Article

Initial-Boundary Value Problems for Nonlinear Dispersive Equations of Higher Orders Posed on Bounded Intervals with General Boundary Conditions

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Abstract: The present article concerns general mixed problems for nonlinear dispersive equations of any odd-orders posed on bounded intervals. The results on existence, uniqueness and exponential decay of solutions are presented.

1. Introduction

In this work, we formulate mixed problems with general boundary conditions for the following dispersive equation:

$$u_t + \sum_{j=1}^{l}(-1)^{j+1}D_x^{2j+1}u + uD_xu = 0, \ x \in (0,L); \ t > 0, \tag{1}$$

where L is an arbitrary real positive number and $l \in \mathbb{N}$. We propose Equation (1) because it includes classical models such as the Korteweg-de Vries (KdV) equation, when $l = 1$ [1–4] and the Kawahara equation, when $l = 2$ [5–8]. Dispersive equations posed on bounded and unbounded intervals with the Dirichlet type boundary conditions were studied in [9–19]. It is known that the KdV and Kawahara equations were deduced on the whole real line, however, approximating the line either by bounded or unbounded intervals, one needs to consider initial-boundary value problems posed either on finite or semi-finite intervals [2,4,9–11,13–15,17–22].

Last years, publications on dispersive equations of higher orders appeared [14,16,23–26]. Usually, Dirichlet conditions such as $D_x^iu(t,0) = D_x^iu(t,L) = D_x^lu(t,L) = 0, i = 0, \ldots, l-1$; $t > 0$ were imposed for Equation (1), see [25,26]. In [27], general mixed problems for linear multi-dimensional $(2b + 1)$-hyperbolic equations were studied by means of functional analisys methods. In [28], we have studied boundary value problems for the following linear stationary dispersive equations on bounded intervals subject to general boundary conditions at the endpoints of intervals:

$$\lambda u + \sum_{j=1}^{l}(-1)^{j+1}D_x^{2j+1}u = f(x), \ x \in (0,L); \ l \in \mathbb{N}, \tag{2}$$

where $\lambda > 0$ and f is a given function. Equation (2) appears while solving Equation (1) making use of either the semigroup theory or semi-discrete approaches [13]. We formulate

well posed initial-boundary value problems to Equation (1) imposing the same boundary conditions as for Equation (2) [28].

Our goal is to prove the existence, uniqueness of local and global regular solutions for the formulated problems as well as exponential decay for small initial data.

This article has the following structure: Section 2 contains notations and preliminaries. In Section 3, we formulate the initial-boundary value problems. In Section 4, we prove local existence and uniqueness of regular solutions as well as a "smoothing effect" of them similar to one established in [29] for the initial problem of the KdV equation. In Section 5, the global existence and uniqueness of regular solutions have been established for arbitrary initial data. In Section 6, the existence and uniqueness of small global regular solutions as well as their exponential decay have been established. Section 7 is a conclusion.

2. Notations and Auxiliary Facts

For $x \in (0, L)$, symbols $D^i = D_x^i = \frac{\partial^i}{\partial x^i}$, $i \in \mathbb{N}$; $D = D^1$ denote the partial derivatives of order i. By $\| \cdot \|_\infty$ we denote the norm in $L^\infty(0, L)$. In what follows, we denote by $(\cdot, \cdot)$ and $\| \cdot \|$ as the inner product and the norm in $L^2(0, L)$ and $\| \cdot \|_{H^m}$, $m \in \mathbb{N}$ stands for the norm in L^2-based Sobolev spaces [30].

Lemma 1 (See [26], Lemma 2.2). *Let u belong to $H_0^1(0, L)$, then the following inequality holds:*

$$\|u\|_\infty \leq \sqrt{2}\|Du\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}. \tag{3}$$

Lemma 2 (See [31], p. 125). *Suppose u and $D^m u$, $m \in \mathbb{N}$ belong to $L^2(0, L)$. Then for the derivatives $D^i u$, $0 \leq i < m$, the following inequality holds:*

$$\|D^i u\| \leq C_1\|D^m u\|^{\frac{i}{m}}\|u\|^{1-\frac{i}{m}} + C_2\|u\|, \tag{4}$$

where C_1, C_2 are constants depending only on L, m, i.

Lemma 3 (See [32]). *Let u belong to $H_0^1(0, L)$, then*

$$\|u\| \leq \frac{L}{\pi}\|Du\|. \tag{5}$$

3. Formulation of the Problem

Consider the following evolution equation:

$$u_t + \sum_{j=1}^{l}(-1)^{j+1}D^{2j+1}u + uDu = 0, \quad x \in (0, L); \ t > 0 \tag{6}$$

subject to initial data

$$u(0, x) = u_0(x), \quad x \in (0, L), \tag{7}$$

where u_0 is a given function. In [28], formulation of boundary value problems for the stationary linear equation Equation (2) on the interval $(0, L)$ has been proposed. In the present work, we will use the same formulation for Equations (6) and (7):
l = 1:

$$u(t, 0) = u(t, L) = Du(t, L) = 0, \quad t > 0, \tag{8}$$

l ≥ 2:

$$u(t,0) = u(t,L) = 0, \quad t > 0, \tag{9}$$

$$D^i u(t,0) = \sum_{j=1}^{l} a_{ij} D^j u(t,0), \quad i = l+1,\ldots,2l-1; \quad t > 0, \tag{10}$$

$$D^i u(t,L) = \sum_{j=1}^{l-1} b_{ij} D^j u(t,L), \quad i = l,\ldots,2l-1; \quad t > 0, \tag{11}$$

where a_{ij}, b_{ij} are real constants. Assumptions on the coefficients imply that the L^2-norm of the solutions of Equation (6) is decreasing. Multiplying Equation (6) by u and integrating over $(0,L)$, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2(t) + \sum_{j=1}^{l}(-1)^{j+1}(D^{2j+1}u,u)(t) = 0.$$

A way to obtain $\frac{d}{dt}\|u\|^2(t) \leq 0, t > 0$ is to choose a_{ij}, b_{ij} such that $\sum_{j=1}^{l}(-1)^{j+1}(D^{2j+1}u,u)(t) \geq 0, t > 0$. Making use of integration by parts, finite induction and Young's inequality, we prove that the coefficients a_{ij}, b_{ij} satisfy the following conditions, see [28]:

For **l = 2**:

$$B_1 = b_{31} - \frac{1}{2} - \frac{b_{21}^2}{2} > 0, \quad A_1 = -a_{31} + \frac{1}{2} - a_{32}^2 > 0, \quad A_2 = \frac{1}{4}. \tag{12}$$

This implies that $b_{31} > \frac{1}{2}$, $a_{31} < \frac{1}{2}$, and $|a_{32}|$, $|b_{21}|$ should be sufficiently small or zero.

For **l = 3**:

$$B_1 = b_{31} - b_{51} - \frac{1}{2} - b_{31}^2 - \frac{1}{2}(|b_{32}| + |b_{52}| + |b_{41}|) > 0,$$

$$B_2 = b_{42} - \frac{1}{2} - b_{32}^2 - \frac{1}{2}(|b_{32}| + |b_{52}| + |b_{41}|) > 0,$$

$$A_1 = a_{51} - \frac{1}{2} - \frac{1}{2}(|a_{52}| + |a_{41}| + |a_{53}|) > 0, \tag{13}$$

$$A_2 = -a_{42} + \frac{1}{2} - \frac{1}{2}(|a_{52}| + |a_{41}| + |a_{43}|) > 0,$$

$$A_3 = \frac{1}{4} - \frac{1}{2}(|a_{53}| + |a_{43}|) > 0.$$

This implies that $b_{51} < -\frac{1}{2}$, $b_{42} > \frac{1}{2}$, $a_{51} > \frac{1}{2}$, $a_{42} < \frac{1}{2}$ and the remaining coefficients in Inequality (13) should be sufficiently small or zero.

For **l ≥ 4**:

$$B_i = \sum_{\substack{k=1 \\ 2k+i \geq l}}^{l-i}(-1)^{k+1}b_{2k+i,i} + (2-l) + \frac{(1-l)}{2}b_{li}^2 - \frac{1}{2}\sum_{\substack{j=1 \\ j \neq i}}^{l-1}\left(\sum_{\substack{k=1 \\ 2k+i \geq l}}^{l-i}|b_{2k+i,j}|\right)^2 > 0, \quad i = 1,\ldots,l-1,$$

$$A_i = \sum_{\substack{k=1 \\ 2k+i \geq l+1}}^{l-i}(-1)^k a_{2k+i,i} + (5-2l) - \frac{1}{2}\sum_{\substack{j=1 \\ j \neq i}}^{l-1}\left(\sum_{\substack{k=1 \\ 2k+i \geq l+1}}^{l-i}|a_{2k+i,j}|\right)^2 - \frac{1}{2}\sum_{\substack{k=1 \\ 2k+i \geq l+1}}^{l-i}|a_{2k+i,l}| > 0, \quad i = 1,\ldots,l-1, \tag{14}$$

$$A_l = \frac{1}{4} - \frac{1}{2}\sum_{i=1}^{l-1}\left(\sum_{\substack{k=1 \\ 2k+i \geq l+1}}^{l-i}|a_{2k+i,l}|\right) > 0.$$

It follows that

$$b_{l+1,l-1} > l - 2, \quad b_{l+j,l-j} > \frac{1}{2}\Big(\sum_{m=1}^{\frac{j-1}{2}} |b_{l+2m-1,l-2m+1}|\Big)^2 + l - 2, \quad \underbrace{j = 3, \ldots, l-1,}_{(j \text{ odd})}$$

$$b_{l+2,l-2} < 2 - l, \quad b_{l+j,l-j} < -\frac{1}{2}\Big(\sum_{m=1}^{\frac{j}{2}-1} |b_{l+2m,l-2m}|\Big)^2 + 2 - l, \quad \underbrace{j = 4, \ldots, l-1,}_{(j \text{ even})}$$

$$a_{l+1,l-1} < 5 - 2l, \quad a_{l+j,l-j} < -\frac{1}{2}\Big(\sum_{m=1}^{\frac{j-1}{2}} |a_{l+2m-1,l-2m+1}|\Big)^2 + 5 - 2l, \quad \underbrace{j = 3, \ldots, l-1,}_{(j \text{ odd})}$$

$$a_{l+2,l-2} > 2l - 5, \quad a_{l+j,l-j} > \frac{1}{2}\Big(\sum_{m=1}^{\frac{j}{2}-1} |a_{l+2m,l-2m}|\Big)^2 + 2l - 5, \quad \underbrace{j = 4, \ldots, l-1}_{(j \text{ even})}$$

(15)

and the remaining coefficients of the Inequality (14) should be sufficiently small or zero.

Assuming these coefficients equal to zero in Inequalities (12)–(14), we get the following boundary conditions for all $l \in \mathbb{N}$, [28]:

$$\begin{aligned}
u(t,0) &= u(t,L) = D^l u(t,L) = 0, \quad t > 0, \\
D^{l+j}u(t,0) &= a_{l+j,l-j}D^{l-j}u(t,0), \quad j = 1, \ldots, l-1; \ t > 0, \\
D^{l+j}u(t,L) &= b_{l+j,l-j}D^{l-j}u(t,L), \quad j = 1, \ldots, l-1; \ t > 0
\end{aligned}$$

(16)

with $b_{31} > \frac{1}{2}$, $a_{31} < \frac{1}{2}$ for $l = 2$; $b_{51} < -\frac{1}{2}$, $b_{42} > \frac{1}{2}$, $a_{51} > \frac{1}{2}$, $a_{42} < \frac{1}{2}$ for $l = 3$ and Inequality (15) for $l \geq 4$.

Remark 1 (See [28], Remark 1). *We call (10) and (11) general boundary conditions because they follow from a more general form:*

$$\sum_{i=1}^{2l-1} \alpha_{ki}D^i u(t,0) = 0, \quad k = 1, \ldots, l-1; \ t > 0,$$

$$\sum_{i=1}^{2l-1} \beta_{ki}D^i u(t,L) = 0, \quad k = 1, \ldots, l; \ t > 0,$$

where α_{ki}, β_{ki} are real numbers.

Remark 2. *In this work, we will study the case $l \geq 2$. For the case $l = 1$ see [26].*

4. Local Regular Solutions

Let T be a real positive number and $Q_T = (0,T) \times (0,L)$. Consider the linear evolution equation

$$u_t + \sum_{j=1}^{l}(-1)^{j+1}D^{2j+1}u = g(t,x) \text{ in } Q_T$$

(17)

subject to initial-boundary conditions Equations (7) and (16), with the coefficients satisfying $b_{31} > \frac{1}{2}$, $a_{31} < \frac{1}{2}$ for $l = 2$; $b_{51} < -\frac{1}{2}$, $b_{42} > \frac{1}{2}$, $a_{51} > \frac{1}{2}$, $a_{42} < \frac{1}{2}$ for $l = 3$ and Inequality (15) for $l \geq 4$, where g is a given function. Define the linear operator in $L^2(0,L)$:

$$Au \equiv \sum_{j=1}^{l}(-1)^{j+1}D^{2j+1}u;$$

$$D(A) \equiv \{u \in H^{2l+1}(0,L) : u \text{ satisfies the boundary conditions } \textit{Equation (16)}\}.$$

Theorem 1 (See [28], Theorem 4.1)**.** *Let $f \in L^2(0,L)$. Then for all $\lambda > 0$ a stationary equation: $\lambda u + Au = f(x)$, $x \in (0,L)$ subject to boundary conditions Equation (16) (omitting t) admits a unique regular solution $u = u(x) \in H^{2l+1}(0,L)$ satisfying*

$$\|u\|_{H^{2l+1}} \leq C\|f\| \tag{18}$$

where C is a constant depending on L, l, λ, $a_{l+j,l-j}$, $b_{l+j,l-j}$, $j = 1, \ldots, l-1$.

Theorem 2. *Let $T > 0$, $u_0 \in D(A)$ and $g \in H^1(0,T;L^2(0,L))$ be given. Then, problem Equations (7), (16) and (17) has a unique solution $u = u(t,x)$:*

$$u \in C([0,T];D(A)) \cap C^1([0,T];L^2(0,L)).$$

Proof. Due to Theorem 1, the operator $\lambda I + A$ is surjective for all $\lambda > 0$. On the other hand, by in [28], (33), we obtain

$$(Au,u) \geq \sum_{i=1}^{l-1} B_i(D^i u(L))^2 + \sum_{i=1}^{l} A_i(D^i u(0))^2 \geq 0 \text{ for all } u \in D(A). \tag{19}$$

By the semigroup theory, the result is proven. (See [33], Lemma 2.2.3 and Corollary 2.4.2) □

Theorem 3. *Let $u_0 \in D(A)$. Then there exists a real $T_0 \in (0,T)$ such that Equations (6), (7) and (16) has a unique regular solution $u = u(t,x)$:*

$$u \in L^\infty(0,T_0;H^{2l+1}(0,L)) \cap L^2(0,T_0;H^{(2l+1)+l}(0,L));$$
$$u_t \in L^\infty(0,T_0;L^2(0,L)) \cap L^2(0,T_0;H^l(0,L)).$$

Proof. Define $g = -vDv$; $v, v_t \in L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H^l(0,L))$. Making use of Inequality (3), one can see that $g \in H^1(0,T;L^2(0,L))$, then by Theorem 2, we can define an operator P related to Equations (7), (16) and (17) such that $v \mapsto u = Pv$. Define the Banach space

$$E = \{v(t,x) : v, v_t \in L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H^l(0,L)); \ v(0,\cdot) \equiv u_0\}$$

with the norm

$$\|v\|_E^2 = \operatorname*{ess\,sup}_{t \in (0,T)} \{\|v\|^2(t) + \|v_t\|^2(t)\} + \int_0^T \sum_{j=1}^{l} \left[\|D^j v\|^2(t) + \|D^j v_t\|^2(t)\right] dt$$

and consider $1 < R < +\infty$ such that

$$(1+L)(1+l)\left(2\|u_0\|_{H^l}^4 + \|u_0\|_{H^{2l+1}}^2\right)\frac{M_2}{M_1} \leq R^2 \tag{20}$$

and a ball $B_R = \{v \in E : \|v\|_E^2 \leq 8R^2\}$. Here, $M_1 = \min\limits_{i \in \{1,\ldots,l-1\}} \{B_i, A_i, A_l\}$ and M_2 is the maximum among the coefficients of the derivatives $(D^l u(t,0))^2$, $(D^i u(t,0))^2$, $(D^i u(t,L))^2$, $i = 1, \ldots, l-1$; $t > 0$ (see Inequality (19) and [28], p. 389).

Remark 3. *Note that by Inequalities (12)–(14), $A_l = \frac{1}{4}$ for all $l \geq 2$, therefore $M_1 \leq \frac{1}{4}$. On the other hand, $1 \leq M_2 < +\infty$ for all $l \geq 2$. This provides that $\frac{M_1}{M_2} \leq \frac{1}{4}$ for all $l \geq 2$.*

Lemma 4. *There is a real $T_* > 0$ such that $P(B_R) \subset B_R$.*

Proof. Let $v \in B_R$, then due to Inequality (20), we get

$$\|Dv\|^2(t) \leq \|Du_0\|^2 + \int_0^T \left[\|Dv\|^2(s) + \|Dv_s\|^2(s)\right]ds \leq \left(\frac{M_1}{M_2} + 8\right)R^2 \leq 9R^2, \ t \in (0, T). \quad (21)$$

Making use of Inequalities (3) and (21), we find

$$\|v\|_\infty^2(t) \leq (34)R^2, \ \|v_t\|_\infty^2(t) \leq 2[8R^2 + \|Dv_t\|^2(t)], \ t \in (0, T). \quad (22)$$

Estimate 1. Multiplying Equation (17) by $2u$, integrating over $(0, L)$ and making use of Inequalities (21) and (22), we obtain

$$\frac{d}{dt}\|u\|^2(t) + 2M_1 U(t, 0, L) \leq -2(vDv, u)(t) \leq \|vDv\|^2(t) + \|u\|^2(t)$$
$$\leq \|v\|_\infty^2(t)\|Dv\|^2(t) + \|u\|^2(t) \leq C_1 + \|u\|^2(t), \ t \in (0, T), \quad (23)$$

where $U(t, 0, L) \equiv \sum_{i=1}^{l-1}\left[(D^i u(t, L))^2 + (D^i u(t, 0))^2\right] + (D^l u(t, 0))^2$ and $C_1 = 9(34)R^4$. By the Gronwall Lemma and Inequality (20),

$$\|u\|^2(t) \leq e^T\left(\frac{R^2}{2} + C_1 T\right), \ t \in (0, T).$$

For $T_1 = \min\left\{\ln 2, \frac{2M_1 R^2}{C_1 M_2}, \frac{2M_1 R^2}{C_1 M_2 L}\right\}$, we find

$$\|u\|^2(t) \leq 2R^2, \ t \in (0, T_1). \quad (24)$$

Substituting Inequality (24) into Inequality (23), integrating the result over $(0, T_1)$ and making use of Inequality (20), we get

$$\int_0^{T_1} U(t, 0, L)dt \leq \frac{1}{2M_1}\left[3\frac{M_1}{M_2}R^2 + \frac{M_1}{M_2}R^2\right] = \frac{2R^2}{M_2}. \quad (25)$$

Estimate 2. Multiplying Equation (17) by $2xu$ and making use of Inequalities (21), (22) (24), we obtain

$$\frac{d}{dt}(x, u^2)(t) + 2L\sum_{i=1}^{l-1} B_i(D^i u(t, L))^2 - 2M_2 U(t, 0, L) + \sum_{j=1}^{l}(2j+1)\|D^j u\|^2(t)$$
$$\leq L\left[\|vDv\|^2(t) + \|u\|^2(t)\right] \leq L(C_1 + 2R^2), \ t \in (0, T_1). \quad (26)$$

Integrating Inequality (26) over $(0, T_1)$ and making use of Inequalities (20) and (25), we conclude

$$\int_0^{T_1}\sum_{j=1}^{l}\|D^j u\|^2(t)dt \leq 2R^2. \quad (27)$$

Estimate 3. Differentiating Equation (17) with respect to t, multiplying the result by $2u_t$ and making use of Inequalities (21) and (22), one gets for an arbitrary $\varepsilon > 0$

$$\frac{d}{dt}\|u_t\|^2(t) + 2M_1 U_t(t, 0, L) \leq \varepsilon\left(\|v_t Dv\|^2(t) + \|vDv_t\|^2(t)\right) + \frac{2}{\varepsilon}\|u_t\|^2(t)$$
$$\leq \varepsilon C_2\left(1 + \|Dv_t\|^2(t)\right) + \frac{2}{\varepsilon}\|u_t\|^2(t), \ t \in (0, T),$$

where $U_t(t, 0, L) \equiv \sum_{i=1}^{l-1} \left[(D^i u_t(t, L))^2 + (D^i u_t(t, 0))^2 \right] + (D^l u_t(t, 0))^2$ and $C_2 = C_2(R)$ is a fixed positive constant. Taking $\varepsilon = \frac{M_1}{(16)M_2 C_2}$, we reduce it to the inequality

$$\frac{d}{dt} \|u_t\|^2(t) + 2M_1 U_t(t, 0, L) \leq \frac{(32)M_2 C_2}{M_1} \|u_t\|^2(t) + \frac{M_1}{(16)M_2} \left(1 + \|Dv_t\|^2(t) \right), \quad t \in (0, T). \quad (28)$$

By the Gronwall Lemma,

$$\|u_t\|^2(t) \leq e^{\frac{(32)M_2 C_2}{M_1} T} \left[\|u_t\|^2(0) + \frac{M_1}{(16)M_2} \int_0^T \left(1 + \|Dv_t\|^2(t) \right) dt \right].$$

Due to Inequalities (3) and (20),

$$\|u_t\|^2(0) \leq (1+l) \left(\|u_0 D u_0\|^2 + \|u_0\|_{H^{2l+1}}^2 \right) \leq (1+l) \left(2\|u_0\|_{H^l}^4 + \|u_0\|_{H^{2l+1}}^2 \right) \leq \frac{M_1}{(1+L)M_2} R^2 \leq \frac{R^2}{4}.$$

Choosing $T_2 = \min \left\{ \frac{M_1 \ln 2}{(32)M_2 C_2}, \frac{M_1^2}{(64)M_2^2 C_2}, \frac{3M_1}{8(16)M_2 C_2 L^2} \right\}$, we find

$$\|u_t\|^2(t) \leq 2R^2, \quad t \in (0, T_2). \quad (29)$$

Substituting Inequality (29) into Inequality (28), integrating the result over $(0, T_2)$ and making use of Inequality (20), we get

$$\int_0^{T_2} U_t(t, 0, L) dt \leq \frac{1}{2M_1} \left[\frac{M_1}{M_2} R^2 + 2 \frac{M_1}{M_2} R^2 + \frac{M_1}{M_2} R^2 \right] = \frac{2R^2}{M_2}. \quad (30)$$

Estimate 4. Differentiating Equation (17) with respect to t, multiplying the result by $2x u_t$ and making use of Inequalities (21), (22) and (29), we obtain

$$\frac{d}{dt}(x, u_t^2)(t) + 2L \sum_{i=1}^{l-1} B_i (D^i u_t(t, L))^2 - 2M_2 U_t(t, 0, L) + \sum_{j=1}^{l} (2j+1) \|D^j u_t\|^2(t)$$

$$\leq \varepsilon L \left(\|v_t Dv\|^2(t) + \|v Dv_t\|^2(t) \right) + \frac{2L}{\varepsilon} \|u_t\|^2(t) \leq \varepsilon L C_2 \left(1 + \|Dv_t\|^2(t) \right) + \frac{4L}{\varepsilon} R^2, \quad t \in (0, T_2).$$

Taking $\varepsilon = \frac{M_1}{(16)M_2 C_2 L}$, integrating over $(0, T_2)$, and making use of Inequalities (20) and (30), we find

$$\int_0^{T_2} \sum_{j=1}^{l} \|D^j u_t\|^2(t) dt \leq 2R^2. \quad (31)$$

For $T_* = \min\{T_1, T_2\}$, it follows from Inequalities (24), (27), (29) and (31) that $Pv = u \in B_R$. This completes the proof of Lemma 4. $\quad \square$

Lemma 5. *There is a real $T_\star > 0$ such that the mapping P is a contraction in B_R.*

Proof. For $v_1, v_2 \in B_R$, denote $u_i = Pv_i$, $i = 1, 2$, $w = v_1 - v_2$, $z = u_1 - u_2$ and $Z(t, 0, L) \equiv \sum_{i=1}^{l-1} [(D^i z(t, L))^2 + (D^i z(t, 0))^2] + (D^l z(t, 0))^2$. Then z satisfies the equation

$$z_t + \sum_{j=1}^{l} (-1)^{j+1} D^{2j+1} z = -v_1 Dw - w Dv_2 \quad \text{in } Q_T, \quad (32)$$

boundary conditions Equation (16) and initial data $z(0, \cdot) \equiv 0$.

Similar arguments used in the proof of Lemma 4 show that $\|z\|_E \leq \frac{1}{2} \|w\|_E$. Therefore, P is a contraction in B_R. $\quad \square$

According to Lemmas 4 and 5 and the Banach Fixed Point Theorem with $T_0 = \min\{T_*, T_\star\}$, problem Equations (6), (7) and (16) has a unique generalized solution $u = u(t, x)$:

$$u \in L^\infty(0, T_0; H^l(0, L)); \tag{33}$$

$$u_t \in L^\infty(0, T_0; L^2(0, L)) \cap L^2(0, T_0; H^l(0, L)). \tag{34}$$

Rewrite Equation (6) in the form

$$u + \sum_{j=1}^{l}(-1)^{j+1}D^{2j+1}u = u - u_t - uDu := F(t, x). \tag{35}$$

Due to Relations (33) and (34), it follows that $uDu \in H^1(0, T_0; L^2(0, L))$, hence $F \in L^\infty(0, T_0; L^2(0, L))$. Making use of Inequality (18), we get

$$u \in L^\infty(0, T_0; H^{2l+1}(0, L)). \tag{36}$$

Acting as in [26], Lemma 4.3, we find

$$u \in L^2(0, T_0; H^{(2l+1)+l}(0, L)). \tag{37}$$

Combining Relations (34), (36) and (37), we complete the proof of Theorem 3. □

Remark 4. *The local result presented in Theorem 3 can be obtained under the following boundary conditions:*

$$D^i u(t, 0) = \sum_{j=0}^{l} a_{ij}D^j u(t, 0), \quad i = l+1, \ldots, 2l; \ t > 0, \tag{38}$$

$$D^i u(t, L) = \sum_{j=0}^{l-1} b_{ij}D^j u(t, L), \quad i = l, \ldots, 2l; \ t > 0 \tag{39}$$

instead of Equations (8)–(11) (see [28], Remark 3). We also call Equations (38) and (39) general boundary conditions because they follow from a more general form:

$$\sum_{i=0}^{2l} \alpha_{ki}D^i u(t, 0) = 0, \quad k = 1, \ldots, l; \ t > 0$$

$$\sum_{i=0}^{2l} \beta_{ki}D^i u(t, L) = 0, \quad k = 1, \ldots, l+1; \ t > 0,$$

where α_{ki}, β_{ki} are real numbers (see Remark 1).

5. Global Regular Solutions

Theorem 4. *Let $u_0 \in H^{2l+1}(0, L)$ satisfying Equation (16). Then for all $T > 0$, problem Equations (6), (7) and (16) has a unique regular solution $u = u(t, x)$:*

$$u \in L^\infty(0, T; H^{2l+1}(0, L)) \cap L^2(0, T; H^{(2l+1)+l}(0, L));$$

$$u_t \in L^\infty(0, T; L^2(0, L)) \cap L^2(0, T; H^l(0, L)).$$

Proof. We will obtain a priori estimates independent of $t \in (0, T)$.
Estimate 1. Multiplying Equation (6) by $2u$, we obtain

$$2(u_t, u)(t) + 2M_1 U(t, 0, L) \le 0, \tag{40}$$

where $U(t, 0, L) \equiv \sum_{i=1}^{l-1} \left[(D^i u(t, L))^2 + (D^i u(t, 0))^2 \right] + (D^l u(t, 0))^2$ and $M_1 = \min_{i \in \{1, \dots, l-1\}} \{B_i, A_i, A_l\}$.

Consequently,

$$\|u\|(t) \le \|u_0\|, \quad t \in (0, T). \tag{41}$$

Estimate 2. Multiplying Equation (6) by $2xu$, we get

$$2(u_t, xu)(t) + 2L \sum_{i=1}^{l-1} B_i (D^i u(t, L))^2 - 2M_2 U(t, 0, L) + \sum_{j=1}^{l} (2j+1) \|D^j u\|^2(t) + 2(uDu, xu)(t) \le 0, \tag{42}$$

where M_2 is calculated in [28], p. 389. Making use of Inequalities (3) and (41), we estimate

$$2(uDu, xu)(t) = -\frac{2}{3}(u, u^2)(t) \ge -\frac{2\sqrt{2}}{3} \|Du\|^{\frac{1}{2}}(t) \|u\|^{\frac{1}{2}}(t) \|u\|^2(t)$$

$$\ge -\frac{2\sqrt{2}}{3} \|Du\|^{\frac{1}{2}}(t) \|u_0\|^{\frac{5}{2}} \ge -\|Du\|^2(t) - C\|u_0\|^{\frac{10}{3}}, \tag{43}$$

where C is a positive constant. On the other hand, due to Inequality (40) and the fact that $\frac{M_2}{M_1} \ge 1$ for all $l \ge 2$, we get

$$-2M_2 U(t, 0, L) \ge \frac{2M_2}{M_1}(u_t, u)(t) \ge 2(u_t, u)(t). \tag{44}$$

Substituting Inequalities (43) and (44) into Inequality (42), we find

$$\frac{d}{dt}(1 + x, u^2)(t) + 2\|Du\|^2(t) + \sum_{j=2}^{l-1} (2j+1) \|D^j u\|^2(t) \le C\|u_0\|^{\frac{10}{3}}. \tag{45}$$

After integration of Inequality (45) over $(0, T)$, we conclude

$$\int_0^T \sum_{j=1}^{l} \|D^j u\|^2(t) dt \le C\|u_0\|^2 \tag{46}$$

where $C = C(T, L, l, \|u_0\|)$ is a positive constant.

Estimate 3. Differentiate Equation (6) with respect to t, multiply the result by $2u_t$ to obtain

$$2(u_{tt}, u_t)(t) + 2M_1 U_t(t, 0, L) + 2(D[uu_t], u_t)(t) \le 0, \tag{47}$$

where $U_t(t, 0, L) \equiv \sum_{i=1}^{l-1} \left[(D^i u_t(t, L))^2 + (D^i u_t(t, 0))^2 \right] + (D^l u_t(t, 0))^2$. Making use of Inequalities (3) and (41), we estimate for an arbitrary $\varepsilon > 0$

$$2(D[uu_t], u_t)(t) = -2(uu_t, Du_t)(t) \ge -\frac{1}{\varepsilon}(|u|^2, |u_t|^2)(t) - \varepsilon \|Du_t\|^2(t)$$

$$\ge -\frac{2}{\varepsilon} \|Du\|(t) \|u\|(t) \|u_t\|^2(t) - \varepsilon \|Du_t\|^2(t) \ge -\left(\frac{1}{\varepsilon^2} \|u_0\|^2 + \|Du\|^2(t) \right) \|u_t\|^2(t) - \varepsilon \|Du_t\|^2(t). \tag{48}$$

Substituting Inequality (48) into Inequality (47), we find

$$U_t(t, 0, L) \le -\frac{1}{M_1}(u_{tt}, u_t)(t) + \frac{1}{2M_1} \left(\frac{1}{\varepsilon^2} \|u_0\|^2 + \|Du\|^2(t) \right) \|u_t\|^2(t) + \frac{\varepsilon}{2M_1} \|Du_t\|^2(t). \tag{49}$$

Estimate 4. Differentiate Equation (6) with respect to t, multiply the result by $2xu_t$ and integrate over $(0, L)$. The result reads

$$2(u_{tt}, xu_t)(t) + 2L \sum_{i=1}^{l-1} B_i (D^i u_t(t, L))^2 - 2M_2 U_t(t, 0, L)$$

$$+ \sum_{j=1}^{l} (2j+1) \|D^j u_t\|^2(t) + 2(D[uu_t], xu_t)(t) \leq 0. \tag{50}$$

Making use of Inequalities (3) and (41), we estimate

$$2(D[uu_t], xu_t)(t) = -2(uu_t, u_t + xDu_t)(t) \geq -2(|u|, u_t^2)(t) - L^2(u^2, u_t^2)(t) - \|Du_t\|^2(t)$$

$$\geq -2\sqrt{2}\|Du\|^{\frac{1}{2}}(t)\|u_0\|^{\frac{1}{2}}\|u_t\|^2(t) - 2L^2\|Du\|(t)\|u_0\|\|u_t\|^2(t) - \|Du_t\|^2(t)$$

$$\geq -C\Big(1 + \|Du\|^2(t)\Big)\|u_t\|^2(t) - \|Du_t\|^2(t) \tag{51}$$

for some positive constant $C = C(L, \|u_0\|)$. On the other hand, taking into account Inequality (49) with $\varepsilon = \frac{M_1}{M_2}$ and exploiting the relation $\frac{M_2}{M_1} \geq 1$ for all $l \geq 2$, we obtain

$$-2M_2 U_t(t, 0, L) \geq 2(u_{tt}, u_t)(t) - C\Big(1 + \|Du\|^2(t)\Big)\|u_t\|^2(t) - \|Du_t\|^2(t) \tag{52}$$

for some positive constant $C = C(M_1, M_2, \|u_0\|)$. Substituting Inequalities (51) and (52) into Inequality (50), we get

$$\frac{d}{dt}(1 + x, u_t^2)(t) + \|Du_t\|^2(t) + \sum_{j=2}^{l} (2j+1)\|D^j u_t\|^2(t) \leq C\Big(1 + \|Du\|^2(t)\Big)(1 + x, u_t^2)(t). \tag{53}$$

Due to Inequality (46), $1 + \|Du\|^2(t) \in L^1(0, T)$, whence by the Gronwall Lemma,

$$\|u_t\|^2(t) \leq (1 + x, u_t^2)(t) \leq C\Big(\|u_0\|_{H^l}^4 + \|u_0\|_{H^{2l+1}}^2\Big). \tag{54}$$

Substituting Inequality (54) into Inequality (53) and integrating over $(0, T)$, we find

$$\int_0^T \sum_{j=1}^{l} \|D^j u_t\|^2(t)dt \leq C\Big(\|u_0\|_{H^l}^4 + \|u_0\|_{H^{2l+1}}^2\Big) \tag{55}$$

with a positive constant $C = C(T, L, l, a_{l+j,l-j}, b_{l+j,l-j}, \|u_0\|)$, $j = 1, \ldots, l-1$.

Estimates Inequalities (41), (46), (54) and (55) allow us to extend the local solution ensured by Theorem 3 to all $T > 0$ and to prove the existence of a generalized solution $u = u(t, x)$:

$$u \in L^\infty(0, T; H^l(0, L)); \quad u_t \in L^\infty(0, T; L^2(0, L)) \cap L^2(0, T; H^l(0, L)). \tag{56}$$

Acting as by the proof of Theorem 3 and making use of Relation (56), we get

$$u \in L^\infty(0, T; H^{2l+1}(0, L)) \cap L^2(0, T; H^{(2l+1)+l}(0, L)).$$

The existence part of Theorem 4 is proved.

Lemma 6. *A regular solution of Equations (6), (7) and (16) is uniquelly defined.*

Proof. Let u_1 and u_2 be two distinct regular solutions of Equations (6), (7) and (16), then the difference $w = u_1 - u_2$ satisfies the equation

$$w_t + \sum_{j=1}^{l} (-1)^{j+1} D^{2j+1} w + \frac{1}{2} D[u_1^2 - u_2^2] = 0, \tag{57}$$

boundary conditions Equation (16) and initial data $w(0, \cdot) \equiv 0$.

Estimate 5. Multiplying Equation (57) by $2w$, we find

$$2(w_t, w)(t) + 2M_1 W(t, 0, L) + (D[u_1^2 - u_2^2], w)(t) \leq 0, \tag{58}$$

where $W(t, 0, L) \equiv \sum_{i=1}^{l-1} \left[(D^i w(t, L))^2 + (D^i w(t, 0))^2 \right] + (D^l w(t, 0))^2$. Since u_1, u_2 are regular solutions of Equations (6), (7) and (16), then

$$\|u_i\|_{H^l}(t) \leq C < +\infty, \quad i = 1, 2 \quad \text{for a.e. } t \in (0, T), \tag{59}$$

where $C = C(L, l, a_{l+j,l-j}, b_{l+j,l-j})$, $j = 1, \ldots, 1 - 1$. Making use of Inequalities (3) and (59), we estimate for an arbitrary $\varepsilon > 0$

$$
\begin{aligned}
(D[u_1^2 - u_2^2], w)(t) &\geq -(|u_1 + u_2||w|, |Dw|)(t) \geq -\|u_1 + u_2\|_\infty(t) \left(\frac{1}{2\varepsilon} \|w\|^2(t) + \frac{\varepsilon}{2} \|Dw\|^2(t) \right) \\
&\geq -2\sqrt{2}C \left(\frac{1}{2\varepsilon} \|w\|^2(t) + \frac{\varepsilon}{2} \|Dw\|^2(t) \right).
\end{aligned}
\tag{60}
$$

Substituting Inequality (60) into Inequality (58), we obtain

$$W(t, 0, L) \leq -\frac{1}{M_1}(w_t, w)(t) + \frac{\sqrt{2}C}{M_1} \left(\frac{1}{2\varepsilon} \|w\|^2(t) + \frac{\varepsilon}{2} \|Dw\|^2(t) \right). \tag{61}$$

Estimate 6. Multiplying Equation (57) by $2xw$, we get

$$
\begin{aligned}
&2(w_t, xw_t)(t) + 2L \sum_{i=1}^{l-1} B_i (D^i w(t, L))^2 - 2M_2 W(t, 0, L) \\
&\quad + \sum_{j=1}^{l} (2j+1) \|D^j w\|^2(t) + 2(D[u_1^2 - u_2^2], xw)(t) \leq 0.
\end{aligned}
\tag{62}
$$

Making use of Inequalities (3) and (59), we estimate for an arbitrary $\varepsilon > 0$

$$
\begin{aligned}
2(D[u_1^2 - u_2^2], xw)(t) &\geq -(|u_1 + u_2||w|, |w| + x|Dw|)(t) \\
&\geq -\|u_1 + u_2\|_\infty(t) \left(\|w\|^2(t) + \frac{1}{2\varepsilon}(x, w^2)(t) + \frac{\varepsilon L}{2} \|Dw\|^2(t) \right) \\
&\geq -2\sqrt{2}C \left[\left(1 + \frac{1}{2\varepsilon} \right)(1 + x, w^2)(t) + \frac{\varepsilon L}{2} \|Dw\|^2(t) \right].
\end{aligned}
\tag{63}
$$

Substituting Inequalities (61) and (63) into Inequality (62), we reduce it to the inequality

$$
\frac{d}{dt}(1 + x, w^2)(t) + \left[3 - \varepsilon \left(\frac{\sqrt{2}CM_2}{M_1} + \sqrt{2}CL \right) \right] \|Dw\|^2(t) + \sum_{j=2}^{l} (2j+1) \|D^j w\|^2(t)
$$

$$
\leq \left[\frac{\sqrt{2}CM_2}{M_1 \varepsilon} + 2\sqrt{2}C \left(1 + \frac{1}{2\varepsilon} \right) \right] (1 + x, w^2)(t).
$$

Taking $\varepsilon > 0$ such that $3 - \varepsilon \left(\frac{\sqrt{2}CM_2}{M_1} + \sqrt{2}CL \right) > 0$ and applying the Gronwall Lemma, we obtain $\|w\|(t) \equiv 0$, $t \in (0, T)$. This completes the proof of Lemma 6. $\quad\square$

Uniqueness part of Theorem 4 is thereby proved. □

6. Exponential Decay of Small Regular Solutions

Theorem 5. *Let $u_0 \in H^{2l+1}(0, L)$ satisfy Equation (16) and*

$$\|u_0\| < \min\{m_1, m_2\}, \tag{64}$$

where

$$m_1 = \left(\frac{\pi^2}{C_0 L^2}\right)^{3/4}, \quad m_2 = \left[\frac{\pi^2}{L^2}\left(2\sqrt{2}K_0(1 + C_1 K_0)\right)^{-1}\right]^2,$$

$$K_0 = \left((1 + L)(1 + l)\left(2\|u_0\|_{H^l}^4 + \|u_0\|_{H^{2l+1}}^2\right) + (2 + L + C_0\|u_0\|^{4/3})\|u_0\|^2\right)^{1/3},$$

$$C_0 = 2^{-2/3} \cdot 3^{-1/3}, \quad C_1 = \left(\frac{2L^2}{\sqrt{2}} + \frac{2M_2^2}{\sqrt{2}M_1^2}\right)\|u_0\|^{1/2}.$$

Then Equations (6), (7) and (16) has a unique global regular solution $u = u(t, x)$:

$$u \in L^\infty((0, +\infty); H^{2l+1}(0, L)) \cap L^2((0, +\infty); H^{(2l+1)+l}(0, L));$$

$$u_t \in L^\infty((0, +\infty); L^2(0, L)) \cap L^2((0, +\infty); H^l(0, L))$$

satisfying the inequalities:

$$\|u\|^2(t) \le Ce^{-\theta t}, \quad \|u_t\|^2(t) \le Ce^{-\theta t}, \quad \|u\|_{H^{2l+1}}^2(t) \le Ce^{-\theta t},$$

where $\theta = \pi^2/(1 + L)L^2$.

Proof. We need global in t a priori estimates of local solutions in order to prolong them for all $t > 0$.

Estimate 1. Estimates Inequalities (40) and (41) are valid in our case:

$$U(t, 0, L) \le -\frac{(u_t, u)}{M_1}(t), \quad \|u\|(t) \le \|u_0\| \quad t > 0, \tag{65}$$

where $U(t, 0, L) \equiv \sum_{i=1}^{l-1}\left[(D^i u(t, L))^2 + (D^i u(t, 0))^2\right] + (D^l u(t, 0))^2$ and $M_1 = \min_{i \in \{1, \dots, l-1\}}\{B_i, A_i, A_l\}$.

Estimate 2. Multiply Equation (6) by $2xu$ and integrate over $(0, L)$ to obtain

$$2(u_t, xu)(t) + 2L\sum_{i=1}^{l-1} B_i(D^i u(t, L))^2 - 2M_2 U(t, 0, L) + \sum_{j=1}^{l}(2j+1)\|D^j u\|^2(t) + 2(uDu, xu)(t) \le 0, \tag{66}$$

where M_2 is calculated in [28], p. 389. Making use of Inequalities (3) and (65), we estimate

$$2(uDu, xu)(t) = -\frac{2}{3}(u, u^2)(t) \ge -\frac{2\sqrt{2}}{3}\|Du\|^{\frac{1}{2}}(t)\|u\|^{\frac{1}{2}}(t)\|u\|^2(t)$$

$$\ge -\|Du\|^2(t) - C_0\|u_0\|^{\frac{4}{3}}\|u\|^2(t), \tag{67}$$

where $C_0 = 2^{-2/3} \cdot 3^{-1/3}$. Substituting Inequalities (65) and (67) into Inequality (66) and using Equation (5), we get

$$\frac{d}{dt}(1 + x, u^2)(t) + \|Du\|^2(t) + \sum_{j=2}^{l}(2j+1)\|D^j u\|^2(t) + \frac{1}{1+L}\left(\frac{\pi^2}{L^2} - C_0\|u_0\|^{\frac{4}{3}}\right)(1 + x, u^2)(t) \le 0.$$

Due to Inequalities (5) and (64),

$$\frac{d}{dt}(1+x,u^2)(t) + \theta(1+x,u^2)(t) \leq 0,$$

where $\theta = \pi^2/(1+L)L^2$. Consequently

$$\|u\|^2(t) \leq Ce^{-\theta t}. \tag{68}$$

Estimate 3. By Inequalities (3) and (65), Inequality (66) becomes

$$2L\sum_{i=1}^{l-1} B_i(D^i u(t,L))^2 + \sum_{j=1}^{l}(2j+1)\|D^j u\|^2(t) \leq -2(u_t,(1+x)u)(t) - 2(uDu,xu)(t)$$

$$\leq 2(1+L)\|u_t\|(t)\|u\|(t) + \frac{2\sqrt{2}}{3}\|Du\|^{\frac{1}{2}}(t)\|u\|^{\frac{1}{2}}(t)\|u\|^2(t)$$

$$\leq \|Du\|^2(t) + (1+L)\|u_t\|^2(t) + (1+L+C_0\|u_0\|^{\frac{4}{3}})\|u\|^2(t).$$

Thus

$$\|u\|^2_{H^l}(t) \leq (1+L)\|u_t\|^2(t) + (2+L+C_0\|u_0\|^{\frac{4}{3}})\|u\|^2(t). \tag{69}$$

Estimate 4. Differentiate Equation (6) with respect to t, multiply the result by $2u_t$ and integrate over $(0,L)$ to obtain

$$2(u_{tt},u_t)(t) + 2M_1 U_t(t,0,L) + 2(D[uu_t],u_t)(t) \leq 0, \tag{70}$$

where $U_t(t,0,L) \equiv \sum_{i=1}^{l-1}\left[(D^i u_t(t,L))^2 + (D^i u_t(t,0))^2\right] + (D^l u_t(t,0))^2$. Repeating arguments used to prove Inequality (48), we estimate for an arbitrary $\varepsilon > 0$

$$2(D[uu_t],u_t)(t) \geq -\frac{2}{\varepsilon}\|Du\|(t)\|u\|(t)\|u_t\|^2(t) - \varepsilon\|Du_t\|^2(t). \tag{71}$$

Substituting Inequality (71) into Inequality (70) and using Inequality (65), we find

$$U_t(t,0,L) \leq -\frac{1}{M_1}(u_{tt},u_t)(t) + \frac{1}{\varepsilon M_1}\|u_0\|\|Du\|(t)\|u_t\|^2(t) + \frac{\varepsilon}{2M_1}\|Du_t\|^2(t). \tag{72}$$

Estimate 5. Differentiate Equation (6) with respect to t, multiply the result by $2xu_t$ to obtain

$$2(u_{tt},xu_t)(t) + 2L\sum_{i=1}^{l-1} B_i(D^i u_t(t,L))^2 - \underbrace{2M_2 U_t(t,0,L)}_{I_1}$$

$$+ \sum_{j=1}^{l}(2j+1)\|D^j u_t\|^2(t) + \underbrace{2(D[uu_t],xu_t)(t)}_{I_2} \leq 0. \tag{73}$$

Taking into account Inequality (72) with $\varepsilon = \frac{M_1}{2M_2}$ and exploiting the relation $\frac{M_2}{M_1} \geq 1$ for all $l \geq 2$, we obtain

$$I_1 \geq 2(u_{tt},u_t)(t) - \frac{4M_2^2}{M_1^2}\|u_0\|\|Du\|(t)\|u_t\|^2(t) - \frac{1}{2}\|Du_t\|^2(t).$$

On the other hand, repeating arguments used to prove Inequality (51), we estimate

$$I_2 \geq -2\sqrt{2}\|u_0\|^{\frac{1}{2}}\|Du\|^{\frac{1}{2}}(t)\left[1 + \frac{2L^2}{\sqrt{2}}\|u_0\|^{\frac{1}{2}}\|Du\|^{\frac{1}{2}}(t)\right]\|u_t\|^2(t) - \frac{1}{2}\|Du_t\|^2(t).$$

Making use of Inequalities (65) and (69), we find

$$I_1 + I_2 \geq 2(u_{tt}, u_t)(t) - 2\sqrt{2}\|u_0\|^{\frac{1}{2}}\|Du\|^{\frac{1}{2}}(t)\Big[1 + \Big(\frac{2L^2}{\sqrt{2}} + \frac{2M_2^2}{\sqrt{2}M_1^2}\Big)\|u_0\|^{\frac{1}{2}}\|Du\|^{\frac{1}{2}}(t)\Big]\|u_t\|^2(t) - \|Du_t\|^2(t)$$

$$\geq 2(u_{tt}, u_t)(t) - 2\sqrt{2}\|u_0\|^{\frac{1}{2}}\Big((1+L)\|u_t\|^2(t) + (2+L+C_0\|u_0\|^{\frac{4}{3}})\|u_0\|^2\Big)^{\frac{1}{4}}\Big[1 + \Big(\frac{2L^2}{\sqrt{2}} + \frac{2M_2^2}{\sqrt{2}M_1^2}\Big)\|u_0\|^{\frac{1}{2}}$$

$$\times \Big((1+L)\|u_t\|^2(t) + (2+L+C_0\|u_0\|^{\frac{4}{3}})\|u_0\|^2\Big)^{\frac{1}{4}}\Big]\|u_t\|^2(t) - \|Du_t\|^2(t).$$

Substituting $I_1 + I_2$ into Inequality (73), we get

$$\frac{d}{dt}(1+x, u_t^2)(t) + \|Du_t\|^2(t) + \sum_{j=2}^{l}(2j+1)\|D^j u_t\|^2(t)$$

$$+ \|Du_t\|^2(t) - 2\sqrt{2}\|u_0\|^{\frac{1}{2}}K(t)\Big(1 + C_1 K(t)\Big)\|u_t\|^2(t) \leq 0. \tag{74}$$

Here

$$K(t) = \Big((1+L)\|u_t\|^2(t) + (2+L+C_0\|u_0\|^{4/3})\|u_0\|^2\Big)^{1/3}, \quad C_1 = \Big(\frac{2L^2}{\sqrt{2}} + \frac{2M_2^2}{\sqrt{2}M_1^2}\Big)\|u_0\|^{\frac{1}{2}}.$$

Using Inequality (5), this inequality can be rewritten as

$$\frac{d}{dt}(1+x, u_t^2)(t) + \|Du_t\|^2(t) + \sum_{j=2}^{l}(2j+1)\|D^j u_t\|^2(t)$$

$$+ \frac{1}{1+L}\Big[\frac{\pi^2}{L^2} - 2\sqrt{2}\|u_0\|^{\frac{1}{2}}K(t)\Big(1 + C_1 K(t)\Big)\Big](1+x, u_t^2)(t) \leq 0. \tag{75}$$

Taking into account Inequality (64), the fact that $K(0) \leq K_0$ and standard arguments, see [14], we reduce it to the form

$$\frac{d}{dt}(1+x, u_t^2)(t) + \theta(1+x, u_t^2)(t) \leq 0,$$

where $\theta = \pi^2/(1+L)L^2$. This implies

$$\|u_t\|^2(t) \leq (1+x, u_t^2)(t) \leq Ce^{-\theta t}. \tag{76}$$

Returning to Inequality (74) with Inequality (76) and integrating over $(0, +\infty)$, we obtain

$$u_t \in L^2((0, +\infty); H^l(0, L)).$$

Finally, substituting Inequalities (68) and (76) into Inequality (69), we find

$$\|u\|_{H^l}^2(t) \leq Ce^{-\theta t}. \tag{77}$$

Estimate 6. (Regularity) Rewrite Equation (6) in the form

$$(-1)^{l+1}D^{2l+1}u = -u_t - \sum_{j=1}^{l-1}(-1)^{j+1}D^{2j+1}u - uDu.$$

We estimate

$$\|D^{2l+1}u\|(t) \leq \|u_t\|(t) + \sum_{2j\leq l-1}\|D^{2j+1}u\|(t) + \sum_{l-1<2j<2l}\|D^{2j+1}u\|(t) + \|uDu\|(t). \tag{78}$$

For $l = 2$, we have $\sum_{2j \leq l-1} \|D^{2j+1}u\|(t) = 0$ and for $l \geq 3$, due to Inequality (77),

$$\sum_{2j \leq l-1} \|D^{2j+1}u\|(t) \leq l\|u\|_{H^l}(t) \leq Ce^{-\frac{\theta}{2}t}. \tag{79}$$

Making use of Inequalities (3) and (77), we obtain

$$\|uDu\|(t) \leq \sqrt{2}\|u\|_{H^l}^2(t) \leq Ce^{-\theta t}. \tag{80}$$

On the other hand, Inequality (4) implies

$$\|D^{2j+1}u\|(t) \leq C_1^j \|D^{2l+1}u\|^{\alpha^j}(t)\|u\|^{1-\alpha^j}(t) + C_2^j\|u\|(t) \ (l-1 < 2j < 2l),$$

where $\alpha^j = \frac{2j+1}{2l+1}$ and C_1^j, C_2^j are constants depending on L, l. Making use of the Young inequality with an arbitrary $\varepsilon > 0$, we get

$$\|D^{2j+1}u\|(t) \leq \varepsilon\|D^{2l+1}u\|(t) + (C^j(\varepsilon) + C_2^j)\|u\|(t).$$

Summing over $l - 1 < 2j < 2l$ and taking into account Inequality (68), we find

$$\sum_{l-1 < 2j < 2l} \|D^{2j+1}u\|(t) \leq l\varepsilon\|D^{2l+1}u\|(t) + C(\varepsilon)e^{-\frac{\theta}{2}t}. \tag{81}$$

Substituting Inequalities (76), (79), (80) and (81) into Inequality (78) and taking $\varepsilon = \frac{1}{2l}$, we obtain

$$\|D^{2l+1}u\|(t) \leq C\left(3e^{-\frac{\theta}{2}t} + e^{-\theta t}\right) \leq Ce^{-\frac{\theta}{2}t}. \tag{82}$$

Again by Inequality (4), for all $i = l + 1, \ldots, 2l$, there are constants C_1^i, C_2^i depending only on L, l such that

$$\|D^i u\|(t) \leq C_1^i \|D^{2l+1}u\|^{\alpha^i}(t)\|u\|^{1-\alpha^i}(t) + C_2^i\|u\|(t) \text{ with } \alpha^i = \frac{i}{2l+1}.$$

By the Young inequality and Inequalities (68) and (82), we get

$$\|D^i u\|(t) \leq Ce^{-\frac{\theta}{2}t}, \quad i = l+1, \ldots, 2l. \tag{83}$$

According to Inequalities (77), (82) and (83), we conclude that

$$\|u\|_{H^{2l+1}}^2(t) \leq Ce^{-\theta t} \tag{84}$$

with $\theta = \pi^2/(1+L)L^2$. Repeating the arguments that appears in the proof of Lemma 4.3 in [26], and taking into account Inequality (84), we establish a "smoothing effect":

$$u \in L^2((0, +\infty); H^{(2l+1)+l}(0, L)).$$

Similar arguments used in the proof of Lemma 6 with Inequality (77) instead of Inequality (59), show the uniqueness of the solution. The proof of Theorem 5 is complete. □

7. Conclusions

Making use of the formulation of a linear stationary version of Equation (1) in [28], we prove in Theorem 3 local existence and uniqueness of regular solutions. In Theorem 4, we prove global in $t \in (0, T)$ existence and uniqueness of regular solution for arbitrary smooth initial data and arbitrary $T > 0$. In Theorem 5, we prove global in $t \in (0, +\infty)$ existence and uniqueness of regular solutions as well as their exponential decay of $\|u\|(t)$, $\|u_t\|(t)$ and $\|u\|_{H^{2l+1}}(t)$ for small initial data. A smoothing effect has been established:

if $u_0 \in H^{2l+1}(0, L)$, then $u \in L^2((0, +\infty); H^{(2l+1)+l}(0, L))$. Our results can be used for constructing of numerical schemes while studying various models of initial-boundary value problems for higher-order dispersive equations.

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