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On the Reciprocal Sums of Products of Two Generalized Bi-Periodic Fibonacci Numbers

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Abstract: This paper concerns the properties of the generalized bi-periodic Fibonacci numbers $\{G_n\}$ generated from the recurrence relation: $G_n = aG_{n-1} + G_{n-2}$ (n is even) or $G_n = bG_{n-1} + G_{n-2}$ (n is odd). We derive general identities for the reciprocal sums of products of two generalized bi-periodic Fibonacci numbers. More precisely, we obtain formulas for the integer parts of the numbers $\left(\sum_{k=n}^{\infty} \frac{(a/b)^{\xi(k+1)}}{G_k G_{k+m}}\right)^{-1}$, $m = 0, 2, 4, \cdots$, and $\left(\sum_{k=n}^{\infty} \frac{1}{G_k G_{k+m}}\right)^{-1}$, $m = 1, 3, 5, \cdots$.

Keywords: bi-periodic Fibonacci numbers; reciprocal; floor function

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1. Introduction

As is well known, the Fibonacci sequence $\{F_n\}$ is generated from the recurrence relation $F_n = F_{n-1} + F_{n-2}$ ($n \geq 2$) with the initial conditions $F_0 = 0$ and $F_1 = 1$. The Fibonacci numbers possess many interesting properties and appear in a variety of application fields [1].

Many authors tried to generalize the Fibonacci sequence. For example, Falcon and Plaza [2] considered the k-Fibonacci sequence. Edson and Yayenie [3] introduced the bi-periodic Fibonacci sequence $\{f_n\}$ defined by

$$f_0 = 0,\ f_1 = 1,\ f_n = \begin{cases} af_{n-1} + f_{n-2}, & \text{if } n \in \mathbb{N}_e; \\ bf_{n-1} + f_{n-2}, & \text{if } n \in \mathbb{N}_o, \end{cases} \quad (n \geq 2), \tag{1}$$

where $\mathbb{N}_e$ ($\mathbb{N}_o$, respectively) denotes the set of positive even (odd, respectively) integers. Filipponi [4] defined the incomplete Fibonacci sequence, and Ramírez [5] introduced the bi-periodic incomplete Fibonacci sequence.

In the remainder of this paper, we use the notation $\{G_n\}_{n=0}^{\infty} = S(G_0, G_1, a, b)$ to denote the generalized bi-periodic Fibonacci numbers $\{G_n\}$ generated from the recurrence relation

$$G_n = \begin{cases} aG_{n-1} + G_{n-2}, & \text{if } n \in \mathbb{N}_e; \\ bG_{n-1} + G_{n-2}, & \text{if } n \in \mathbb{N}_o, \end{cases} \quad (n \geq 2),$$

with initial conditions G_0 and G_1, where G_0 is a nonnegative integer, G_1, a and b are positive integers.

Recently, Ohtsuka and Nakamura [6] reported an interesting property of the Fibonacci numbers $\{F_n\} = S(0,1,1,1)$ and proved the following identities:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_n - F_{n-1}, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_n - F_{n-1} - 1, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o, \end{cases} \tag{2}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_{n-1}F_n, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o, \end{cases} \tag{3}$$

where $\lfloor \cdot \rfloor$ is the floor function.

Following the work of Ohtsuka and Nakamura, diverse results for the numbers of the form $\{G_n\} = S(G_0, G_1, a, a)$ have been reported in the literature (see [7–15] and references cited therein).

On the other hand, reciprocal sums of the generalized bi-periodic numbers were considered in [16,17]. In [16], Basbuk and Yazlik proved the following identity for $\{G_n\} = S(0, 1, a, b)$:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{(a/b)^{\psi(k)}}{G_k} \right)^{-1} \right\rfloor = \begin{cases} G_n - G_{n-1}, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ G_n - G_{n-1} - 1, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o, \end{cases} \tag{4}$$

where

$$\psi(k) = \xi(k+1) - \xi(n+1) - (-1)^n \left\lfloor \frac{k-n}{2} \right\rfloor,$$

and $\xi(n)$ is the parity function, such that

$$\xi(n) = \begin{cases} 0, & \text{if } n \in \{0\} \cup \mathbb{N}_e; \\ 1, & \text{if } n \in \mathbb{N}_o. \end{cases}$$

For $\{G_n\} = S(G_0, G_1, a, b)$, Choi and Choo [17] identified the integer parts for the numbers

$$\left(\sum_{k=n}^{\infty} \frac{(a/b)^{\xi(k+1)}}{G_k^2} \right)^{-1}.$$

In this paper, we extend the results in [17] by considering the reciprocal sums of products of two generalized bi-periodic Fibonacci numbers. More precisely, we obtain general identities for the numbers

$$\left(\sum_{k=n}^{\infty} \frac{(a/b)^{\xi(k+1)}}{G_k G_{k+m}} \right)^{-1}, \quad m \in \{0\} \cup \mathbb{N}_e,$$

and

$$\left(\sum_{k=n}^{\infty} \frac{1}{G_k G_{k+m}} \right)^{-1}, \quad m \in \mathbb{N}_0.$$

2. Results

2.1. The Case where $m \in \{0\} \cup \mathbb{N}_e$

Lemma 1 below will be used to prove the results for the case where $m \in \mathbb{N}_e$.

Lemma 1. *Assume that $m \in \{0\} \cup \mathbb{N}_e$. Then, for $\{G_n\} = S(G_0, G_1, a, b)$, (a)–(e) below hold:*

(a) $G_{n+2}G_{n+m+1} - G_n G_{n+m-1} = a^{\xi(n)} b^{\xi(n+1)} G_n G_{n+m} + a^{\xi(n+1)} b^{\xi(n)} G_{n+1} G_{n+m+1}$.
(b) $G_{n+1}G_{n+m} - G_{n+2}G_{n+m-1} = (-1)^n (G_m G_3 - G_{m+1} G_2)$.
(c) $a^{\xi(n+1)} b^{\xi(n)} G_{n-1} G_{n+m+1} - a^{\xi(n)} b^{\xi(n+1)} G_n G_{n+m} = (-1)^n (a G_{m+1} G_1 - b G_{m+2} G_0)$.
(d) $G_{n+1}G_{n+m-2} - G_n G_{n+m-1} = (-1)^n (G_m G_3 - G_{m+1} G_2)$.
(e) $G_{n-1}G_{n+m} - G_n G_{n+m-1} = (-1)^n (G_m G_1 - G_{m+1} G_0)$.

Proof. Since

$$G_n = a^{\xi(n-1)} b^{\xi(n)} G_{n-1} + G_{n-2},$$

then, (a) follows from the identity

$$\begin{aligned} G_n G_{n+m+1} &= (G_{n+2} - a^{\xi(n+1)} b^{\xi(n)} G_{n+1}) G_{n+m+1} \\ &= G_n (a^{\xi(n)} b^{\xi(n+1)} G_{n+m} + G_{n+m-1}). \end{aligned}$$

(b)–(e) are special cases of ([18], Theorem 2.2). □

Theorem 1. *Consider the generalized bi-periodic Fibonacci numbers* $\{G_n\} = S(G_0, G_1, a, b)$ *and let*

$$\Phi_m := b(G_m G_3 - G_{m+1} G_2).$$

If $m \in \{0\} \cup \mathbb{N}_e$, *then (a) and (b) below hold:*

(a) *If*

$$\frac{\Phi_m}{ab+2} \notin \mathbb{Z},$$

define

$$g_m := \left\lfloor \frac{\Phi_m}{ab+2} \right\rfloor + \Delta,$$

where

$$\Delta = \begin{cases} 1, & \text{if } \Phi_m > 0; \\ 0, & \text{if } \Phi_m < 0. \end{cases}$$

(i) If $\Phi_m > 0$, *then there exist positive integers* n_0 *and* n_1 *such that*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\xi(k+1)}}{G_k G_{k+m}} \right)^{-1} \right\rfloor = \begin{cases} bG_n G_{n+m-1} + g_m - 1, & \text{if } n \geq n_0 \text{ and } n \in \mathbb{N}_e; \\ bG_n G_{n+m-1} - g_m, & \text{if } n \geq n_1 \text{ and } n \in \mathbb{N}_o. \end{cases} \tag{5}$$

(ii) If $\Phi_m < 0$, *then there exist positive integers* n_2 *and* n_3 *such that*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\xi(k+1)}}{G_k G_{k+m}} \right)^{-1} \right\rfloor = \begin{cases} bG_n G_{n+m-1} + g_m, & \text{if } n \geq n_2 \text{ and } n \in \mathbb{N}_e; \\ bG_n G_{n+m-1} - g_m - 1, & \text{if } n \geq n_3 \text{ and } n \in \mathbb{N}_o. \end{cases} \tag{6}$$

(b) *If*

$$\frac{\Phi_m}{ab+2} \in \mathbb{Z},$$

define

$$\hat{g}_m := \frac{\Phi_m}{ab+2},$$

and

$$\Gamma_m := \hat{g}_m b(bG_{m+2}G_0 - aG_{m+1}G_1) - \hat{g}_m^2.$$

(i) If $\Gamma_m \geq 0$, *then there exist positive integers* n_4 *and* n_5, *such that*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\xi(k+1)}}{G_k G_{k+m}} \right)^{-1} \right\rfloor = \begin{cases} bG_n G_{n+m-1} + \hat{g}_m, & \text{if } n \geq n_4 \text{ and } n \in \mathbb{N}_e; \\ bG_n G_{n+m-1} - \hat{g}_m, & \text{if } n \geq n_5 \text{ and } n \in \mathbb{N}_o. \end{cases} \tag{7}$$

(ii) If $\Gamma_m < 0$, *then there exist positive integers* n_4 *and* n_5, *such that*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\xi(k+1)}}{G_k G_{k+m}} \right)^{-1} \right\rfloor = \begin{cases} bG_n G_{n+m-1} + \hat{g}_m - 1, & \text{if } n \geq n_6 \text{ and } n \in \mathbb{N}_e; \\ bG_n G_{n+m-1} - \hat{g}_m - 1, & \text{if } n \geq n_7 \text{ and } n \in \mathbb{N}_o. \end{cases} \tag{8}$$

Proof. (a) To prove (5), assume that $\Phi_m > 0$. Then

$$\Phi_m - g_m(ab+2) < 0.$$

Firstly, consider

$$\begin{aligned} X_1 &= \frac{1}{bG_nG_{n+m-1}+(-1)^n g_m} - \frac{1}{bG_{n+2}G_{n+m+1}+(-1)^n g_m} - \frac{\left(\frac{a}{b}\right)^{\xi(n+1)}}{G_nG_{n+m}} - \frac{\left(\frac{a}{b}\right)^{\xi(n)}}{G_{n+1}G_{n+m+1}} \\ &= \frac{Y_1}{(bG_nG_{n+m-1}+(-1)^n g_m)(bG_{n+2}G_{n+m+1}+(-1)^n g_m)G_nG_{n+1}G_{n+m}G_{n+m+1}}, \end{aligned}$$

where, by Lemma 1 (a)

$$Y_1 = \left\{ \left(\frac{a}{b}\right)^{\xi(n)} G_nG_{n+m} + \left(\frac{a}{b}\right)^{\xi(n+1)} G_{n+1}G_{n+m+1} \right\} \hat{Y}_1,$$

with

$$\hat{Y}_1 = b^2(G_nG_{n+1}G_{n+m}G_{n+m+1} - G_nG_{n+2}G_{n+m-1}G_{n+m+1}) - (-1)^n g_m b(G_nG_{n+m-1} + G_{n+2}G_{n+m+1}) - g_m^2.$$

By Lemma 1 (b,c), we have

$$\begin{aligned} G_nG_{n+1}G_{n+m}G_{n+m+1} - G_nG_{n+2}G_{n+m-1}G_{n+m+1} &= (G_{n+1}G_{n+m} - G_{n+2}G_{n+m-1})G_nG_{n+m+1} \\ &= (-1)^n(G_mG_3 - G_{m+1}G_2)G_nG_{n+m+1}, \end{aligned}$$

and

$$\begin{aligned} & G_nG_{n+m-1} + G_{n+2}G_{n+m+1} \\ =\ & G_n(G_{n+m+1} - a^{\xi(n+m)}b^{\xi(n+m+1)}G_{n+m}) + (a^{\xi(n+1)}b^{\xi(n)}G_{n+1} + G_n)G_{n+m+1} \\ =\ & 2G_nG_{n+m+1} - a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m} + a^{\xi(n+1)}b^{\xi(n)}(a^{\xi(n)}b^{\xi(n+1)}G_n + G_{n-1})G_{n+m+1} \\ =\ & (ab+2)G_nG_{n+m+1} - a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m} + a^{\xi(n+1)}b^{\xi(n)}G_{n-1}G_{n+m+1} \\ =\ & (ab+2)G_nG_{n+m+1} + (-1)^n(aG_{m+1} - bG_{m+2}G_0). \end{aligned}$$

Then

$$\begin{aligned} \hat{Y}_1 &= (-1)^n b^2(G_mG_3 - G_{m+1}G_2)G_nG_{n+m+1} \\ &\quad -(-1)^n g_m b\{(ab+2)G_nG_{n+m+1} + (-1)^n(aG_{m+1} - bG_{m+2}G_0)\} - g_m^2 \\ &= (-1)^n b\{\Phi_m - g(ab+2)\}G_nG_{n+m+1} + g_m b(bG_{m+2}G_0 - aG_{m+1}G_1) - g_m^2. \end{aligned}$$

If $n \in \mathbb{N}_e$, then there exists a positive integer m_0 such that, for $n \geq m_0$, $X_1 < 0$, and

$$\frac{1}{bG_nG_{n+m-1}+g_m} - \frac{1}{bG_{n+2}G_{n+m+1}+g_m} < \frac{\left(\frac{a}{b}\right)^{\xi(n+1)}}{G_nG_{n+m}} + \frac{\left(\frac{a}{b}\right)^{\xi(n)}}{G_{n+1}G_{n+m+1}}.$$

Repeatedly applying the above inequality, we have

$$\frac{1}{bG_nG_{n+m-1}+g_m} < \sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\xi(k+1)}}{G_kG_{k+m}}, \text{ if } n \geq m_0 \text{ and } n \in \mathbb{N}_e. \tag{9}$$

Similarly, we obtain, for some positive integer m_1,

$$\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\xi(k+1)}}{G_kG_{k+m}} < \frac{1}{bG_nG_{n+m-1}-g_m}, \text{ if } n \geq m_1 \text{ and } n \in \mathbb{N}_o. \tag{10}$$

Next, consider

$$\begin{aligned} X_2 &= \frac{1}{bG_nG_{n+m-1}+(-1)^n g_m-1} - \frac{1}{bG_{n+1}G_{n+m}+(-1)^{n+1}g_m-1} - \frac{\left(\frac{a}{b}\right)^{\xi(n+1)}}{G_nG_{n+m}} \\ &= \frac{Y_2}{(bG_nG_{n+m-1}+(-1)^n g_m-1)(bG_{n+1}G_{n+m}+(-1)^{n+1}g_m-1)G_nG_{n+m}}, \end{aligned}$$

where

$$\begin{aligned} Y_2 &= bG_nG_{n+1}G_{n+m}^2 - a^{\xi(n+1)}b^{1+\xi(n)}G_nG_{n+1}G_{n+m-1}G_{n+m} - bG_n^2G_{n+m-1}G_{n+m+1} \\ &\quad -(-1)^n g_m\{2G_nG_{n+m}+a^{\xi(n+1)}b^{\xi(n)}(G_{n+1}G_{n+m}-G_nG_{n+m-1})\} \\ &\quad +a^{\xi(n+1)}b^{\xi(n)}(G_nG_{n+m-1}+G_{n+1}G_{n+m}) + a^{\xi(n+1)}b^{\xi(n)-1}(g_m^2-1). \end{aligned}$$

Using Lemma 1 (d,e), we have

$$\begin{aligned} &bG_nG_{n+1}G_{n+m}^2 - a^{\xi(n+1)}b^{1+\xi(n)}G_nG_{n+1}G_{n+m-1}G_{n+m} - bG_n^2G_{n+m-1}G_{n+m} \\ =\ & bG_nG_{n+1}G_{n+m}(G_{n+m}-a^{\xi(n+1)}b^{\xi(n)}G_{n+m-1}) - bG_n^2G_{n+m-1}G_{n+m} \\ =\ & bG_nG_{n+m}(G_{n+1}G_{n+m-2}-G_nG_{n+m-1}) \\ =\ & (-1)^n b(G_mG_3-G_{m+1}G_2)G_nG_{n+m}, \end{aligned}$$

and

$$\begin{aligned} &2G_nG_{n+m}+a^{\xi(n+1)}b^{\xi(n)}(G_{n+1}G_{n+m}-G_nG_{n+m-1}) \\ =\ & 2G_nG_{n+m}+a^{\xi(n+1)}b^{\xi(n)}(a^{\xi(n)}b^{\xi(n+1)}G_n+G_{n-1})G_{n+m}-a^{\xi(n+1)}b^{\xi(n)}G_nG_{n+m-1}) \\ =\ & (ab+2)G_nG_{n+m}+a^{\xi(n+1)}b^{\xi(n)}(G_{n-1}G_{n+m}-G_nG_{n+m-1}) \\ =\ & (ab+2)G_nG_{n+m}+(-1)^n a^{\xi(n+1)}b^{\xi(n)}(G_mG_1-G_{m+1}G_0). \end{aligned}$$

Hence we obtain

$$\begin{aligned} Y_2 &= (-1)^n\{\Phi_m - g_m(ab+2)\}G_nG_{n+m} + a^{\xi(n+1)}b^{\xi(n)}(G_nG_{n+m-1}+G_{n+1}G_{n+m}) \\ &\quad -a^{\xi(n+1)}b^{\xi(n)}g_m(G_mG_1-G_{m+1}G_0) + a^{\xi(n+1)}b^{\xi(n)-1}(g_m^2-1), \end{aligned}$$

and there exists a positive integer m_2 such that, for $n \geq m_2$, $X_2 > 0$, and

$$\frac{\left(\frac{a}{b}\right)^{\xi(n+1)}}{G_nG_{n+m}} < \frac{1}{bG_nG_{n+m-1}+(-1)^n g_m-1} - \frac{1}{bG_{n+1}G_{n+m}+(-1)^{n+1}g_m-1}.$$

Repeatedly applying the above inequality, we have

$$\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\xi(k+1)}}{G_kG_{k+m}} < \frac{1}{bG_nG_{n+m-1}+(-1)^n g_m-1}, \text{ if } n \geq m_2. \tag{11}$$

Similarly, consider

$$\begin{aligned} X_3 &= \frac{1}{bG_nG_{n+m-1}+(-1)^n g_m+1} - \frac{1}{bG_{n+1}G_{n+m}+(-1)^{n+1}g_m+1} - \frac{\left(\frac{a}{b}\right)^{\xi(n+1)}}{G_nG_{n+m}} \\ &= \frac{Y_3}{(bG_nG_{n+m-1}+(-1)^n g_m+1)(bG_{n+1}G_{n+m}+(-1)^{n+1}g_m+1)G_nG_{n+m}}, \end{aligned}$$

where

$$
\begin{aligned}
Y_3 &= Y_2 - 2a^{\xi(n+1)}b^{\xi(n)}(G_nG_{n+m-1} + G_{n+1}G_{n+m}) \\
&= (-1)^n\{\Phi_m - g_m(ab+2)\}G_nG_{n+m} - a^{\xi(n+1)}b^{\xi(n)}(G_nG_{n+m-1} + G_{n+1}G_{n+m}) \\
&\quad -a^{\xi(n+1)}b^{\xi(n)}g_m(G_mG_1 - G_{m+1}G_0) + a^{\xi(n+1)}b^{\xi(n)-1}(g_m^2 - 1).
\end{aligned}
$$

There exists a positive integer m_3 such that, for $n \geq m_3$, $X_3 < 0$, and

$$
\frac{1}{bG_nG_{n+m-1} + (-1)^n g_m + 1} - \frac{1}{bG_{n+1}G_{n+m} + (-1)^{n+1} g_m + 1} < \frac{\left(\frac{a}{b}\right)^{\xi(n+1)}}{G_nG_{n+m}},
$$

from which we have

$$
\frac{1}{bG_nG_{n+m-1} + (-1)^n g_m + 1} < \sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\xi(k+1)}}{G_kG_{k+m}}, \text{ if } n \geq m_3. \tag{12}
$$

Then (5) follows from (9)–(12).

Now, suppose that $\Phi_m < 0$. In this case, we have

$$
\Phi_m - g_m(ab+2) > 0,
$$

and (9)–(12) are respectively modified as

$$
\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\xi(k+1)}}{G_kG_{k+m}} < \frac{1}{bG_nG_{n+m-1} + g_m}, \text{ if } n \geq m_4 \text{ and } n \in \mathbb{N}_e, \tag{13}
$$

$$
\frac{1}{bG_nG_{n+m-1} - g_m} < \sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\xi(k+1)}}{G_kG_{k+m}}, \text{ if } n \geq m_5 \text{ and } n \in \mathbb{N}_o, \tag{14}
$$

$$
\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\xi(k+1)}}{G_kG_{k+m}} < \frac{1}{bG_nG_{n+m-1} + (-1)^n g_m - 1}, \text{ if } n \geq m_6, \tag{15}
$$

and

$$
\frac{1}{bG_nG_{n+m-1} + (-1)^n g_m + 1} < \sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\xi(k+1)}}{G_kG_{k+m}}, \text{ if } n \geq m_7. \tag{16}
$$

Then, (6) easily follows and the proof of (a) is completed.

(b) Suppose that

$$
\frac{\Phi_m}{ab+2} \in \mathbb{Z}.
$$

We recall the proof of (a). If $\Gamma_m \geq 0$, then replacing g_m by $\hat{g}_m$, we have $\hat{Y}_1 = \Gamma_m \geq 0$, and there exists a positive integer m_8 such that $X_1 > 0$ if $n \geq m_8$. Hence, we obtain

$$
\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\xi(k+1)}}{G_kG_{k+m}} < \frac{1}{bG_nG_{n+m-1} + (-1)^n \hat{g}_m}, \text{ if } n \geq m_8. \tag{17}
$$

Similarly, there exists a positive integer m_9 such that $X_3 < 0$ if $n \geq m_9$, from which we have

$$
\frac{1}{bG_nG_{n+m} + (-1)^n \hat{g} + 1} < \sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\xi(k+1)}}{G_kG_{k+m}}, \text{ if } n \geq m_9. \tag{18}
$$

Then, (7) follows from (17) and (18). (8) can be proved similarly, and details are omitted. □

If $a = b$, then Theorem 1 reduces ([7], Theorem 2.1) with $m \in \{0\} \cup \mathbb{N}_e$ and $\{G_n\} = \{H_n\} = S(G_0, G_1, a, a)$.

If $m = 0$, then Theorem 1 reduces ([17], Theorem 2).

2.2. The Case Where $m \in \mathbb{N}_o$

To deal with the case where $m \in \mathbb{N}_o$, we need the following lemma.

Lemma 2. *Assume that $m \in \mathbb{N}_o$. Then, for $\{G_n\} = S(G_0, G_1, a, b)$, (a)-(e) below hold:*

(a) $G_{n+2}G_{n+m+1} - G_nG_{n+m-1} = a^{\xi(n+1)}b^{\xi(n)}(G_nG_{n+m} + G_{n+1}G_{n+m+1})$.

(b) $a^{\xi(n+1)}b^{\xi(n)}G_{n+1}G_{n+m} - a^{\xi(n)}b^{\xi(n+1)}G_{n+2}G_{n+m-1} = (-1)^n(aG_mG_3 - bG_{m+1}G_2)$.

(c) $G_{n-1}G_{n+m+1} - G_nG_{n+m} = (-1)^n(G_{m+2}G_0 - G_{m+1}G_1)$.

(d) $a^{\xi(n+1)}b^{\xi(n)}G_{n+1}G_{n+m-2} - a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1} = (-1)^n(aG_mG_3 - bG_{m+1}G_2)$.

(e) $a^{\xi(n+1)}b^{\xi(n)}G_{n-1}G_{n+m} - a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1} = (-1)^n(aG_mG_1 - bG_{m+1}G_0)$.

Proof. (a) follows from the identity

$$\begin{aligned} G_nG_{n+m+1} &= (G_{n+2} - a^{\xi(n+1)}b^{\xi(n)}G_{n+1})G_{n+m+1} \\ &= G_n(a^{\xi(n+m)}b^{\xi(n+m+1)}G_{n+m} + G_{n+m-1}). \end{aligned}$$

(b)–(e) are special cases of ([18], Theorem 2.2). □

Theorem 2. *Consider the generalized bi-periodic Fibonacci numbers $\{G_n\} = S(G_0, G_1, a, b)$ and let*

$$\Delta_m := aG_mG_3 - bG_{m+1}G_2.$$

If $m \in \mathbb{N}_0$, then (a) and (b) below hold:

(a) *If*

$$\frac{\Delta_m}{ab+2} \notin \mathbb{Z},$$

define

$$h_m := \left\lfloor \frac{\Delta_m}{ab+2} \right\rfloor + \Delta,$$

where

$$\Delta = \begin{cases} 1, & \text{if } \Delta_m > 0; \\ 0, & \text{if } \Delta_m < 0. \end{cases}$$

(i) If $\Delta_m > 0$, then there exist positive integers n_0 and n_1 such that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_kG_{k+m}} \right)^{-1} \right\rfloor = \begin{cases} bG_nG_{n+m-1} + h_m - 1, & \text{if } n \geq n_0 \text{ and } n \in \mathbb{N}_e; \\ aG_nG_{n+m-1} - h_m, & \text{if } n \geq n_1 \text{ and } n \in \mathbb{N}_o. \end{cases} \quad (19)$$

(ii) If $\Delta_m < 0$, then there exist positive integers n_2 and n_3 such that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_kG_{k+m}} \right)^{-1} \right\rfloor = \begin{cases} bG_nG_{n+m-1} + h_m, & \text{if } n \geq n_2 \text{ and } n \in \mathbb{N}_e; \\ aG_nG_{n+m-1} - h_m - 1, & \text{if } n \geq n_3 \text{ and } n \in \mathbb{N}_o. \end{cases} \quad (20)$$

(b) *If*

$$\frac{\Delta_m}{ab+2} \in \mathbb{Z},$$

define

$$\hat{h}_m := \frac{\Delta_m}{ab+2},$$

and

$$\Psi_m := \hat{h}_m ab(G_{m+1}G_1 - G_{m+2}G_0) - \hat{h}_m^2.$$

(i) If $\Psi_m \geq 0$*, then there exist positive integers* n_4 *and* n_5 *such that*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_k G_{k+m}} \right)^{-1} \right\rfloor = \begin{cases} aG_nG_{n+m-1} + \hat{h}_m, & \text{if } n \geq n_4 \text{ and } n \in \mathbb{N}_e; \\ bG_nG_{n+m-1} - \hat{h}_m, & \text{if } n \geq n_5 \text{ and } n \in \mathbb{N}_o. \end{cases} \tag{21}$$

(ii) If $\Psi_m < 0$*, then there exist positive integers* n_6 *and* n_7 *such that*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_k G_{k+m}} \right)^{-1} \right\rfloor = \begin{cases} aG_nG_{n+m-1} + \hat{h}_m - 1, & \text{if } n \geq n_6 \text{ and } n \in \mathbb{N}_e; \\ bG_nG_{n+m-1} - \hat{h}_m - 1, & \text{if } n \geq n_7 \text{ and } n \in \mathbb{N}_o. \end{cases} \tag{22}$$

Proof. (a) To prove (19), assume that $\Delta_m > 0$. Then

$$\Delta_m - h_m(ab+2) < 0.$$

Firstly, consider

$$\begin{aligned} X_1 &= \frac{1}{a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1}+(-1)^n h_m} - \frac{1}{a^{\xi(n)}b^{\xi(n+1)}G_{n+2}G_{n+m+1}+(-1)^n h_m} - \frac{1}{G_nG_{n+m}} - \frac{1}{G_{n+1}G_{n+m+1}} \\ &= \frac{Y_1}{(a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1}+(-1)^n h_m)(a^{\xi(n)}b^{\xi(n+1)}G_{n+2}G_{n+m+1}+(-1)^n h_m)G_nG_{n+1}G_{n+m}G_{n+m+1}}, \end{aligned}$$

where, by Lemma 2 (a)

$$Y_1 = (G_nG_{n+m} + G_{n+1}G_{n+m+1})\hat{Y}_1,$$

with

$$\begin{aligned} \hat{Y}_1 &= abG_nG_{n+1}G_{n+m}G_{n+m+1} - a^{2\xi(n)}b^{2\xi(n+1)}G_nG_{n+2}G_{n+m-1}G_{n+m+1} \\ &\quad -(-1)^n h_m a^{\xi(n)}b^{\xi(n+1)}(G_nG_{n+m-1} + G_{n+2}G_{n+m+1}) - h_m^2. \end{aligned}$$

By Lemma 2 (b,c), we have

$$\begin{aligned} & abG_nG_{n+1}G_nG_{n+m} - a^{2\xi(n)}b^{2\xi(n+1)}G_nG_{n+2}G_{n+m-1}G_{n+m+1} \\ =\ & (abG_{n+1}G_{n+m} - a^{2\xi(n)}b^{2\xi(n+1)}G_{n+2}G_{n+m-1})G_nG_{n+m+1} \\ =\ & a^{\xi(n)}b^{\xi(n+1)}(a^{\xi(n+1)}b^{\xi(n)}G_{n+1}G_{n+m} - a^{\xi(n)}b^{\xi(n+1)}G_{n+2}G_{n+m-1})G_nG_{n+m+1} \\ =\ & (-1)^n a^{\xi(n)}b^{\xi(n+1)}(aG_mG_3 - bG_{m+1}G_2)G_nG_{n+m+1}, \end{aligned}$$

and

$$\begin{aligned} & G_nG_{n+m-1} + G_{n+2}G_{n+m+1} \\ =\ & G_n(G_{n+m+1} - a^{\xi(n+m)}b^{\xi(n+m+1)}G_{n+m}) + (a^{\xi(n+1)}b^{\xi(n)}G_{n+1} + G_n)G_{n+m+1} \\ =\ & 2G_nG_{n+m+1} - a^{\xi(n+1)}b^{\xi(n)}G_nG_{n+m} + a^{\xi(n+1)}b^{\xi(n)}(a^{\xi(n)}b^{\xi(n+1)}G_n + G_{n-1})G_{n+m+1} \\ =\ & (ab+2)G_nG_{n+m+1} + a^{\xi(n+1)}b^{\xi(n)}(G_{n-1}G_{n+m+1} - G_nG_{n+m}) \\ =\ & (ab+2)G_nG_{n+m+1} + (-1)^n a^{\xi(n+1)}b^{\xi(n)}(G_{m+2}G_0 - G_{m+1}G_1). \end{aligned}$$

Then

$$\begin{aligned}\hat{Y}_1 &= (-1)^n a^{\xi(n)} b^{\xi(n+1)} (aG_m G_3 - bG_{m+1}G_2) G_n G_{n+m+1} \\ &\quad -(-1)^n h_m a^{\xi(n)} b^{\xi(n+1)} \{(ab+2) G_n G_{n+m+1} + (-1)^n a^{\xi(n+1)} b^{\xi(n)} (G_{m+2}G_0 - G_{m+1}G_1)\} - h_m^2 \\ &= (-1)^n a^{\xi(n)} b^{\xi(n+1)} \{\Delta_m - h_m(ab+2)\} G_n G_{n+m+1} + h_m ab (G_{m+1}G_1 - G_{m+2}G_0) - h_m^2.\end{aligned}$$

If $n \in \mathbb{N}_e$, then there exists a positive integer m_0 such that, for $n \geq m_0$, $X_1 < 0$, and

$$\frac{1}{a^{\xi(n)} b^{\xi(n+1)} G_n G_{n+m-1} + h_m} - \frac{1}{a^{\xi(n)} b^{\xi(n+1)} G_{n+2} G_{n+m+1} + h_m} < \frac{1}{G_n G_{n+m}} + \frac{1}{G_{n+1} G_{n+m+1}}.$$

Repeatedly applying the above inequality, we have

$$\frac{1}{a^{\xi(n)} b^{\xi(n+1)} G_n G_{n+m-1} + h_m} < \sum_{k=n}^{\infty} \frac{1}{G_k G_{k+m}}, \text{ if } n \geq m_0 \text{ and } n \in \mathbb{N}_e. \tag{23}$$

Similarly, we obtain, for some positive integer m_1,

$$\sum_{k=n}^{\infty} \frac{1}{G_k G_{k+m}} < \frac{1}{a^{\xi(n)} b^{\xi(n+1)} G_n G_{n+m-1} - h_m}, \text{ if } n \geq m_1 \text{ and } n \in \mathbb{N}_o. \tag{24}$$

Next, consider

$$\begin{aligned}X_2 &= \frac{1}{a^{\xi(n)} b^{\xi(n+1)} G_n G_{n+m-1} + (-1)^n h_m - 1} - \frac{1}{a^{\xi(n+1)} b^{\xi(n)} G_{n+1} G_{n+m} + (-1)^{n+1} h_m - 1} - \frac{1}{G_n G_{n+m}} \\ &= \frac{Y_2}{(a^{\xi(n)} b^{\xi(n+1)} G_n G_{n+m-1} + (-1)^n h_m - 1)(a^{\xi(n+1)} b^{\xi(n)} G_{n+1} G_{n+m} + (-1)^{n+1} h_m - 1) G_n G_{n+m}},\end{aligned}$$

where

$$\begin{aligned}Y_2 &= a^{\xi(n+1)} b^{\xi(n)} G_n G_{n+1} G_{n+m}^2 - ab G_n G_{n+1} G_{n+m-1} G_{n+m} - a^{\xi(n)} b^{\xi(n+1)} G_n^2 G_{n+m-1} G_{n+m+1} \\ &\quad -(-1)^n h_m (2G_n G_{n+m} + a^{\xi(n+1)} b^{\xi(n)} G_{n+1} G_{n+m} - a^{\xi(n)} b^{\xi(n+1)} G_n G_{n+m-1}) \\ &\quad + a^{\xi(n)} b^{\xi(n+1)} G_n G_{n+m-1} + a^{\xi(n+1)} b^{\xi(n)} G_{n+1} G_{n+m}) + h_m^2 - 1.\end{aligned}$$

Using Lemma 2 (d,e), we have

$$\begin{aligned}& a^{\xi(n+1)} b^{\xi(n)} G_n G_{n+1} G_{n+m}^2 - ab G_n G_{n+1} G_{n+m-1} G_{n+m} - a^{\xi(n)} b^{\xi(n+1)} G_n^2 G_{n+m-1} G_{n+m+1} \\ &= a^{\xi(n+1)} b^{\xi(n)} G_n G_{n+1} G_{n+m} (G_{n+m} - a^{\xi(n)} b^{\xi(n+1)} G_{n+m-1}) - a^{\xi(n)} b^{\xi(n+1)} G_n^2 G_{n+m-1} G_{n+m} \\ &= G_n G_{n+m} (a^{\xi(n+1)} b^{\xi(n)} G_{n+1} G_{n+m-2} - a^{\xi(n)} b^{\xi(n+1)} G_n G_{n+m-1}) \\ &= (-1)^n (aG_m G_3 - bG_{m+1} G_2) G_n G_{n+m},\end{aligned}$$

and

$$\begin{aligned}& 2G_n G_{n+m} + a^{\xi(n+1)} b^{\xi(n)} G_{n+1} G_{n+m} - a^{\xi(n)} b^{\xi(n+1)} G_n G_{n+m-1}) \\ &= 2G_n G_{n+m} + a^{\xi(n+1)} b^{\xi(n)} (a^{\xi(n)} b^{\xi(n+1)} G_n + G_{n-1}) G_{n+m} - a^{\xi(n)} b^{\xi(n+1)} G_n G_{n+m-1}) \\ &= (ab+2) 2G_n G_{n+m} + a^{\xi(n+1)} b^{\xi(n)} G_{n-1} G_{n+m} - a^{\xi(n)} b^{\xi(n+1)} G_n G_{n+m-1}) \\ &= (ab+2) 2G_n G_{n+m} + (-1)^n (aG_m G_1 - bG_{m+1} G_0).\end{aligned}$$

Hence we obtain

$$\begin{aligned}Y_2 &= (-1)^n \{\Delta_m - h_m(ab+2)\} G_n G_{n+m} + a^{\xi(n)} b^{\xi(n+1)} G_n G_{n+m-1} + a^{\xi(n+1)} b^{\xi(n)} G_{n+1} G_{n+m}) \\ &\quad - h_m (aG_m G_1 - bG_{m+1} G_0) + h_m^2 - 1,\end{aligned}$$

and there exists a positive integer m_2 such that, for $n \geq m_2$, $X_2 > 0$, and

$$\frac{1}{G_nG_{n+m}} < \frac{1}{a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1}+(-1)^nh_m-1} - \frac{1}{a^{\xi(n+1)}b^{\xi(n)}G_{n+1}G_{n+m}+(-1)^{n+1}h_m-1}.$$

Repeatedly applying the above inequality, we have

$$\sum_{k=n}^{\infty}\frac{1}{G_kG_{k+m}} < \frac{1}{a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1}+(-1)^nh_m-1}, \text{ if } n \geq m_2. \tag{25}$$

Finally, consider

$$\begin{aligned} X_3 &= \frac{1}{a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1}+(-1)^nh_m+1} - \frac{1}{a^{\xi(n+1)}b^{\xi(n)}G_{n+1}G_{n+m}+(-1)^{n+1}h_m+1} - \frac{1}{G_nG_{n+m}} \\ &= \frac{Y_3}{(a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1}+(-1)^nh_m+1)(a^{\xi(n+1)}b^{\xi(n)}G_{n+1}G_{n+m}+(-1)^{n+1}h_m+1)G_nG_{n+m}}, \end{aligned}$$

where

$$\begin{aligned} Y_3 &= Y_2 - 2(a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1} + a^{\xi(n+1)}b^{\xi(n)}G_{n+1}G_{n+m}) \\ &= (-1)^n\{\Delta_m - h_m(ab+2)\}G_nG_{n+m} - a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1} - a^{\xi(n+1)}b^{\xi(n)}G_{n+1}G_{n+m}) \\ &\quad -h_m(aG_mG_1 - bG_{m+1}G_0) + h_m^2 - 1. \end{aligned}$$

There exists a positive integer m_3 such that, for $n \geq m_3$, $X_3 < 0$, and

$$\frac{1}{a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1}+(-1)^nh_m+1} - \frac{1}{a^{\xi(n+1)}b^{\xi(n)}G_{n+1}G_{n+m}+(-1)^{n+1}h_m+1} < \frac{1}{G_nG_{n+m}}.$$

Repeatedly applying the above inequality, we have

$$\frac{1}{a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1}+(-1)^nh_m+1} < \sum_{k=n}^{\infty}\frac{1}{G_kG_{k+m}}, \text{ if } n \geq m_2. \tag{26}$$

Then (19) follows from (23)–(26).

Now suppose that $\Delta_m < 0$. In this case, we have

$$\Delta_m - h_m(ab+2) > 0,$$

and (23)–(26) are respectively modified as

$$\sum_{k=n}^{\infty}\frac{1}{G_kG_{k+m}} < \frac{1}{a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1}+h_m}, \text{ if } n \geq m_4 \text{ and } n \in \mathbb{N}_e, \tag{27}$$

$$\frac{1}{a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1}-h_m} < \sum_{k=n}^{\infty}\frac{1}{G_kG_{k+m}}, \text{ if } n \geq m_5 \text{ and } n \in \mathbb{N}_o, \tag{28}$$

$$\sum_{k=n}^{\infty}\frac{1}{G_kG_{k+m}} < \frac{1}{a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1}+(-1)^nh_m-1}, \text{ if } n \geq m_6, \tag{29}$$

and

$$\frac{1}{a^{\xi(n)}b^{\xi(n+1)}G_nG_{n+m-1}+(-1)^nh_m+1} < \sum_{k=n}^{\infty}\frac{1}{G_kG_{k+m}}, \text{ if } n \geq m_7. \tag{30}$$

Then, (20) easily follows and the proof of (a) is completed.

(b) (21) and (22) can be proved as in Theorem 1, and details are omitted. □

If $a = b$, then Theorem 2 reduces ([7], (Theorem 2.1) with $m \in \mathbb{N}_o$ and $\{G_n\} = \{H_n\} = S(G_0, G_1, a, a)$.

3. Discussion

This paper concerned the properties of the generalized bi-periodic Fibonacci numbers $\{G_n\}$ generated from the recurrence relation: $G_n = aG_{n-1} + G_{n-2}$ (n is even) or $G_n = bG_{n-1} + G_{n-2}$ (n is odd). We derived quite a general identities related to reciprocal sums of products of two generalized bi-periodic Fibonacci numbers. More precisely, we obtained formulas for the integer parts of the numbers

$$\left(\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\xi(k+1)}}{G_k G_{k+m}} \right)^{-1}, \quad m = 0, 2, 4, \cdots,$$

and

$$\left(\sum_{k=n}^{\infty} \frac{1}{G_k G_{k+m}} \right)^{-1}, \quad m = 1, 3, 5, \cdots.$$

The identities obtained in this paper include many existing results as special cases. As already noted in [16], an open problem is whether we can obtain similar results for the same numbers of higher order. It seems that we can also derive similar identities for the numbers of the form

$$\left(\sum_{k=n}^{\infty} \frac{1}{G_k H_{k+m}} \right)^{-1}, \quad m = 0, 1, 2, 3, \cdots,$$

where $\{G_n\} = S(G_0, G_1, a, b)$ and $\{H_n\} = S(H_0, H_1, a, b)$, which is left as another open problem.

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