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Review

Some Applications of the Wright Function in Continuum Physics: A Survey

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Abstract: The Wright function is a generalization of the exponential function and the Bessel functions. Integral relations between the Mittag–Leffler functions and the Wright function are presented. The applications of the Wright function and the Mainardi function to description of diffusion, heat conduction, thermal and diffusive stresses, and nonlocal elasticity in the framework of fractional calculus are discussed.

Keywords: fractional calculus; Caputo derivative; Mittag–Leffler functions; Wright function; Mainardi function; Laplace transform; Fourier transform; nonperfect thermal contact; nonlocal elasticity; fractional nonlocal elasticity

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1. Introduction

The fractional calculus (the theory of integrals and derivatives of non-integer order) has attracted considerable interest of researchers and has many applications in physics, chemistry, rheology, geology, hydrology, medicine, engineering, finance, etc. (see, for example, West–Bologna–Grigolini [1], Magin [2], Povstenko [3], Tarasov [4], Povstenko [5], Uchaikin [6], Atanacković–Pilipović–Stanković–Zorica [7], Herrmann [8], Povstenko [9], Datsko–Gafiychuk–Podlubny [10], West [11], Skiadas [12], Tarasov [13], Kumar–Singh [14], Su [15] and references therein). The Mittag–Leffler functions and the Wright function appear in solutions of various types of equations with fractional operators. The Mittag–Leffler function in one parameter $E_\alpha(z)$ was introduced in [16,17]. The generalized Mittag–Leffler function in two parameters $E_{\alpha,\beta}(z)$ was considered in [18,19]. A comprehensive treatment of properties of the Mittag–Leffler functions can be found in Erdélyi–Magnus–Oberhettinger–Tricomi [20], Gorenflo–Mainardi [21], Podlubny [22], Kilbas–Srivastava–Trujillo [23], Gorenflo–Kilbas–Mainardi–Rogosin [24]. Numerical algorithms for calculation of the Mittag–Leffler functions were proposed in [25] and implemented in [26]. The Wright function was presented in [27,28] and later on discussed by Erdélyi–Magnus–Oberhettinger–Tricomi [20], Gorenflo–Mainardi [21], Podlubny [22], Kilbas–Srivastava–Trujillo [23], Gorenflo–Kilbas–Mainardi–Rogosin [24], Luchko [29], among others. Numerical algorithms for calculating the Wright function were suggested in [30].

In 1996, Mainardi [31,32] solved the diffusion-wave equation with the Caputo fractional derivative of the order α

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\frac{\partial^2 T}{\partial x^2}, \qquad 0 < \alpha \le 2, \tag{1}$$

on a real line (the Cauchy problem) and a half-line (the signaling problem). The solutions were obtained in terms of the Mainardi function $M\left(z; \frac{\alpha}{2}\right)$ [33], where

$$z = \frac{|x|}{\sqrt{a}t^{\alpha/2}} \tag{2}$$

is the similarity variable, a can be treated as the generalized thermal diffusivity coefficient.

Equation (1) in the limiting case $\alpha \to 0$ corresponds to the Helmholtz equation (localized diffusion); the subdiffusion regime is characterized by the values $0 < \alpha < 1$. For $1 < \alpha < 2$, the diffusion-wave Equation (1) interpolates between the diffusion equation ($\alpha = 1$) and the wave equation ($\alpha = 2$).

Applications of fractional calculus to viscoelasticity have been studied by many authors. The historical notes and the extensive bibliography on this subject can be found in the book of Mainardi [34]. According to the Scott–Blair stress-strain law, the dependence between the stress $\sigma(x,t)$ and the strain $\epsilon(x,t)$ can be written as [34,35]

$$\sigma(x,t) = \rho a \frac{\partial^\nu \epsilon(x,t)}{\partial t^\nu}, \qquad 0 \le \nu \le 1. \tag{3}$$

The constitutive Equation (3) characterizes a viscoelastic material intermediate between a perfectly elastic solid (the Hooke law for the value $\nu = 0$) and a perfectly viscous fluid (the Newton law when $\nu = 1$) with the corresponding interpretations of the coefficient a in terms of the elasticity constant or the kinematic viscosity. The relation (3) leads to the evolution Equation (1) with $\alpha = 2 - \nu$.

The book [36] presents a picture of the state-of-the-art for solutions of the diffusion-wave equation with one, two, and three space variables in Cartesian, cylindrical, and spherical coordinates under different kinds of boundary conditions.

In the present survey article, we briefly discuss the properties of the Mittag–Leffler functions and Wright function and present the integral relations between the Mittag–Leffler functions and the Wright function. The applications of the Wright function and the Mainardi function to the description of diffusion, heat conduction, thermal and diffusive stresses, and nonlocal elasticity in the framework of fractional calculus are reviewed.

2. Mathematical Preliminaries

2.1. Integrals and Derivatives of Fractional Order

The Riemann–Liouville integral of fractional order α is defined as [21–23]:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, \mathrm{d}\tau, \qquad \alpha > 0, \tag{4}$$

where $\Gamma(\alpha)$ is the gamma function.

The Riemann–Liouville derivative of fractional order α has the form

$$D^\alpha_{RL} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left[\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, \mathrm{d}\tau \right], \qquad n-1 < \alpha < n, \tag{5}$$

whereas the Caputo fractional derivative is written as

$$D^\alpha_C f(t) \equiv \frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\, \mathrm{d}\tau, \qquad n-1 < \alpha < n. \tag{6}$$

The fractional operators have the following Laplace transform rules:

$$\mathcal{L}\{I^\alpha f(t)\} = \frac{1}{s^\alpha} f^*(s), \tag{7}$$

$$\mathcal{L}\{\, D^\alpha_{RL} f(t)\} = s^\alpha f^*(s) - \sum_{k=0}^{n-1} D^k I^{n-\alpha} f(0^+) s^{n-1-k}, \qquad n-1 < \alpha < n, \tag{8}$$

$$\mathcal{L}\left\{ \frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha} \right\} = s^\alpha f^*(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}, \qquad n-1 < \alpha < n. \tag{9}$$

Here, the asterisk denotes the transform, and s is the Laplace transform variable.

2.2. Mittag–Leffler Functions

The Mittag–Leffler function in one parameter α

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad \alpha > 0,\ z \in C, \tag{10}$$

can be considered as the extension of the exponential function $e^z = E_1(z)$, whereas the generalized Mittag–Leffler function in two parameters α and β is defined by the series representation

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha > 0,\ \beta > 0,\ z \in C. \tag{11}$$

In the general case, the parameters α and β can be treated as complex numbers with some limitations on their real parts [24], but we restrict ourselves to positive values of α and β.

The following recurrence relations [20,24]

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z). \tag{12}$$

$$E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{\mathrm{d}E_{\alpha,\beta+1}(z)}{\mathrm{d}z} \tag{13}$$

are valid for the Mittag–Leffler functions.

For investigation of the convergence of integrals containing the Mittag–Leffler functions, their asymtotic representations for large negative values of argument are useful. For $x \to \infty$, we have

$$E_\alpha(-x) \sim \frac{1}{\Gamma(1-\alpha)x}, \tag{14}$$

$$E_{\alpha,2}(-x) \sim \frac{1}{\Gamma(2-\alpha)x}, \tag{15}$$

$$E_{\alpha,\alpha}(-x) \sim -\frac{1}{\Gamma(-\alpha)x^2}, \tag{16}$$

$$E_{\alpha,\beta}(-x) \sim \frac{1}{\Gamma(\beta-\alpha)x}. \tag{17}$$

The essential role of the Mittag–Leffler functions in fractional calculus is connected with the formula for the inverse Laplace transform (see Gorenflo–Mainardi [21], Podlubny [22], Kilbas–Srivastava–Trujillo [23], Gorenflo–Kilbas–Mainardi–Rogosin [24]):

$$\mathcal{L}^{-1}\left\{ \frac{s^{\alpha-\beta}}{s^\alpha + b} \right\} = t^{\beta-1} E_{\alpha,\beta}(-bt^\alpha). \tag{18}$$

2.3. Wright Function and Mainardi Function

The Wright function is a generalization of the exponential function and the Bessel functions and is defined as [27,28] (see also refs. [20–24,31,32,37–39])

$$W(\alpha,\beta;z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(\alpha k + \beta)}, \qquad \alpha > -1, \quad \beta \in C, \quad z \in C. \tag{19}$$

The Wright function satisfies the recurrence equations [20]

$$\alpha z W(\alpha,\alpha+\beta;z) = W(\alpha,\beta-1;z) + (1-\beta)W(\alpha,\beta;z), \tag{20}$$

$$\frac{\mathrm{d}W(\alpha,\beta;z)}{\mathrm{d}z} = W(\alpha,\alpha+\beta;z). \tag{21}$$

The Mainardi function $M(\alpha;z)$ [22,31–33] is a particular case of the Wright function

$$M(\alpha;z) = W(-\alpha, 1-\alpha; -z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\,\Gamma[-\alpha k + (1-\alpha)]}, \qquad 0 < \alpha < 1, \quad z \in C. \tag{22}$$

The Wright function and the Mainardi function appear in formulae for the inverse Laplace transform (see Mainardi [31,32], Stanković [40], Gajić–Stanković [41]):

$$\mathcal{L}^{-1}\{\exp(-\lambda s^{\alpha})\} = \frac{\alpha\lambda}{t^{\alpha+1}} M\big(\alpha; \lambda t^{-\alpha}\big), \quad 0 < \alpha < 1,\ \lambda > 0, \tag{23}$$

$$\mathcal{L}^{-1}\Big\{s^{\alpha-1}\exp(-\lambda s^{\alpha})\Big\} = \frac{1}{t^{\alpha}} M\big(\alpha; \lambda t^{-\alpha}\big), \quad 0 < \alpha < 1,\ \lambda > 0, \tag{24}$$

$$\mathcal{L}^{-1}\Big\{s^{-\beta}\exp(-\lambda s^{\alpha})\Big\} = t^{\beta-1} W\big(-\alpha, \beta; -\lambda t^{-\alpha}\big), \quad 0 < \alpha < 1,\ \lambda > 0. \tag{25}$$

2.4. The Integral Transform Relations between the Mittag–Leffler Function and Wright Function

The Laplace transform of the Wright function is expressed in terms of the Mittag–Leffler function [20,22,23]

$$\mathcal{L}\{W(\alpha,\beta;t)\} = \frac{1}{s} E_{\alpha,\beta}\left(\frac{1}{s}\right), \quad \alpha > 0,\ \beta > 0, \tag{26}$$

and [37]

$$\mathcal{L}\{W(\alpha,\beta;-t)\} = E_{-\alpha,\beta-\alpha}(-s), \quad -1 < \alpha < 0,\ \beta > 0, \tag{27}$$

whereas, for the Mainardi function, the corresponding relation takes the form

$$\mathcal{L}\{M(\alpha;t)\} = E_{\alpha}(-s), \quad 0 < \alpha < 1. \tag{28}$$

The Mittag–Leffler functions and the Wright function are related by the Fourier cosine transform (Povstenko [36,42]):

$$\int_0^{\infty} E_{\alpha}(-\xi^2)\cos(x\xi)\,\mathrm{d}\xi = \frac{\pi}{2} M\Big(\frac{\alpha}{2};x\Big), \qquad 0 < \alpha < 2,\ x > 0, \tag{29}$$

$$\int_0^{\infty} E_{\alpha,2}(-\xi^2)\cos(x\xi)\,\mathrm{d}\xi = \frac{\pi}{2} W\Big(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -x\Big), \quad 0 < \alpha < 2,\ x > 0, \tag{30}$$

$$\int_0^{\infty} E_{\alpha,\alpha}(-\xi^2)\cos(x\xi)\,\mathrm{d}\xi = \frac{\pi}{2} W\Big(-\frac{\alpha}{2}, \frac{\alpha}{2}; -x\Big), \quad 0 < \alpha < 2,\ x > 0, \tag{31}$$

$$\int_0^{\infty} E_{\alpha,\beta}(-\xi^2)\cos(x\xi)\,\mathrm{d}\xi = \frac{\pi}{2} W\Big(-\frac{\alpha}{2}, \beta-\frac{\alpha}{2}; -x\Big), \quad 0 < \alpha < 2,\ \beta > 0,\ x > 0, \tag{32}$$

as well as by the Fourier sine transform

$$\int_0^{\infty} \xi\, E_{\alpha}(-\xi^2)\sin(x\xi)\,\mathrm{d}\xi = \frac{\pi}{2} W\Big(-\frac{\alpha}{2}, 1-\alpha; -x\Big), \qquad 0 < \alpha < 2,\ x > 0, \tag{33}$$

$$\int_0^{\infty} \xi\, E_{\alpha,2}(-\xi^2)\sin(x\xi)\,\mathrm{d}\xi = \frac{\pi}{2} W\Big(-\frac{\alpha}{2}, 2-\alpha; -x\Big), \quad 0 < \alpha < 2,\ x > 0, \tag{34}$$

$$\int_0^{\infty} \xi\, E_{\alpha,\alpha}(-\xi^2)\sin(x\xi)\,\mathrm{d}\xi = \frac{\pi}{2} W\Big(-\frac{\alpha}{2}, 0; -x\Big) = \frac{\alpha\pi}{4}\, x\, M\Big(\frac{\alpha}{2};x\Big), \quad 0 < \alpha < 2,\ x > 0, \tag{35}$$

$$\int_0^{\infty} \xi\, E_{\alpha,\beta}(-\xi^2)\sin(x\xi)\,\mathrm{d}\xi = \frac{\pi}{2} W\Big(-\frac{\alpha}{2}, \beta-\alpha; -x\Big), \quad 0 < \alpha < 2,\ \beta > 0\ , x > 0. \tag{36}$$

Due to (16), we can also obtain for $E_{\alpha,\alpha}\left(-\xi^2\right)$

$$\int_0^\infty \xi^2\, E_{\alpha,\alpha}(-\xi^2)\, \cos(x\xi)\, \mathrm{d}\xi = -\frac{\pi}{2}\, W\Big(-\frac{\alpha}{2}, -\frac{\alpha}{2}; -x\Big), \quad 0 < \alpha < 2, \;\; x > 0, \tag{37}$$

$$\int_0^\infty \xi^3\, E_{\alpha,\alpha}(-\xi^2)\, \sin(x\xi)\, \mathrm{d}\xi = -\frac{\pi}{2}\, W\Big(-\frac{\alpha}{2}, -\alpha; -x\Big), \quad 0 < \alpha < 2, \;\; x > 0. \tag{38}$$

The equations presented above allow us to obtain additional integral relations between the Mittag–Leffler functions and the Wright function, which can be helpful when solving problems in polar or cylindrical coordinates using the Hankel transform of order zero. Taking into account the integral representation of the Bessel function $J_0(x)$ (Watson [43], Abramowitz-Stegun [44])

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\mathrm{d}\theta, \tag{39}$$

$$J_0(x) = \frac{2}{\pi}\int_0^\infty \sin(x\cosh t)\mathrm{d}t, \quad x > 0, \tag{40}$$

$$J_0(x) = \frac{2}{\pi}\int_1^\infty \frac{\sin(xt)}{\sqrt{t^2-1}}\,\mathrm{d}t, \quad x > 0, \tag{41}$$

we get

$$\int_0^\infty E_\alpha\Big(-\xi^2\Big) J_0(r\xi)\mathrm{d}\xi = \frac{1}{2}\int_0^\pi M\Big(\frac{\alpha}{2}; r\sin\theta\Big)\mathrm{d}\theta, \quad 0 < \alpha \le 2, \;\; r > 0, \tag{42}$$

$$\int_0^\infty E_{\alpha,2}\Big(-\xi^2\Big) J_0(r\xi)\mathrm{d}\xi = \frac{1}{2}\int_0^\pi W\Big(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -r\sin\theta\Big)\mathrm{d}\theta, \quad 0 < \alpha \le 2, \;\; r > 0, \tag{43}$$

$$\int_0^\infty E_{\alpha,\alpha}\Big(-\xi^2\Big) J_0(r\xi)\mathrm{d}\xi = \frac{1}{2}\int_0^\pi W\Big(-\frac{\alpha}{2}, \frac{\alpha}{2}; -r\sin\theta\Big)\mathrm{d}\theta, \quad 0 < \alpha \le 2, \;\; r > 0, \tag{44}$$

$$\int_0^\infty E_{\alpha,\beta}\Big(-\xi^2\Big) J_0(r\xi)\mathrm{d}\xi = \frac{1}{2}\int_0^\pi W\Big(-\frac{\alpha}{2}, \beta-\frac{\alpha}{2}; -r\sin\theta\Big)\mathrm{d}\theta, \quad 0 < \alpha \le 2, \;\; r > 0. \tag{45}$$

Similarly,

$$\int_0^\infty E_\alpha\Big(-\xi^2\Big) J_0(r\xi)\xi\,\mathrm{d}\xi = \int_0^\infty W\Big(-\frac{\alpha}{2}, 1-\alpha; -r\cosh t\Big)\mathrm{d}t, \quad 0 < \alpha \le 2, \;\; r > 0, \tag{46}$$

$$\int_0^\infty E_{\alpha,2}\Big(-\xi^2\Big) J_0(r\xi)\xi\,\mathrm{d}\xi = \int_0^\infty W\Big(-\frac{\alpha}{2}, 2-\alpha; -r\cosh t\Big)\mathrm{d}t, \quad 0 < \alpha \le 2, \;\; r > 0, \tag{47}$$

$$\int_0^\infty E_{\alpha,\alpha}\Big(-\xi^2\Big) J_0(r\xi)\xi\,\mathrm{d}\xi = \int_0^\infty W\Big(-\frac{\alpha}{2}, 0; -r\cosh t\Big)\mathrm{d}t, \quad 0 < \alpha \le 2, \;\; r > 0, \tag{48}$$

$$\int_0^\infty E_{\alpha,\beta}\Big(-\xi^2\Big) J_0(r\xi)\xi\,\mathrm{d}\xi = \int_0^\infty W\Big(-\frac{\alpha}{2}, \beta-\alpha; -r\cosh t\Big)\mathrm{d}t, \quad 0 < \alpha \le 2, \;\; r > 0, \tag{49}$$

and

$$\int_0^\infty E_\alpha\Big(-\xi^2\Big) J_0(r\xi)\xi\,\mathrm{d}\xi = \int_1^\infty W\Big(-\frac{\alpha}{2}, 1-\alpha; -rt\Big)\frac{1}{\sqrt{t^2-1}}\,\mathrm{d}t, \quad 0 < \alpha \le 2, \;\; r > 0, \tag{50}$$

$$\int_0^\infty E_{\alpha,2}\Big(-\xi^2\Big) J_0(r\xi)\xi\,\mathrm{d}\xi = \int_1^\infty W\Big(-\frac{\alpha}{2}, 2-\alpha; -rt\Big)\frac{1}{\sqrt{t^2-1}}\,\mathrm{d}t, \quad 0 < \alpha \le 2, \;\; r > 0, \tag{51}$$

$$\int_0^\infty E_{\alpha,\alpha}\left(-\xi^2\right)J_0(r\xi)\xi\,\mathrm{d}\xi = \int_1^\infty W\left(-\frac{\alpha}{2},0;-rt\right)\frac{1}{\sqrt{t^2-1}}\,\mathrm{d}t$$
$$= \frac{\alpha r}{2}\int_0^\infty M\left(\frac{\alpha}{2};r\sqrt{1+u^2}\right)\mathrm{d}u, \quad 0<\alpha\le 2,\ r>0, \tag{52}$$

$$\int_0^\infty E_{\alpha,\beta}\left(-\xi^2\right)J_0(r\xi)\xi\,\mathrm{d}\xi = \int_1^\infty W\left(-\frac{\alpha}{2},\beta-\alpha;-rt\right)\frac{1}{\sqrt{t^2-1}}\,\mathrm{d}t, \quad 0<\alpha\le 2,\ r>0. \tag{53}$$

In addition,

$$\int_0^\infty E_{\alpha,\alpha}\left(-\xi^2\right)J_0(r\xi)\xi^2\,\mathrm{d}\xi = -\frac{1}{2}\int_0^\pi W\left(-\frac{\alpha}{2},-\frac{\alpha}{2};-r\sin\theta\right)\mathrm{d}\theta, \quad 0<\alpha\le 2,\ r>0, \tag{54}$$

$$\int_0^\infty E_{\alpha,\alpha}\left(-\xi^2\right)J_0(r\xi)\xi^3\,\mathrm{d}\xi = -\int_0^\infty W\left(-\frac{\alpha}{2},-\alpha;-r\cosh t\right)\mathrm{d}t$$
$$= -\int_1^\infty W\left(-\frac{\alpha}{2},-\alpha;-rt\right)\frac{1}{\sqrt{t^2-1}}\,\mathrm{d}t, \quad 0<\alpha\le 2,\ r>0. \tag{55}$$

3. Applications of the Wright Function

3.1. Fractional Heat Conduction in Nonhomogeneous Media under Perfect Thermal Contact

Time-fractional heat conduction in two joint half-lines was considered by Povstenko [36,45,46]. In the general case, the heat conduction equation with the Caputo derivative of the order $0<\alpha\le 2$ in one half-line

$$\frac{\partial^\alpha T_1}{\partial t^\alpha} = a_1\frac{\partial^2 T_1}{\partial x^2}, \quad x>0, \tag{56}$$

and the corresponding equation with the Caputo derivative of the order $0<\beta\le 2$ in another half-line

$$\frac{\partial^\beta T_2}{\partial t^\beta} = a_2\frac{\partial^2 T_2}{\partial x^2}, \quad x<0, \tag{57}$$

were treated under the boundary conditions of perfect thermal contact which state that two bodies must have the same temperature at the contact point and the heat fluxes through the contact point must be the same:

$$T_1(x,t)\Big|_{x=0^+} = T_2(x,t)\Big|_{x=0^-}, \tag{58}$$

$$k_1 D_{RL}^{1-\alpha}\frac{\partial T_1(x,t)}{\partial x}\bigg|_{x=0^+} = k_2 D_{RL}^{1-\beta}\frac{\partial T_2(x,t)}{\partial x}\bigg|_{x=0^-}, \quad 0<\alpha\le 2, \quad 0<\beta\le 2. \tag{59}$$

In the condition (59), k_1 and k_2 are the generalized thermal conductivities of two bodies; the Riemann–Liouville fractional derivative of the negative order $D_{RL}^{-\alpha}(f(t))$ is understood as the Riemann–Liouville fractional integral $I^\alpha(f(t))$.

Here, we present the fundamental solution to the first Cauchy problem with the initial condition

$$t=0:\ T_1 = p_0\,\delta(x-\varrho), \quad x>0, \quad \varrho>0, \tag{60}$$

for the case $\alpha=\beta$ (for details see Povstenko [46]):

$$T_1(x,t) = \frac{p_0}{2\sqrt{a_1}t^{\alpha/2}}\left[M\left(\frac{\alpha}{2};\frac{|x-\varrho|}{\sqrt{a_1}t^{\alpha/2}}\right) + \frac{\varepsilon-1}{\varepsilon+1}M\left(\frac{\alpha}{2};\frac{x+\varrho}{\sqrt{a_1}t^{\alpha/2}}\right)\right], \quad x\ge 0, \tag{61}$$

$$T_2(x,t) = \frac{\varepsilon p_0}{(\varepsilon+1)\sqrt{a_1}t^{\alpha/2}}M\left(\frac{\alpha}{2};\frac{|x|}{\sqrt{a_2}t^{\alpha/2}} + \frac{\rho}{\sqrt{a_1}t^{\alpha/2}}\right), \quad x\le 0, \tag{62}$$

where

$$\varepsilon = \frac{k_1\sqrt{a_2}}{k_2\sqrt{a_1}}. \tag{63}$$

For the corresponding problem with uniform initial temperature T_0 in one of half-lines [45], in the particular case $\alpha = \beta$, we have:

$$T_1 = T_0 - \frac{T_0}{(1+\varepsilon)}\, W\left(-\frac{\alpha}{2}, 1; -\frac{x}{\sqrt{a_1}t^{\alpha/2}}\right), \qquad x > 0, \tag{64}$$

$$T_2 = \frac{\varepsilon T_0}{(1+\varepsilon)}\, W\left(-\frac{\alpha}{2}, 1; -\frac{|x|}{\sqrt{a_2}t^{\alpha/2}}\right), \qquad x < 0. \tag{65}$$

The time-fractional heat conduction equations with the Caputo derivatives in a semi-infinite medium composed of a region $0 < x < L$ and a region $L < x < \infty$ under the boundary conditions of perfect thermal contact at $x = L$ and the insulated boundary condition at $x = 0$ with uniform initial temperature in a layer were investigated in [47]. The approximate solution of the considered problem for small values of time is obtained based on Tauberian theorems for the Laplace transform. For $\alpha = \beta$, this solution reads

$$T_1 \simeq T_0 - \frac{T_0}{1+\varepsilon}\, W\left(-\frac{\alpha}{2}, 1; -\frac{L-x}{\sqrt{a_1}t^{\alpha/2}}\right), \quad 0 \le x \le L, \tag{66}$$

$$T_2 \simeq \frac{\varepsilon T_0}{1+\varepsilon}\, W\left(-\frac{\alpha}{2}, 1; -\frac{x-L}{\sqrt{a_2}t^{\alpha/2}}\right), \quad L \le x < \infty. \tag{67}$$

Fractional heat conduction in an infinite medium with a spherical inclusion when a sphere $0 \le r < R$ is at the initial uniform temperature T_0 and a matrix $R < r < \infty$ is at a zero initial temperature was considered by Povstenko [36,48]. In the case of perfect thermal contact at the boundary $r = R$,

$$r = R: \quad T_1(r,t) = T_2(r,t), \tag{68}$$

$$k_1 D_{RL}^{1-\alpha}\frac{\partial T_1(r,t)}{\partial r} = k_2 D_{RL}^{1-\beta}\frac{\partial T_2(r,t)}{\partial r}, \quad 0 < \alpha \le 2, \quad 0 < \beta \le 2, \tag{69}$$

the approximate solution for small values of time has the following form (we present only the solution for $\alpha = \beta$):

$$\begin{aligned} T_1(r,t) \;\simeq\; & T_0 - \frac{RT_0k_2}{(k_2-k_1)r}\left[W\left(-\frac{\alpha}{2}, 1; -\frac{R-r}{\sqrt{a_1}t^{\alpha/2}}\right) - W\left(-\frac{\alpha}{2}, 1; -\frac{R+r}{\sqrt{a_1}t^{\alpha/2}}\right)\right] \\ & + \frac{cRT_0}{r}\int_0^t \frac{(t-\tau)^{\alpha/2-1}}{\tau^{\alpha/2}}\left[M\left(\frac{\alpha}{2}; \frac{R-r}{\sqrt{a_1}\tau^{\alpha/2}}\right)\right. \\ & \left. - M\left(\frac{\alpha}{2}; \frac{R+r}{\sqrt{a_1}\tau^{\alpha/2}}\right)\right] E_{\alpha/2,\alpha/2}\left[-b(t-\tau)^{\alpha/2}\right] \mathrm{d}\tau, \end{aligned} \tag{70}$$

$$\begin{aligned} T_2(r,t) \simeq & -\frac{RT_0k_1}{(k_2-k_1)r}\, W\left(-\frac{\alpha}{2}, 1; -\frac{r-R}{\sqrt{a_2}t^{\alpha/2}}\right) + \frac{cRT_0}{r}\int_0^t \frac{(t-\tau)^{\alpha/2-1}}{\tau^{\alpha/2}} \\ & \times M\left(\frac{\alpha}{2}; \frac{r-R}{\sqrt{a_2}\tau^{\alpha/2}}\right) E_{\alpha/2,\alpha/2}\left[-b(t-\tau)^{\alpha/2}\right] \mathrm{d}\tau, \end{aligned} \tag{71}$$

where

$$b = \frac{(k_2-k_1)\sqrt{a_1a_2}}{R(k_1\sqrt{a_1}+k_2\sqrt{a_2})}, \qquad c = \frac{k_1k_2(\sqrt{a_1}+\sqrt{a_2})}{(k_2-k_1)\,(k_1\sqrt{a_1}+k_2\sqrt{a_2})}. \tag{72}$$

It should be mentioned that, for the classical heat conduction, the method of analysis of the solution for small values of time was described by Luikov [49] and Özişik [50]. In the case of fractional diffusion equation, the decay rate at large values of time was analyzed by Sakamoto–Yamamoto [51].

3.2. Fractional Heat Conduction in Nonhomogeneous Media under Nonperfect Thermal Contact

Near the interface between two solids, a transition region arises whose state differs from the state of contacting media owing to different conditions of material–particle interaction. The transition region has its own physical, mechanical, and chemical properties, and processes occurring in it differ from those in the bulk. Small thickness of the intermediate region between two solids allows us to reduce a three-dimensional problem to a two-dimensional one for median surface endowed with equivalent physical properties. There are several approaches to reducing three-dimensional equations to the corresponding two-dimensional equations for the median surface. For example, introducing the mixed coordinate system (ξ, η, z), where ξ and η are the curvilinear coordinates in the median surface and z is the normal coordinate, the linear or polynomial dependence of the considered functions on the normal coordinate can be assumed. This assumption is often used in the theory of elastic shells.

For the classical heat conduction equation, which is based on the conventional Fourier law, the reduction of the three-dimensional problem to the simplified two-dimensional one was pioneered by Marguerre [52,53] and later on developed by many authors. In this case, the assumption of linear or polynomial dependence of temperature on the normal coordinate or more general operator method were used. An extensive literature on this subject can be found, for example, in [9]. For time-fractional heat conduction, the reduction of the three-dimensional equation to the two-dimensional one was carried out by Povstenko [9,54,55].

A solution to the problem (56), (57) with uniform initial temperature in one of half-lines under conditions of nonperfect thermal contact was obtained in [56]. In the particular case $\alpha = \beta$, the solution reads

$$T_1 = T_0 - \frac{T_0}{(1+\varepsilon)} W\left(-\frac{\alpha}{2}, 1; -\frac{x}{\sqrt{a_1} t^{\alpha/2}}\right) + \frac{T_0(1-\varepsilon)}{2(1+\varepsilon)} \int_0^t \frac{(t-\tau)^{\alpha/2-1}}{\tau^{\alpha/2}} \times M\left(\frac{\alpha}{2}; \frac{x}{\sqrt{a_1}\tau^{\alpha/2}}\right) E_{\alpha/2,\alpha/2}\left[-b_\Sigma (t-\tau)^{\alpha/2}\right] \mathrm{d}\tau, \qquad x > 0, \tag{73}$$

$$T_2 = \frac{\varepsilon T_0}{(1+\varepsilon)} W\left(-\frac{\alpha}{2}, 1; -\frac{|x|}{\sqrt{a_2} t^{\alpha/2}}\right) + \frac{T_0(1-\varepsilon)}{2(1+\varepsilon)} \int_0^t \frac{(t-\tau)^{\alpha/2-1}}{\tau^{\alpha/2}} \times M\left(\frac{\alpha}{2}; \frac{|x|}{\sqrt{a_2}\tau^{\alpha/2}}\right) E_{\alpha/2,\alpha/2}\left[-b_\Sigma (t-\tau)^{\alpha/2}\right] \mathrm{d}\tau, \qquad x < 0, \tag{74}$$

where ε is defined by (63),

$$b_\Sigma = \frac{k_1\sqrt{a_2} + k_2\sqrt{a_1}}{C_\Sigma \sqrt{a_1 a_2}}, \tag{75}$$

C_Σ is the reduced heat capacity of the median surface of the transition region. When $C_\Sigma \to 0$, the solutions (73), (74) coincide with the solutions (64), (65).

3.3. Fractional Heat Conduction under Time-Harmonic Impact

Ångström [57] was the first to investigate the standard parabolic heat conduction equation under time-harmonic impact. An extensive review of literature in this field in the case of classical diffusion equation can be found in the book by Mandelis [58].

Fractional heat conduction with a source varying harmonically in time was studied by Povstenko [59]. Equation (1) with a source term

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \frac{\partial^2 T}{\partial x^2} + Q_0\, \delta(x) \mathrm{e}^{i\omega t}, \qquad 0 < \alpha \le 2, \tag{76}$$

was solved in the domain $-\infty < x < \infty$ under zero initial conditions. Temperature is expressed as

$$T(x,t) = \frac{Q_0}{2\sqrt{a}} \int_0^t \tau^{\alpha/2-1}\, W\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -\frac{|x|}{\sqrt{a}\tau^{\alpha/2}}\right) \mathrm{e}^{i\omega(t-\tau}\, \mathrm{d}\tau. \tag{77}$$

The corresponding problem in the central symmetric case

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) + Q_0\, \frac{\delta(r)}{4\pi r^2}\, \mathrm{e}^{i\omega t}, \qquad 0 < r < \infty, \quad 0 < \alpha \le 2, \tag{78}$$

has the solution

$$T(x,t) = \frac{\alpha Q_0}{8\pi a^{3/2}} \int_0^t \frac{1}{\tau^{1+\alpha/2}}\, M\left(\frac{\alpha}{2}; \frac{r}{\sqrt{a}\tau^{\alpha/2}}\right) \mathrm{e}^{i\omega(t-\tau}\, \mathrm{d}\tau. \tag{79}$$

3.4. Fractional Nonlocal Elasticity

Nonlocal continuum physics assumes integral constitutive equations. In the nonlocal theory of the continuum mechanics, stresses at the reference point $\mathbf{x}$ of an elastic solid at time t depend not only on the strains at this point at this time, but also on strains at all the points $\mathbf{x}'$ of a body and all the times prior to and at time t:

$$\mathbf{t}(\mathbf{x}, t, \epsilon_L, \epsilon_T) = \int_0^t \int_V \gamma(|\mathbf{x}-\mathbf{x}'|, t-t', \epsilon_L, \epsilon_T)\, \boldsymbol{\sigma}(\mathbf{x}', t')\, \mathrm{d}v(\mathbf{x}')\, \mathrm{d}t', \tag{80}$$

$$\boldsymbol{\sigma}(\mathbf{x}', t') = 2\mu\, \mathbf{e}(\mathbf{x}', t') + \lambda\, \mathrm{tr}\, \mathbf{e}(\mathbf{x}', t')\mathbf{I}, \tag{81}$$

where $\mathbf{t}$ and $\boldsymbol{\sigma}$ are the nonlocal and classical stress tensors, $\mathbf{x}$ and $\mathbf{x}'$ are the reference and running points, $\mathbf{e}$ the linear strain tensor, λ and μ are Lamé constants, $\mathbf{I}$ stands for the unit tensor. The volume integral in (80) is over the region occupied by the solid. The time-non-locality describes memory effects, distributed lag (distributed time delay), and frequency dispersion; the space-non-locality deals with the long-range interaction. The weight function (the non-locality kernel) $\gamma(|\mathbf{x}-\mathbf{x}'|, t-t', \epsilon_L, \epsilon_T)$ depends on two basic non-locality parameters (see Eringen [60]): the characteristic length ratio

$$\epsilon_L = \frac{\text{Internal characteristic length}}{\text{External characteristic length}}$$

and the characteristic time ratio

$$\epsilon_T = \frac{\text{Internal characteristic time}}{\text{External characteristic time}}.$$

When $\epsilon_T \to 0$, the memory effects are eliminated; for $\epsilon_L \to 0$ the space-non-locality disappears.

In the pioneering works by Podstrigach [61,62], a new nontraditional thermodynamic pair (the chemical potential tensor $\boldsymbol{\varphi}$ and the concentration tensor $\mathbf{c}$) was introduced (see also [63,64]). The tensor character of the chemical potential means that, for solids, the work of bringing the substance into a point in a body depends on the direction. In this case, the diffusion equation, split into the mean and deviatoric parts, has the form

$$\rho\, \frac{\partial(\mathrm{tr}\, \mathbf{c})}{\partial t} = 3a\, \Delta\, (\mathrm{tr}\, \boldsymbol{\varphi}), \tag{82}$$

$$\rho\, \frac{\partial(\mathrm{dev}\, \mathbf{c})}{\partial t} = 2a_1\, \Delta\, (\mathrm{dev}\, \boldsymbol{\varphi}), \tag{83}$$

where ρ is the mass density, and a and a_1 are the corresponding diffusion coefficients.

Starting from interrelated equations describing elasticity and diffusion, Podstrigach [65] eliminated the chemical potential tensor from the constitutive equation for the stress tensor and obtained the stress–strain relation containing spatial and time derivatives. In the infinite medium, this relation can be integrated using the Fourier and Laplace integral transforms, and the final result, written for the mean and deviatoric parts, has the nonlocal integral form:

$$\begin{aligned}\mathrm{tr}\,\boldsymbol{\sigma} \;&=\; 3K_c\,\mathrm{tr}\,\mathbf{e} + 3\,\frac{K_\varphi - K_c}{p}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\gamma_{(p)}\big(x-x',y-y',z-z',t-t'\big)\\ &\times\;\mathrm{tr}\,\mathbf{e}\big(x',y',z',t'\big)\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'\,\mathrm{d}t',\end{aligned} \tag{84}$$

$$\begin{aligned}\mathrm{dev}\,\boldsymbol{\sigma} \;&=\; 2\mu_c\,\mathrm{dev}\,\mathbf{e} + 2\,\frac{\mu_\varphi - \mu_c}{q}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\gamma_{(q)}\big(x-x',y-y',z-z',t-t'\big)\\ &\times\;\mathrm{dev}\,\mathbf{e}\big(x',y',z',t'\big)\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'\,\mathrm{d}t'.\end{aligned} \tag{85}$$

Here, K_c, K_φ, μ_c, μ_φ, p, and q are material constants (for details, see [42,65]). The kernel $\gamma_{(p)}(x,y,z,t)$ has the following form:

$$\gamma_{(p)}(x,y,z,t) = \left(\frac{p}{2t}\right)^{5/2}\left(3 - \frac{p\,r^2}{2t}\right)\exp\left(-\frac{p\,r^2}{4t}\right), \tag{86}$$

where $r = \sqrt{x^2+y^2+z^2}$; the kernel $\gamma_{(q)}(x,y,z,t)$ is obtained from the kernel $\gamma_{(p)}(x,y,z,t)$ substituting p by q.

The results of Podstrigach [65] were generalized by Povstenko [42] for the case of fractional diffusion equations

$$\rho\,\frac{\partial^\alpha(\mathrm{tr}\,\mathbf{c})}{\partial t^\alpha} = 3a\,\Delta\,(\mathrm{tr}\,\boldsymbol{\varphi}), \tag{87}$$

$$\rho\,\frac{\partial^\alpha(\mathrm{dev}\,\mathbf{c})}{\partial t^\alpha} = 2a_1\,\Delta\,(\mathrm{dev}\,\boldsymbol{\varphi}). \tag{88}$$

The kernel $\gamma_{(p)}(x,y,z,t)$ in the fractional generalization of the constitutive Equation (84) for the mean part of the stress tensor is expressed in terms of the Wright function:

$$\gamma_{(p)}(x,y,z,t) = -\frac{\sqrt{\pi}p^2}{\sqrt{2}\,t^{\alpha+1}\,r}\,W\left(-\frac{\alpha}{2},-\alpha;-\sqrt{p}\,\frac{r}{t^{\alpha/2}}\right). \tag{89}$$

The kernel $\gamma_{(q)}(x,y,z,t)$ in the fractional generalization of the constitutive Equation (85) for the deviatoric part of the stress tensor is obtained by substituting p with q.

In the case of only space-non-locality, the constitutive equation for the stress tensor reads

$$\mathbf{t}(\mathbf{x},\epsilon_L) = \int_V \gamma\big(|\mathbf{x}-\mathbf{x}'|,\epsilon_L\big)\boldsymbol{\sigma}\big(\mathbf{x}'\big)\mathrm{d}v(\mathbf{x}'). \tag{90}$$

The space-nonlocal elasticity reduces to the classical theory of elasticity in the long wave-length limit and to the atomic lattice theory in the short wave-length limit. Several versions of nonlocal elasticity based on various assumptions were proposed by different authors (see, for example, Podstrigach [65], Eringen [66,67], Kunin [68,69] and references therein).

In the case of space-nonlocal constitutive Equation (90), the nonlocal kernel $\gamma(|\mathbf{x}-\mathbf{x}'|,\epsilon_L)$ is a delta sequence and in the classical elasticity limit $\epsilon_L \to 0$ becomes the Dirac delta function. For example, slightly changing the notation, the nonlocal kernel $\gamma(|\mathbf{x}-\mathbf{x}'|,\tau)$ can be considered as the Green function of the Cauchy problem for the diffusion operator (see Eringen [67,70]):

$$\frac{\partial\gamma(\mathbf{x},\tau)}{\partial\tau} - a\Delta\,\gamma(\mathbf{x},\tau) = 0, \tag{91}$$

$$\tau = 0: \quad \gamma(\mathbf{x},\tau) = \delta(\mathbf{x}), \tag{92}$$

which results in the kernel

$$\gamma(\mathbf{x},\tau) = \frac{1}{(2\sqrt{\pi a \tau})^n} \exp\left(-\frac{|\mathbf{x}|^2}{4a\tau}\right) \tag{93}$$

for $n = 1, 2, 3$ space variables. In this case, the nonlocal stress tensor is a solution of the corresponding Cauchy problem:

$$\frac{\partial \mathbf{t}(\mathbf{x},\tau)}{\partial \tau} - a\Delta\, \mathbf{t}(\mathbf{x},\tau) = 0, \tag{94}$$

$$\tau = 0: \quad \mathbf{t}(\mathbf{x},\tau) = \boldsymbol{\sigma}(\mathbf{x}). \tag{95}$$

It should be emphasized that in, the formal sense, τ in the initial-value problems (91), (92) and (94), (95) looks like time, but in fact τ is a non-locality parameter related to the space-non-locality characteristic ratio ϵ_L.

In the paper [71], the nonlocal kernel $\gamma(|\mathbf{x}-\mathbf{x}'|,\tau)$ was considered as the Green function of the Cauchy problem for the fractional diffusion operator

$$\frac{\partial^\alpha \gamma(\mathbf{x},\tau)}{\partial \tau^\alpha} - a\Delta\, \gamma(\mathbf{x},\tau) = 0, \quad 0 < \alpha \leq 1, \tag{96}$$

$$\tau = 0: \quad \gamma(\mathbf{x},\tau) = \delta(\mathbf{x}). \tag{97}$$

In the framework of this approach, instead of the Cauchy problem (94)–(95), we obtain

$$\frac{\partial^\alpha \mathbf{t}(\mathbf{x},\tau)}{\partial \tau^\alpha} - a\Delta\, \mathbf{t}(\mathbf{x},\tau) = 0, \quad 0 < \alpha \leq 1, \tag{98}$$

$$\tau = 0: \quad \mathbf{t}(\mathbf{x},\tau) = \boldsymbol{\sigma}(\mathbf{x}). \tag{99}$$

In the case of one spatial coordinate, the nonlocal kernel takes the form

$$\gamma(x,\tau) = \frac{1}{2\sqrt{a\tau^{\alpha/2}}}\, M\left(\frac{\alpha}{2}; \frac{|x|}{\sqrt{a\tau^{\alpha/2}}}\right), \qquad 0 \leq \alpha \leq 1, \tag{100}$$

and in the central symmetric case

$$\gamma(r,\tau) = \frac{1}{4\pi a \tau^\alpha r}\, W\left(-\frac{\alpha}{2}, 1-\alpha; -\frac{r}{\sqrt{a\tau^{\alpha/2}}}\right), \qquad 0 \leq \alpha \leq 1. \tag{101}$$

4. Conclusions

In this survey, we have reviewed the main applications of the Wright function and the Mainardi function in continuum physics based essentially on the author's works. We have presented the integral relations between the Mittag–Leffler functions and the Wright function, which can be useful when solving fractional differential equations. We have restricted ourselves to the standard Mittag–Leffler functions and Wright function. The interested reader is referred to publications on further generalizations of the Mittag–Leffler functions [24,72–75] and of the Wright function [24,29,76–78].

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