the previous result $(2.7)$.

## 2.3 First Order Rational Solutions

We have the following result at order $N = 1$:

**Proposition 2.1.** *The function* $v$ *defined by*

$$v(x,t) = 8 \frac{{a_1}^2}{(2\, a_1 x - d_1 + c_1)^2}, \tag{2.8}$$

*is a solution to the KdV equation (1.1) with* $a_1$, $c_1$, $d_1$ *arbitrarily real parameters.*

*Remark* 2.1. This solution independent of $t$ does not present any interest.

## 2.4 Second Order Rational Solutions

**Proposition 2.2.** *The function* $v$ *defined by*

$$v(x,t) = -2 \frac{n(x,t)}{d(x,t)^2}, \tag{2.9}$$

*with*
$n(x,t) = -192\, {a_1}^8 x^4 + 12\, {a_1}^3 (32\, {a_1}^4 d_1 - 32\, {a_1}^4 c_1) x^3 + 12\, {a_1}^3 (-24\, {a_1}^3 {c_1}^2 + 48\, {a_1}^3 c_1 d_1 - 24\, {a_1}^3 {d_1}^2) x^2 + 12\, {a_1}^3 (96\, {a_1}^5 t - 48\, a_1 a_2 d_1 - 8\, {a_1}^2 {c_1}^3 + 24\, {a_1}^2 d_1 {c_1}^2 - 48\, {a_1}^2 c_2 + 8\, {a_1}^2 {d_1}^3 - 24\, {a_1}^2 c_1 {d_1}^2 + 48\, {a_1}^2 d_2 + 48\, a_1 a_2 c_1) x + 12\, {a_1}^3 (4\, c_1 a_1 {d_1}^3 - 24\, a_1 d_1 d_2 + 24\, a_2 {d_1}^2 + 24\, a_2 {c_1}^2 + 48\, t {a_1}^4 c_1 - a_1 {c_1}^4 - a_1 {d_1}^4 + 4\, d_1 a_1 {c_1}^3 - 48\, t {a_1}^4 d_1 - 24\, a_1 c_1 c_2 - 48\, c_1 a_2 d_1 + 24\, c_1 d_2 a_1 + 24\, d_1 c_2 a_1 - 6\, {c_1}^2 a_1 {d_1}^2)$,

$d(x,t) = 8\, {a_1}^4 x^3 + (-12\, {a_1}^3 d_1 + 12\, {a_1}^3 c_1) x^2 + (6\, {a_1}^2 {c_1}^2 - 12\, c_1 {a_1}^2 d_1 + 6\, {a_1}^2 {d_1}^2) x - 12\, c_2 a_1 + 12\, d_2 a_1 - 3\, d_1 a_1 {c_1}^2 + a_1 {c_1}^3 + 3\, c_1 a_1 {d_1}^2 + 12\, a_2 c_1 - a_1 {d_1}^3 + 24\, t {a_1}^4 - 12\, a_2 d_1$

*is a rational solution to the KdV equation (1.1), quotient of two polynomials with numerator of degree* $4$ *in* $x$, $1$ *in* $t$, *and denominator of degree* $6$ *in* $x$, $2$ *in* $t$.

## 2.5 Rational Solutions of Order Three

We get the following rational solutions given by:

**Proposition 2.3.** *The function* $v$ *defined by*

$$v(x,t) = -2 \frac{n(x,t)}{d(x,t)^2}, \tag{2.10}$$

*with*
$n(x,t) = -24576\, {a_1}^{16} x^{10} - 24\, {a_1}^2 (-5120\, {a_1}^{13} d_1 + 5120\, {a_1}^{13} c_1) x^9 - 24\, {a_1}^2 (11520\, {a_1}^{12} {c_1}^2 - 23040\, {a_1}^{12} c_1 d_1 + 11520\, {a_1}^{12} {d_1}^2) x^8 - 24\, {a_1}^2 (-15360\, {a_1}^{11} {d_1}^3 + 46080\, {a_1}^{11} c_1 {d_1}^2 + 15360\, {a_1}^{11} {c_1}^3 - 46080\, {a_1}^{11} {c_1}^2 d_1) x^7 - 24\, {a_1}^2 (-53760\, {a_1}^{10} {c_1}^3 d_1 + 80640\, {a_1}^{10} {c_1}^2 {d_1}^2 + 13440\, {a_1}^{10} {d_1}^4 - 53760\, {a_1}^{10} c_1 {d_1}^3 + 13440\, {a_1}^{10} {c_1}^4) x^6 - 24\, {a_1}^2 (-207360\, {a_1}^7 {a_2}^2 c_1 + 40320\, {a_1}^9 c_1 {d_1}^4 - 80640\, {a_1}^9 {c_1}^2 {d_1}^3 - 69120\, {a_1}^8 a_3 d_1 + 207360\, {a_1}^7 {a_2}^2 d_1 - 8064\, {a_1}^9 {d_1}^5 - 69120\, {a_1}^9 c_3 - 40320\, {a_1}^9 {c_1}^4 d_1 + 80640\, {a_1}^9 {c_1}^3 {d_1}^2 + 8064\, {a_1}^9 {c_1}^5 - 207360\, {a_1}^8 a_2 d_2 + 69120\, {a_1}^9 d_3 + 207360\, {a_1}^8 a_2 c_2 + 69120\, {a_1}^8 a_3 c_1) x^5 - 24\, {a_1}^2 (3360\, {a_1}^8 {c_1}^6 + 3360\, {a_1}^8 {d_1}^6 + 345600\, t^2 {a_1}^{14} + 86400\, {a_1}^8 {c_2}^2 + 86400\, {a_1}^8 {d_2}^2 - 345600\, t {a_1}^{11} c_2 + 345600\, t {a_1}^{11} d_2 - 20160\, {a_1}^8 {c_1}^5 d_1 + 50400\, {a_1}^8 {c_1}^4 {d_1}^2 - 67200\, {a_1}^8 {c_1}^3 {d_1}^3 + 50400\, {a_1}^8 {c_1}^2 {d_1}^4 - 20160\, {a_1}^8 c_1 {d_1}^5 - 172800\, {a_1}^8 c_1 c_3 + 172800\, {a_1}^8 c_1 d_3 - 172800\, {a_1}^8 c_2 d_2 + 172800\, {a_1}^8 c_3 d_1 - 172800\, {a_1}^8 d_1 d_3 + 172800\, {a_1}^7 a_3 {c_1}^2 + 172800\, {a_1}^7 a_3 {d_1}^2 - 432000\, {a_1}^6 {a_2}^2 {c_1}^2 - 432000\, {a_1}^6 {a_2}^2 {d_1}^2 + 345600\, t {a_1}^{10} a_2 c_1 - 345600\, t {a_1}^{10} a_2 d_1 + 345600\, {a_1}^7 a_2 c_1 c_2 - 345600\, {a_1}^7 a_2 c_1 d_2 - 345600\, {a_1}^7 a_2 c_2 d_1 + 345600\, {a_1}^7 a_2 d_1 d_2 - 345600\, {a_1}^7 a_3 c_1 d_1 + 864000\, {a_1}^6 {a_2}^2 c_1 d_1) x^4 - 24\, {a_1}^2 (960\, {a_1}^7 {c_1}^7 - 960\, {a_1}^7 {d_1}^7 + 691200\, t^2 {a_1}^{13} c_1 - 691200\, t^2 {a_1}^{13} d_1 - 6720\, {a_1}^7 {c_1}^6 d_1 + 20160\, {a_1}^7 {c_1}^5 {d_1}^2 - 33600\, {a_1}^7 {c_1}^4 {d_1}^3 + 33600\, {a_1}^7 {c_1}^3 {d_1}^4 - 20160\, {a_1}^7 {c_1}^2 {d_1}^5 + 6720\, {a_1}^7 c_1 {d_1}^6 + 172800\, {a_1}^7 c_1 {c_2}^2 + 172800\, {a_1}^7 c_1 {d_2}^2 - 172800\, {a_1}^7 {c_2}^2 d_1 - 172800\, {a_1}^7 d_1 {d_2}^2 -$

$172800\,a_1^7c_1^2c_3+172800\,a_1^7c_1^2d_3-172800\,a_1^7c_3d_1^2+172800\,a_1^7d_1^2d_3+172800\,a_1^6a_3c_1^3-172800\,a_1^6a_3d_1^3-345600\,a_1^5a_2^2c_1^3+$
$345600\,a_1^5a_2^2d_1^3-691200\,ta_1^{10}c_1c_2+691200\,ta_1^{10}c_1d_2+691200\,ta_1^{10}c_2d_1-691200\,ta_1^{10}d_1d_2+691200\,ta_1^9a_2c_1^2+$
$691200\,ta_1^9a_2d_1^2+345600\,a_1^7c_1c_3d_1-345600\,a_1^7c_1d_1d_3+172800\,a_1^6a_2c_1^2c_2-172800\,a_1^6a_2c_1^2d_2+172800\,a_1^6a_2c_2d_1^2-$
$172800\,a_1^6a_2d_1^2d_2-518400\,a_1^6a_3c_1^2d_1+518400\,a_1^6a_3c_1d_1^2+1036800\,a_1^5a_2^2c_1^2d_1-1036800\,a_1^5a_2^2c_1d_1^2-345600\,a_1^7c_1c_2d_2+$
$345600\,a_1^7c_2d_1d_2-345600\,a_1^6a_2c_1c_2d_1+345600\,a_1^6a_2c_1d_1d_2-1382400\,ta_1^9a_2c_1d_1)x^3-24\,a_1^2(180\,a_1^6c_1^8+180\,a_1^6d_1^8+$
$518400\,t^2a_1^{12}c_1^2+518400\,t^2a_1^{12}d_1^2-1440\,a_1^6c_1^7d_1+5040\,a_1^6c_1^6d_1^2-10080\,a_1^6c_1^5d_1^3+12600\,a_1^6c_1^4d_1^4-10080\,a_1^6c_1^3d_1^5+$
$5040\,a_1^6c_1^2d_1^6-1440\,a_1^6c_1d_1^7-86400\,a_1^6c_1^3c_3+86400\,a_1^6c_1^3d_3+86400\,a_1^6c_3d_1^3-86400\,a_1^6d_1^3d_3+86400\,a_1^5a_3c_1^4+$
$86400\,a_1^5a_3d_1^4-129600\,a_1^4a_2^2c_1^4-129600\,a_1^4a_2^2d_1^4+129600\,a_1^6c_1^2c_2^2+129600\,a_1^6c_1^2d_2^2+129600\,a_1^6c_2^2d_1^2+$
$129600\,a_1^6d_1^2d_2^2-259200\,a_1^6c_1c_2^2d_1-259200\,a_1^6c_1d_1d_2^2-259200\,a_1^6c_2d_1^2d_2-1036800\,t^2a_1^{12}c_1d_1-518400\,ta_1^9c_1^2c_2+$
$518400\,ta_1^9c_1^2d_2-518400\,ta_1^9c_2d_1^2+518400\,ta_1^9d_1^2d_2+518400\,ta_1^8a_2c_1^3-518400\,ta_1^8a_2d_1^3+259200\,a_1^6c_1^2c_3d_1-$
$259200\,a_1^6c_1^2d_1d_3-259200\,a_1^6c_1c_3d_1^2+259200\,a_1^6c_1d_1^2d_3-345600\,a_1^5a_3c_1^3d_1+518400\,a_1^5a_3c_1^2d_1^2-345600\,a_1^5a_3c_1d_1^3+$
$518400\,a_1^4a_2^2c_1^3d_1-777600\,a_1^4a_2^2c_1^2d_1^2+518400\,a_1^4a_2^2c_1d_1^3-259200\,a_1^6c_1^2c_2d_2+518400\,a_1^6c_1c_2d_1d_2+1036800\,ta_1^9c_1c_2d_1-$
$1036800\,ta_1^9c_1d_1d_2-1555200\,ta_1^8a_2c_1^2d_1+1555200\,ta_1^8a_2c_1d_1^2)x^2-24\,a_1^2(-86400\,a_1^5c_2^3+86400\,a_1^5d_2^3+691200\,t^3a_1^{14}+$
$20\,a_1^5c_1^9-20\,a_1^5d_1^9+518400\,ta_1^8c_1^2c_2d_1-518400\,ta_1^8c_1^2d_1d_2-518400\,ta_1^8c_1c_2d_1^2+518400\,ta_1^8c_1d_1^2d_2-691200\,ta_1^7a_2c_1^3d_1+$
$1036800\,ta_1^7a_2c_1^2d_1^2-691200\,ta_1^7a_2c_1d_1^3-1036800\,ta_1^7a_2c_1c_2+1036800\,ta_1^7a_2c_1d_2+1036800\,ta_1^7a_2c_2d_1-1036800\,ta_1^7a_2d_1d_2+$
$172800\,t^2a_1^{11}c_1^3-172800\,t^2a_1^{11}d_1^3+259200\,a_1^5c_2^2d_2-259200\,a_1^5c_2d_2^2+86400\,a_1^2a_2^3c_1^3-86400\,a_1^2a_2^3d_1^3-1036800\,t^2a_1^{11}c_2+$
$1036800\,t^2a_1^{11}d_2-180\,a_1^5c_1^8d_1+720\,a_1^5c_1^7d_1^2-1680\,a_1^5c_1^6d_1^3+2520\,a_1^5c_1^5d_1^4-2520\,a_1^5c_1^4d_1^5+1680\,a_1^5c_1^3d_1^6-$
$720\,a_1^5c_1^2d_1^7+180\,a_1^5c_1d_1^8+518400\,ta_1^8c_2^2+518400\,ta_1^8d_2^2-21600\,a_1^5c_1^4c_3+21600\,a_1^5c_1^4d_3-21600\,a_1^5c_3d_1^4+$
$21600\,a_1^5d_1^4d_3+21600\,a_1^4a_3c_1^5-21600\,a_1^4a_3d_1^5-21600\,a_1^3a_2^2c_1^5+21600\,a_1^3a_2^2d_1^5+43200\,a_1^5c_1^3c_2^2+43200\,a_1^5c_1^3d_2^2-$
$43200\,a_1^5c_2^2d_1^3-43200\,a_1^5d_1^3d_2^2-86400\,a_1^5c_1^3c_2d_2-129600\,a_1^5c_1^2c_2^2d_1-129600\,a_1^5c_1^2d_1d_2^2+129600\,a_1^5c_1c_2^2d_1^2+$
$129600\,a_1^5c_1d_1^2d_2^2+86400\,a_1^5c_2d_1^3d_2-518400\,t^2a_1^{11}c_1^2d_1+518400\,t^2a_1^{11}c_1d_1^2-172800\,ta_1^8c_1^3c_2+172800\,ta_1^8c_1^3d_2+$
$172800\,ta_1^8c_2d_1^3-172800\,ta_1^8d_1^3d_2+172800\,ta_1^7a_2c_1^4+172800\,ta_1^7a_2d_1^4-1036800\,ta_1^8c_2d_2+518400\,ta_1^6a_2^2c_1^2+$
$518400\,ta_1^6a_2^2d_1^2+86400\,a_1^5c_1^3c_3d_1-86400\,a_1^5c_1^3d_1d_3-129600\,a_1^5c_1^2c_3d_1^2+129600\,a_1^5c_1^2d_1^2d_3+86400\,a_1^5c_1c_3d_1^3-$
$86400\,a_1^5c_1d_1^3d_3-21600\,a_1^4a_2c_1^4c_2+21600\,a_1^4a_2c_1^4d_2-21600\,a_1^4a_2c_2d_1^4+21600\,a_1^4a_2d_1^4d_2-108000\,a_1^4a_3c_1^4d_1+$
$216000\,a_1^4a_3c_1^3d_1^2-216000\,a_1^4a_3c_1^2d_1^3+108000\,a_1^4a_3c_1d_1^4+108000\,a_1^3a_2^2c_1^4d_1-216000\,a_1^3a_2^2c_1^3d_1^2+216000\,a_1^3a_2^2c_1^2d_1^3-$
$108000\,a_1^3a_2^2c_1d_1^4+259200\,a_1^4a_2c_1c_2^2+259200\,a_1^4a_2c_1d_2^2-259200\,a_1^4a_2c_2^2d_1-259200\,a_1^4a_2d_1d_2^2-259200\,a_1^3a_2^2c_1^2c_2+$
$259200\,a_1^3a_2^2c_1^2d_2-259200\,a_1^3a_2^2c_2d_1^2+259200\,a_1^3a_2^2d_1^2d_2-259200\,a_1^2a_2^3c_1^2d_1+259200\,a_1^2a_2^3c_1d_1^2+1036800\,t^2a_1^{10}a_2c_1-$
$1036800\,t^2a_1^{10}a_2d_1+259200\,a_1^5c_1^2c_2d_1d_2-259200\,a_1^5c_1c_2d_1^2d_2+86400\,a_1^4a_2c_1^3c_2d_1-86400\,a_1^4a_2c_1^3d_1d_2-129600\,a_1^4a_2c_1^2c_2d_1^2+$
$129600\,a_1^4a_2c_1^2d_1^2d_2+86400\,a_1^4a_2c_1c_2d_1^3-86400\,a_1^4a_2c_1d_1^3d_2-518400\,a_1^4a_2c_1c_2d_2+518400\,a_1^4a_2c_2d_1d_2+518400\,a_1^3a_2^2c_1c_2d_1-$
$518400\,a_1^3a_2^2c_1d_1d_2-1036800\,ta_1^6a_2^2c_1d_1)x-24\,a_1^2(518400\,a_1^3a_2c_1c_2d_1d_2+1036800\,ta_1^6a_2c_1c_2d_1-1036800\,ta_1^6a_2c_1d_1d_2+$
$518400\,a_1^2a_2a_3c_1c_2-518400\,a_1^2a_2a_3c_1d_2-518400\,a_1^2a_2a_3c_2d_1+518400\,a_1^2a_2a_3d_1d_2+1036800\,a_1a_2^2a_3c_1d_1+43200\,a_1^4c_1^3c_2d_1d_2-$
$64800\,a_1^4c_1^2c_2d_1^2d_2+43200\,a_1^4c_1c_2d_1^3d_2-259200\,a_1^3a_2c_1^2c_2d_2-259200\,a_1^3a_2c_1c_2^2d_1-259200\,a_1^3a_2c_1d_1d_2^2-259200\,a_1^3a_2c_2d_1^2d_2-$
$86400\,ta_1^7c_1^3d_1d_2-129600\,ta_1^7c_1^2c_2d_1^2+129600\,ta_1^7c_1^2d_1^2d_2+86400\,ta_1^7c_1c_2d_1^3-86400\,ta_1^7c_1d_1^3d_2-108000\,ta_1^6a_2c_1^4d_1+$
$216000\,ta_1^6a_2c_1^3d_1^2-216000\,ta_1^6a_2c_1^2d_1^3+108000\,ta_1^6a_2c_1d_1^4-777600\,ta_1^5a_2^2c_1^2d_1+777600\,ta_1^5a_2^2c_1d_1^2+86400\,ta_1^7c_1^3c_2d_1-$
$518400\,ta_1^7c_1c_2d_2+518400\,ta_1^7c_2d_1d_2-518400\,ta_1^6a_2c_1^2c_2+518400\,ta_1^6a_2c_1^2d_2-518400\,ta_1^6a_2c_2d_1^2+518400\,ta_1^6a_2d_1^2d_2+$
$21600\,a_1^3a_2c_1^4c_2d_1-21600\,a_1^3a_2c_1^4d_1d_2-43200\,a_1^3a_2c_1^3c_2d_1^2+43200\,a_1^3a_2c_1^3d_1^2d_2+43200\,a_1^3a_2c_1^2c_2d_1^3-43200\,a_1^3a_2c_1^2d_1^3d_2-$
$21600\,a_1^3a_2c_1c_2d_1^4+21600\,a_1^3a_2c_1d_1^4d_2+388800\,a_1^2a_2^2c_1^2c_2d_1-388800\,a_1^2a_2^2c_1^2d_1d_2-388800\,a_1^2a_2^2c_1c_2d_1^2+$
$388800\,a_1^2a_2^2c_1d_1^2d_2-1036800\,t^2a_1^9a_2c_1d_1+518400\,a_1^3a_2c_3d_2-518400\,a_1^3a_2d_2d_3-172800\,a_1^3a_3c_1c_3+172800\,a_1^3a_3c_1d_3+$
$172800\,a_1^3a_3c_3d_1-172800\,a_1^3a_3d_1d_3+518400\,a_1^2a_2^2c_1c_3-518400\,a_1^2a_2^2c_1d_3-1555200\,a_1^2a_2^2c_2d_2-518400\,a_1^2a_2^2c_3d_1+$
$518400\,a_1^2a_2^2d_1d_3-172800\,a_1^2a_3^2c_1d_1-1555200\,a_1a_2^3c_1c_2+1555200\,a_1a_2^3c_1d_2+1555200\,a_1a_2^3c_2d_1-1555200\,a_1a_2^3d_1d_2-$
$518400\,a_1a_2^2a_3c_1^2-518400\,a_1a_2^2a_3d_1^2+259200\,ta_1^5a_2^2c_1^3-259200\,ta_1^5a_2^2d_1^3-10800\,a_1^4c_1^4c_2d_2-21600\,a_1^4c_1^3c_2^2d_1-$
$21600\,a_1^4c_1^3d_1d_2^2+32400\,a_1^4c_1^2c_2^2d_1^2+32400\,a_1^4c_1^2d_1^2d_2^2-21600\,a_1^4c_1c_2^2d_1^3-21600\,a_1^4c_1d_1^3d_2^2-10800\,a_1^4c_2d_1^4d_2+$
$129600\,a_1^3a_2c_1^2c_2^2+129600\,a_1^3a_2c_1^2d_2^2+129600\,a_1^3a_2c_2^2d_1^2+129600\,a_1^3a_2d_1^2d_2^2-518400\,a_1^3a_2c_2c_3+518400\,a_1^3a_2c_2d_3-$
$21600\,ta_1^7c_1^4c_2+21600\,ta_1^7c_1^4d_2-21600\,ta_1^7c_2d_1^4+21600\,ta_1^7d_1^4d_2+21600\,ta_1^6a_2c_1^5-21600\,ta_1^6a_2d_1^5-172800\,a_1a_2^3c_1^3d_1+$
$259200\,a_1a_2^3c_1^2d_1^2-172800\,a_1a_2^3c_1d_1^3-86400\,t^2a_1^{10}c_1^3d_1+129600\,t^2a_1^{10}c_1^2d_1^2-86400\,t^2a_1^{10}c_1d_1^3+21600\,a_1^2a_2^2c_1^3d_1^3-$
$16200\,a_1^2a_2^2c_1^2d_1^4+6480\,a_1^2a_2^2c_1d_1^5+129600\,a_1^4c_1c_2^2d_2-129600\,a_1^4c_1c_2d_2^2-129600\,a_1^4c_2^2d_1d_2+129600\,a_1^4c_2d_1d_2^2-$
$518400\,t^2a_1^{10}c_1c_2+518400\,t^2a_1^{10}c_1d_2+518400\,t^2a_1^{10}c_2d_1-518400\,t^2a_1^{10}d_1d_2+518400\,t^2a_1^9a_2c_1^2+518400\,t^2a_1^9a_2d_1^2-$
$129600\,a_1^2a_2^2c_1^3c_2+129600\,a_1^2a_2^2c_1^3d_2+129600\,a_1^2a_2^2c_2d_1^3-129600\,a_1^2a_2^2d_1^3d_2+259200\,ta_1^7c_1c_2^2+259200\,ta_1^7c_1d_2^2-$
$259200\,ta_1^7c_2^2d_1-259200\,ta_1^7d_1d_2^2+10800\,a_1^4c_1^4c_3d_1-10800\,a_1^4c_1^4d_1d_3-21600\,a_1^4c_1^3c_3d_1^2+21600\,a_1^4c_1^3d_1^2d_3+$
$21600\,a_1^4c_1^2c_3d_1^3-21600\,a_1^4c_1^2d_1^3d_3-10800\,a_1^4c_1c_3d_1^4+10800\,a_1^4c_1d_1^4d_3-4320\,a_1^3a_2c_1^5c_2+4320\,a_1^3a_2c_1^5d_2+$
$4320\,a_1^3a_2c_2d_1^5-4320\,a_1^3a_2d_1^5d_2-12960\,a_1^3a_3c_1^5d_1+32400\,a_1^3a_3c_1^4d_1^2-43200\,a_1^3a_3c_1^3d_1^3+32400\,a_1^3a_3c_1^2d_1^4-$

$12960\,{a_1}^3 a_3 c_1 {d_1}^5 + 6480\,{a_1}^2 {a_2}^2 {c_1}^5 d_1 - 16200\,{a_1}^2 {a_2}^2 {c_1}^4 {d_1}^2 + 5400\,{a_1}^4 {c_1}^4 {c_2}^2 + 5400\,{a_1}^4 {c_1}^4 {d_2}^2 + 5400\,{a_1}^4 {c_2}^2 {d_1}^4 +$
$5400\,{a_1}^4 {d_1}^4 {d_2}^2 - 172800\,{a_1}^4 c_3 d_3 + 777600\,{a_1}^2 {a_2}^2 {c_2}^2 + 777600\,{a_1}^2 {a_2}^2 {d_2}^2 + 86400\,{a_1}^2 {a_3}^2 {c_1}^2 + 86400\,{a_1}^2 {a_3}^2 {d_1}^2 - 1555200\,{a_2}^4 c_1 d_1 +$
$45\,{a_1}^4 {c_1}^8 {d_1}^2 - 120\,{a_1}^4 {c_1}^7 {d_1}^3 + 210\,{a_1}^4 {c_1}^6 {d_1}^4 - 252\,{a_1}^4 {c_1}^5 {d_1}^5 + 210\,{a_1}^4 {c_1}^4 {d_1}^6 - 120\,{a_1}^4 {c_1}^3 {d_1}^7 + 45\,{a_1}^4 {c_1}^2 {d_1}^8 - 10\,{a_1}^4 c_1 {d_1}^9 -$
$2160\,{a_1}^4 {c_1}^5 c_3 + 2160\,{a_1}^4 {c_1}^5 d_3 + 2160\,{a_1}^4 c_3 {d_1}^5 - 2160\,{a_1}^4 {d_1}^5 d_3 + 2160\,{a_1}^3 a_3 {c_1}^6 + 2160\,{a_1}^3 a_3 {d_1}^6 - 1080\,{a_1}^2 {a_2}^2 {c_1}^6 -$
$1080\,{a_1}^2 {a_2}^2 {d_1}^6 - 43200\,{a_1}^4 c_1 {c_2}^3 + 43200\,{a_1}^4 c_1 {d_2}^3 + 43200\,{a_1}^4 {c_2}^3 d_1 - 43200\,{a_1}^4 d_1 {d_2}^3 + 345600\,t^3 {a_1}^{13} c_1 - 345600\,t^3 {a_1}^{13} d_1 -$
$10\,{a_1}^4 {c_1}^9 d_1 + 43200\,a_1 {a_2}^3 {c_1}^4 + 43200\,a_1 {a_2}^3 {d_1}^4 + 21600\,t^2 {a_1}^{10} {c_1}^4 + 21600\,t^2 {a_1}^{10} {d_1}^4 + {a_1}^4 {c_1}^{10} + {a_1}^4 {d_1}^{10} + 86400\,{a_1}^4 {c_3}^2 +$
$86400\,{a_1}^4 {d_3}^2 + 777600\,{a_2}^4 {c_1}^2 + 777600\,{a_2}^4 {d_1}^2),$

$d(x,t) = -64\,{a_1}^8 x^6 + (192\,{a_1}^7 d_1 - 192\,{a_1}^7 c_1) x^5 + (-240\,{a_1}^6 {d_1}^2 - 240\,{a_1}^6 {c_1}^2 + 480\,{a_1}^6 c_1 d_1) x^4 + (-160\,{a_1}^5 {c_1}^3 + 480\,{a_1}^5 c_2 -$
$480\,{a_1}^5 d_2 + 160\,{a_1}^5 {d_1}^3 + 480\,{a_1}^5 {c_1}^2 d_1 - 960\,t{a_1}^8 + 480\,{a_1}^4 a_2 d_1 - 480\,{a_1}^4 a_2 c_1 - 480\,{a_1}^5 c_1 {d_1}^2) x^3 + (-1440\,t{a_1}^7 c_1 -$
$60\,{a_1}^4 {d_1}^4 - 720\,{a_1}^4 c_1 d_2 - 60\,{a_1}^4 {c_1}^4 + 1440\,{a_1}^3 a_2 c_1 d_1 + 720\,{a_1}^4 d_1 d_2 - 720\,{a_1}^4 c_2 d_1 - 720\,{a_1}^3 a_2 {d_1}^2 + 720\,{a_1}^4 c_1 c_2 +$
$240\,{a_1}^4 c_1 {d_1}^3 + 1440\,t{a_1}^7 d_1 + 240\,{a_1}^4 {c_1}^3 d_1 - 360\,{a_1}^4 {c_1}^2 {d_1}^2 - 720\,{a_1}^3 a_2 {c_1}^2) x^2 + (4320\,a_1 {a_2}^2 d_1 - 1440\,{a_1}^3 c_3 + 1440\,{a_1}^3 d_3 -$
$720\,{a_1}^6 {d_1}^2 t - 720\,{a_1}^6 {c_1}^2 t - 4320\,a_1 {a_2}^2 c_1 + 12\,{a_1}^3 {d_1}^5 - 12\,{a_1}^3 {c_1}^5 + 360\,a_2 {a_1}^2 {d_1}^3 - 4320\,a_2 {a_1}^2 d_2 + 4320\,a_2 {a_1}^2 c_2 -$
$360\,a_2 {a_1}^2 {c_1}^3 - 360\,{a_1}^3 {d_1}^2 d_2 + 360\,{a_1}^3 {d_1}^2 c_2 - 120\,{a_1}^3 {d_1}^2 {c_1}^3 + 120\,{a_1}^3 {c_1}^2 {d_1}^3 - 360\,{a_1}^3 {c_1}^2 d_2 + 360\,{a_1}^3 {c_1}^2 c_2 + 1440\,a_3 {a_1}^2 c_1 -$
$1440\,a_3 {a_1}^2 d_1 - 60\,{a_1}^3 {d_1}^4 c_1 + 60\,{a_1}^3 {c_1}^4 d_1 + 1080\,{a_1}^2 {c_1}^2 a_2 d_1 + 720\,{a_1}^3 d_1 d_2 c_1 - 720\,{a_1}^3 c_1 c_2 d_1 + 1440\,t{a_1}^6 c_1 d_1 - 1080\,{a_1}^2 {d_1}^2 a_2 c_1) x +$
$60\,{a_1}^2 {d_1}^3 d_2 + 720\,a_1 a_3 {d_1}^2 + 120\,t{a_1}^5 {d_1}^3 + 2880\,t^2 {a_1}^8 + 720\,a_1 a_2 d_1 d_2 + 2880\,t{a_1}^4 a_2 c_1 - 60\,a_1 a_2 {d_1}^4 - 2880\,t{a_1}^4 d_1 a_2 +$
$720\,a_1 a_3 {c_1}^2 - {d_1}^6 {a_1}^2 + 720\,a_1 a_2 c_1 c_2 - {c_1}^6 {a_1}^2 - 15\,{a_1}^2 {c_1}^2 {d_1}^4 - 15\,{a_1}^2 {c_1}^4 {d_1}^2 + 720\,d_3 {a_1}^2 c_1 + 720\,c_3 {a_1}^2 d_1 + 6\,{d_1}^5 {a_1}^2 c_1 +$
$6\,{c_1}^5 {a_1}^2 d_1 - 60\,c_2 {a_1}^2 {d_1}^3 - 1440\,c_2 {a_1}^2 d_2 + 20\,{d_1}^3 {a_1}^2 {c_1}^3 - 60\,d_2 {a_1}^2 {c_1}^3 - 2880\,t{a_1}^5 c_2 - 720\,{a_1}^2 c_1 c_3 + 60\,{a_1}^2 {c_1}^3 c_2 -$
$120\,t{a_1}^5 {c_1}^3 - 60\,a_1 a_2 {c_1}^4 + 2880\,t{a_1}^5 d_2 - 720\,{a_1}^2 d_1 d_3 - 1440\,{a_2}^2 {d_1}^2 + 720\,{a_1}^2 {c_2}^2 + 720\,{a_1}^2 {d_2}^2 + 2880\,d_1 {a_2}^2 c_1 + 180\,d_1 {a_1}^2 {c_1}^2 d_2 -$
$1440\,d_1 a_3 a_1 c_1 - 360\,a_1 {c_1}^2 a_2 {d_1}^2 + 180\,{a_1}^2 c_1 c_2 {d_1}^2 - 180\,{d_1}^2 d_2 {a_1}^2 c_1 - 180\,{c_1}^2 c_2 {a_1}^2 d_1 - 360\,{d_1}^2 t{a_1}^5 c_1 + 360\,{c_1}^2 t{a_1}^5 d_1 -$
$720\,c_2 a_1 a_2 d_1 + 240\,{d_1}^3 a_1 a_2 c_1 - 720\,d_2 a_1 a_2 c_1 + 240\,{c_1}^3 a_1 a_2 d_1 - 1440\,{a_2}^2 {c_1}^2$

*is a rational solution to the KdV equation (1.1), quotient of two polynomials with the numerator of order $10$ in $x$, $3$ in $t$, the denominator of degree $12$ in $x$, $4$ in $t$.*

## 2.6 Another Orders

For greater orders, the solutions become very complex and we cannot give them here. We only give solutions with parameters $a_j$ equal to $1$ and all other parameters equal to $0$.

**Order** $4$

**Proposition 2.4.** *The function $v$ defined by*

$$v(x,t) = -2\,\frac{n(x,t)}{d(x,t)^2}, \tag{2.11}$$

$n(x,t) = -10\,x^{18} - 360\,tx^{15} - 14175\,t^2x^{12} + 330750\,t^3x^9 + 5953500\,t^4x^6 - 22325625\,t^6,$

$d(x,t) = x^{10} + 45\,tx^7 + 4725\,t^3x$

*is a rational solution to the KdV equation (1.1), quotient of two polynomials with numerator of degree $18$ in $x$, $6$ in $t$, and denominator of degree $20$ in $x$, $6$ in $t$.*

**Order** $5$

**Proposition 2.5.** *The function $v$ defined by*

$$v(x,t) = -2\,\frac{n(x,t)}{d(x,t)^2}, \tag{2.12}$$

$n(x,t) = -15\,x^{28} - 1890\,tx^{25} - 113400\,t^2x^{22} - 84837375\,t^4x^{16} - 7501410000\,t^5x^{13} - 64699661250\,t^6x^{10} -$
$168781725000\,t^7x^7 - 4430520281250\,t^8x^4 + 8861040562500\,t^9x,$

$d(x,t) = -x^{15} - 105\,tx^{12} - 1575\,t^2x^9 - 33075\,t^3x^6 + 992250\,t^4x^3 + 1488375\,t^5$

*is a rational solution to the KdV equation (1.1), quotient of two polynomials with numerator of degree* $28$ *in* $x$, $9$ *in* $t$, *and denominator of degree* $30$ *in* $x$, $10$ *in* $t$.

**Order** $6$

**Proposition 2.6.** *The function* $v$ *defined by*

$$v(x,t) = -2\,\frac{n(x,t)}{d(x,t)^2}, \tag{2.13}$$

$n(x,t) = -21\,x^{40} - 6300\,tx^{37} - 793800\,t^2x^{34} - 39690000\,t^3x^{31} - 1909585125\,t^4x^{28} - 43320642750\,t^5x^{25} + 8174661547500\,t^6x^{22} + 376889591925000\,t^7x^{19} + 5541251715759375\,t^8x^{16} + 60042410851500000\,t^9x^{13} - 1339696292124093750\,t^{10}x^{10} + 2026431366238125000\,t^{11}x^7 - 8865637227291796875 0\,t^{12}x^4 - 1063876467275015625 00\,t^{13}x,$

$d(x,t) = -x^{21} - 210\,tx^{18} - 10395\,t^2x^{15} - 264600\,t^3x^{12} + 5457375\,t^4x^9 - 343814625\,t^5x^6 - 3438146250\,t^6x^3 + 5157219375\,t^7$

*is a rational solution to the KdV equation (1.1), quotient of two polynomials with numerator of degree* $40$ *in* $x$, $13$ *in* $t$, *and denominator of degree* $42$ *in* $x$, $14$ *in* $t$.

# 3 CONCLUSION

We have given three types of representations of solutions to the KdV equation. First, solutions in terms of elementary exponential functions have been constructed. In particular, performing a passage to the limit when one parameter goes to $0$ we get rational solutions to the KdV equation. We give another representation in terms of determinants without the presence of a limit. So we obtain an infinite hierarchy of multi-parametric families of rational solutions to the KdV equation as a quotient of a polynomials depending on $3N$ real parameters.

We can formulate some remarks about the structure of these solutions. For the N-order solution, the numerator of the solution is a polynomial of degree $N(N=1)-2$ in $x$ and the denominator a polynomial of degree $N(N+1)$ in $x$. The structure relative to $t$ seems more complicated. It would be relevant to study these polynomials in more details.

# COMPETING INTERESTS

Author has declared that no competing interests exist.

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