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On Jungck–Branciari–Wardowski Type Fixed Point Results

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Abstract: The terms of $F-$integral contraction as well as $(\varpi, \tilde{\zeta}, F, i)-$integral contraction are introduced. Fixed point and common fixed point theorems are established. For the mapping F we use only the supposition that it is strictly increasing. As a consequence of the main theorems we obtain Jungck–Wardowski, Branciari–Wardowski and Jungck–Branciari type results. Consequently, the results presented in the article enhance and complement some known results in literature.

Keywords: fixed point; banach contraction principle; branciari contraction; jungck contraction; compatible mappings

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1. Introduction and Preliminaries

In 1976, Jungck [1] generalized the principle proposed by Banach [2] as follows:

Theorem 1. *Let $h, i : \Omega \to \Omega$, $ih(a) = hi(a)$, $a \in \Omega$ where (Ω, w) is a complete metric space, and*

$$w(ha, hb) \le \lambda w(ia, ib), \quad a, b \in \Omega, \ \lambda \in (0, 1). \tag{1}$$

If $h(\Omega) \subset i(\Omega)$ and i is continuous then there exists a unique $u \in \Omega$ so that $hu = iu = u$.

Wardowski [3] proposed a new contractive condition that generalizes [2].

Definition 1. *Let (Ω, w) be a metric space and $\mathcal{F}$ be a set of mappings $F : (0, +\infty) \to (-\infty, +\infty)$ satisfying the next three conditions:*

(F1) For all $l_1, l_2 \in (0, +\infty), l_1 < l_2$ yields $F(l_1) < F(l_2)$;
(F2) If $\{a_n\}_{n \in \mathbb{N}} \subset (0, +\infty)$ and $\lim_{n \to +\infty} a_n = 0$, then $\lim_{n \to +\infty} F(a_n) = -\infty$ and vice versa.
(F3) $\lim_{a \to 0^+} a^{\mu} F(a) = 0$ for some $\mu \in (0, 1)$.

A mapping $h : \Omega \to \Omega$ is $F-$contraction (in the sense of D. Wardowski) on (Ω, w) if there exists $\varpi > 0$ such that for all $a, b \in \Omega$, $w(ha, hb) > 0$ yields

$$\varpi + F(w(ha, hb)) \le F(w(a, b)). \tag{2}$$

Theorem 2. *Let (Ω, w) be a complete metric space, and let $h : \Omega \to \Omega$ be a $F-$contraction. Then, there is a $b \in \Omega$, $b = hb$ and it is unique.*

Remark 1. *Based on $F(l-0) \leq F(l) \leq F(l+0)$, $l \in (0,+\infty)$, and (F1) we conclude that there are $\lim_{a\to b^-} F(a) = F(b-0)$ and $\lim_{a\to b^+} F(a) = F(b+0)$. For all particulars see [4,5]. More details of the property (F2) can be found in [6,7]. Likewise, if $F:(0,+\infty)\to(-\infty,+\infty)$ is a strictly increasing function, then either $F(0+0) = \lim_{a\to 0^+} F(a) = m, m \in \mathbb{R}$ or $F(0+0) = \lim_{x\to 0^+} F(a) = -\infty$.*

In the proofs of our results in the follow-up we will use the following known lemmas from ([8,9]).

Lemma 1. *[10] Suppose that $\{a_n\}_{n\in\mathbb{N}}$ which belongs to a metric space (Ω,w) and satisfies $\lim_{n\to+\infty} w(a_n,a_{n+1}) = 0$ is not a Cauchy sequence. Therefore, there exists $\varepsilon > 0$ and sequences of positive integers $\{n_k\}$, $\{m_k\}$, $n_k > m_k > k$ such that the sequences*

$$\{w(a_{n_k},a_{m_k}), w(a_{n_k+1},a_{m_k}), w(a_{n_k},a_{m_k-1}), w(a_{n_k+1},a_{m_k-1}), w(a_{n_k+1},a_{m_k+1})\},$$

tend to ε^+ when $k\to+\infty$.

The second significant Banach contraction principle generalization is established in 2002 by Branciari [11]. Firstly, we recall some necessary notions.

Let Ψ be the class of all functions $\tilde{\zeta}:[0,+\infty)\to[0,+\infty)$ which is Lebesgue integrable, summable on every compact set on $[0,+\infty)$ and $\int_0^{\varepsilon}\tilde{\zeta}(t)dt > 0$ for all $\varepsilon > 0$.

The following lemmas are useful for our main results. We shall also suppose that $\tilde{\zeta}\in\Psi$.

Lemma 2. *[12] Let $\{l_n\}_{n\in\mathbb{N}}$ be a non-negative sequence of real numbers so that $\lim_{n\to+\infty} l_n = l$. Then $\lim_{n\to+\infty}\int_0^{l_n}\tilde{\zeta}(t)dt = \int_0^{l}\tilde{\zeta}(t)dt$.*

Lemma 3. *[12] Let $\{l_n\}_{n\in\mathbb{N}}$ be a non-negative sequence of real numbers. Then $\lim_{n\to+\infty}\int_0^{l_n}\tilde{\zeta}(t)dt = 0$ if and only if $\lim_{n\to+\infty} l_n = 0$.*

Here is the Branciaris theorem [11]:

Theorem 3. *Let h be a mapping from a complete metric space (Ω,w) into itself satisfying*

$$\int_0^{w(ha,hb)}\tilde{\zeta}(t)dt \leq \lambda\int_0^{w(a,b)}\tilde{\zeta}(t)dt,$$

for all $a,b\in\Omega$, where $\lambda\in(0,1)$ is a constant and $\tilde{\zeta}\in\Psi$. Then h has a unique fixed point $b\in\Omega$ such that $\lim_{n\to+\infty} h^n a = b$ for each $a\in\Omega$.

For the further results, it is necessary to define the following terms, see Jungck [13,14], also see Abbas and Jungck [15] (Definition 1.3.).

Let $\Omega\neq\emptyset$ and $h,i:\Omega\to\Omega$. If for some $a\in\Omega$, $b = ha = ia$ then a is a coincidence point and b is a point of coincidence of h and i. A pair (h,i) is compatible in (Ω,w) if $\lim_{n\to+\infty} w(hi(a_n), ih(a_n)) = 0$, for every sequence $\{a_n\}$ in Ω such that $\lim_{n\to+\infty} h(a_n) = \lim_{n\to+\infty} i(a_n) = t$, for some $t\in\Omega$. In addition, a pair (h,i) is weakly compatible if $ha = ia$ implies $hi(a) = ih(a)$, $a\in\Omega$. A sequence $\{a_n\}$ in Ω is a Picard–Jungck sequence of the pair (h,i) (based on a_0) if $b_n = ha_n = ia_{n+1}$ for all $n\in\mathbb{N}\cup\{0\}$.

Proposition 1. *[15] If weakly compatible mappings $h, i : \Omega \to \Omega$ have a point of coincidence which is unique $b = ha = ia$, then b is a unique common fixed point of h and i.*

2. Main Result

In this section we shall combine Jungck's, Braniciari's and Wardowski's results for obtaining common and usual fixed points of some self-mappings on metric space (Ω, w). Our results merge, generalize and refine several recent results in the literature. We commence with the following definition.

Definition 2. *Let (Ω, w) be a metric space and $\mathcal{F}$ be a family of mappings $F : (0, +\infty) \to (-\infty, +\infty)$ which satisfy condition (F1). A mapping $h : \Omega \to \Omega$ is said to be an integral $F-$contraction on (Ω, w) if there exists $\varpi > 0$ such that for all $a, b \in \Omega$, $w(ha, hb) > 0$ we have*

$$\varpi + F(\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt) \leq F(\int_0^{w(a,b)} \tilde{\zeta}(t)dt),\ \tilde{\zeta} \in \Psi. \tag{3}$$

Remark 2. *If $\tilde{\zeta}(t) \equiv 1$ then we have a $F-$contraction.*

Theorem 4. *If $F(a) = \ln a$ then the notion of Branciari contraction and integral $F-$contraction are equivalent.*

Proof. At first, we suppose that the mapping h is Branciari contraction. Then

$$-ln\lambda + \ln(\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt) \leq \ln(\int_0^{w(a,b)} \tilde{\zeta}(t)dt)$$

and accordingly we get integral $F-$contraction for $\varpi = -\ln\lambda > 0$.

If h is integral $F-$contraction then we have the following:

$$\varpi + \ln(\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt) \leq \ln(\int_0^{w(a,b)} \tilde{\zeta}(t)dt).$$

Let $\varpi = \ln\varpi_1$. Then $\varpi_1 > 1$ and $\ln(\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt) \leq \ln(\frac{1}{\varpi_1}\int_0^{w(a,b)} \tilde{\zeta}(t)dt)$. Then h is Branciari contraction for $\lambda = \frac{1}{\varpi_1} < 1$. □

Our first new result on integral $F-$contraction is the following one:

Theorem 5. *Let $h : \Omega \to \Omega$ be an integral $F-$contraction with property (F1) in (Ω, w), where (Ω, w) is a metric space which is completed. Then there exists a unique $a \in \Omega$, $a = ha$.*

Proof. We will initially show that fixed point is unique, under the assumption that such a point exists. We presume opposite i.e., there exist u, v, $u \neq v$ and $u = hu$ and $v = hv$. This assumption is obviously false since $\varpi > 0$, $F(\int_0^{w(u,v)} \tilde{\zeta}(t)dt) \in \mathbb{R}$. By (3) it follows:

$$\varpi + F(\int_0^{w(u,v)} \tilde{\zeta}(t)dt) \leq F(\int_0^{w(u,v)} \tilde{\zeta}(t)dt).$$

Let $a_0 \in \Omega$ and $ha_n = a_{n+1}$. If $a_k = a_{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$, then a_k is a unique fixed point. So, $a_k \neq a_{k+1}$ for every $k \in \mathbb{N} \cup \{0\}$. Then,

$$F(\int_0^{w(a_{n+1},a_n)} \tilde{\zeta}(t)dt) < \varpi + F(\int_0^{w(a_{n+1},a_n)} \tilde{\zeta}(t)dt) \leq F(\int_0^{w(a_n,a_{n-1})} \tilde{\zeta}(t)dt)$$

By (F1) we have that

$$\int_0^{w(a_{n+1},a_n)} \tilde{\zeta}(t)dt < \int_0^{w(a_n,a_{n-1})} \tilde{\zeta}(t)dt$$

and thence $w(a_{n+1}, a_n) < w(a_n, a_{n-1})$ for all $n \in \mathbb{N}$. Sequence $\{w(a_{n+1}, a_n)\}$ is monotone decreasing, bounded from bellow and so there exists $\tilde{\rho}$ such that

$$\lim_{n\to+\infty} w(a_n, a_{n+1}) = \tilde{\rho} \geq 0.$$

In addition, $w(a_n, a_{n+1}) > \tilde{\rho}$ for all $n \in \mathbb{N} \cup \{0\}$. Suppose that $\tilde{\rho} > 0$, then

$$\varpi + F(\int_0^{\tilde{\rho}+0} \tilde{\zeta}(t)dt) \leq F(\int_0^{\tilde{\rho}+0} \tilde{\zeta}(t)dt),$$

so we have contradiction and thus $\tilde{\rho} = 0$. Therefrom we have that $\lim_{n\to+\infty} w(a_n, a_{n+1}) = 0$.

It remains to prove that $\{a_n\}$ is a Cauchy sequence. Suppose the contrary. If we put $a = a_{n_k}$ and $b = a_{m_k}$ in contractive condition (3), we obtain

$$F(\int_0^{w(a_{n_k+1},a_{m_k+1})} \tilde{\zeta}(t)dt) < \varpi + F(\int_0^{w(a_{n_k+1},a_{m_k+1})} \tilde{\zeta}(t)dt) \leq F(\int_0^{w(a_{n_k},a_{m_k})} \tilde{\zeta}(t)dt).$$

By Lemma 1 $w(a_{n_k+1}, a_{m_k+1}) \to \varepsilon^+$ and $w(a_{n_k}, a_{m_k}) \to \varepsilon^+$ as $k \to +\infty$ so we get that

$$F(\int_0^{\varepsilon+0} \tilde{\zeta}(t)dt) < \varpi + F(\int_0^{\varepsilon+0} \tilde{\zeta}(t)dt) \leq F(\int_0^{\varepsilon+0} \tilde{\zeta}(t)dt),$$

i.e., consequently, the sequence $\{a_n\}$ is a Cauchy sequence and there exists $a \in \Omega$ such that $\lim_{n\to+\infty} a_n = a$.

Using (3) we have that $w(ha, hb) < w(a, b)$ and therefore h must be continuous. Then $ha = h(\lim_{n\to+\infty} a_n) = \lim_{n\to+\infty} a_{n+1} = a$. □

Example 1. *Let $\Omega = [0, 1]$ and $w(a, b) = |a - b|$. Then metric space (Ω, w) is complete. Let $h(a) = \frac{a}{2}$, $\tilde{\zeta}(t) = 2t$ and $F(a) = -\frac{1}{a}$. Then*

$$F(\int_0^{w(a,b)} \tilde{\zeta}(t)dt) - F(\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt) = -\frac{1}{(a-b)^2} + \frac{4}{(a-b)^2} = \frac{3}{(a-b)^2} \geq 3.$$

Therefore, all requirements of Theorem 5 are satisfied for $\varpi \in (0, 3]$ and obviously is $h(0) = 0$.

Corollary 1. *Let (Ω, w) be a complete metric space, $h : \Omega \to \Omega$ be a function such that there exists $K_i > 0$, $i = \overline{1,5}$ and for all $a, b \in \Omega$ with $w(ha, hb) > 0$, any of the following contractive conditions hold:*

$$K_1 + \int_0^{w(ha,hb)} \tilde{\zeta}(t)dt \leq \int_0^{w(a,b)} \tilde{\zeta}(t)dt;$$

$$K_2 - \frac{1}{\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt} \leq -\frac{1}{\int_0^{w(a,b)} \tilde{\zeta}(t)dt};$$

$$K_3 - \frac{1}{\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt} + \int_0^{w(ha,hb)} \tilde{\zeta}(t)dt \leq -\frac{1}{\int_0^{w(a,b)} \tilde{\zeta}(t)dt} + \int_0^{w(a,b)} \tilde{\zeta}(t)dt;$$

$$K_4 + \frac{1}{1-\exp(\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt)} \leq \frac{1}{1-\exp(\int_0^{w(a,b)} \tilde{\zeta}(t)dt)};$$

$$K_5 + \frac{1}{\exp(-\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt) - \exp(\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt)} \leq \frac{1}{\exp(-\int_0^{w(a,b)} \tilde{\zeta}(t)dt) - \exp(\int_0^{w(a,b)} \tilde{\zeta}(t)dt)};$$

then in every case h has a fixed point which is unique.

Proof. Proof follows directly from Theorem 5. Indeed, since each of the functions $F(l) = l$, $F(l) = -\frac{1}{l}$, $F(l) = -\frac{1}{l} + l$, $F(l) = \frac{1}{1-\exp(r)}$, $F(l) = \frac{1}{\exp(-l)-\exp(l)}$ is strictly increasing on $(0, +\infty)$ the result follows. □

Remark 3. *If in Theorem 5 instead of the contractive condition (3) we assume the following condition for all $a, b \in \Omega$ and $w(ha, hb) > 0$,*

$$\varpi + F(\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt) \leq F(\int_0^{\mathbf{L}(a,b)} \tilde{\zeta}(t)dt), \, \tilde{\zeta} \in \Psi. \tag{4}$$

where

$$\mathbf{L}(a,b) = \max\{w(a,b), w(a,ha), w(b,hb)\}, \tag{5}$$

$$\mathbf{L}(a,b) = \max\{w(a,b), w(a,ha), w(b,hb), \frac{w(a,hb)+w(ha,b)}{2}\}, \tag{6}$$

$$\mathbf{L}(a,b) = \max\{w(a,b), \frac{w(a,ha)+w(b,hb)}{2}, \frac{w(a,hb)+w(b,ha)}{2}\}. \tag{7}$$

then there exists a unique fixed point of the mapping h with the addition that one of the mappings h or F is continuous.

In the next definition the notion of $(\varpi, \tilde{\zeta}, F, i)-$integral contraction is introduced.

Definition 3. *Let $h, i : \Omega \to \Omega$ where (Ω, w) is a metric space. A mapping h is a $(\varpi, \tilde{\zeta}, F, i)-$integral contraction if there exists a function $\varpi : (0, +\infty) \to (0, +\infty)$ satisfying*

$$\liminf_{s \to t^+} \varpi(s) > 0, \ \textit{for all } t > 0, \tag{8}$$

$\tilde{\zeta} \in \Psi$ function $F : (0, +\infty) \to (-\infty, +\infty)$ with property (F1) such that for all $a, b \in \Omega$ with $ha \neq hb$ and $ia \neq ib$ one has

$$\varpi\Big(\int_0^{w(ia,ib)} \tilde{\zeta}(t)dt\Big) + F\Big(\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt\Big) \leq F\Big(\int_0^{w(ia,ib)} \tilde{\zeta}(t)dt\Big). \tag{9}$$

We now state a new result for the term $(\varpi, \tilde{\zeta}, F, i)-$ integral contraction. We succeed in generalizing results from several manuscripts in existing literature, for instance ([11–34]).

Theorem 6. *Let $h, i : \Omega \to \Omega$, h is a $(\varpi, \tilde{\zeta}, F, i)-$integral contraction where (Ω, w) is a metric space. Presume that there exists a Picard sequence $\{b_n\}_{n\in\mathbb{N}\cup\{0\}}$ of (h, i). Further, suppose that (i) or (ii) holds:*

(i) $(i\Omega, w)$ is complete,
(ii) (Ω, w) is complete, i is continuous and (h, i) is compatible.

Then h and i have a unique point of coincidence.

Proof. We initially prove that there is a unique point of coincidence of h and i, assuming that such a point exists. Let $b_1 \neq b_2$ be points of coincidence for h and i. Using that, we conclude that there exist a_1 and a_2 $(a_1 \neq a_2)$ so that $ha_1 = ia_1 = b_1$ and $ha_2 = ia_2 = b_2$. The condition (9) yields that

$$\varpi\left(\int_0^{w(ia_1,ia_2)} \tilde{\zeta}(t)dt\right) + F\left(\int_0^{w(ha_1,ha_2)} \tilde{\zeta}(t)dt\right) \leq F\left(\int_0^{w(ia_1,ia_2)} \tilde{\zeta}(t)dt\right), \tag{10}$$

i.e.,

$$\varpi\left(\int_0^{w(b_1,b_2)} \tilde{\zeta}(t)dt\right) + F\left(\int_0^{w(b_1,b_2)} \tilde{\zeta}(t)dt\right) \leq F\left(\int_0^{w(b_1,b_2)} \tilde{\zeta}(t)dt\right), \tag{11}$$

which is a contradiction, because $\varpi\left(\int_0^{w(b_1,b_2)} \tilde{\zeta}(t)dt\right) > 0$.

Suppose now that there is a Picard–Jungck sequence $\{b_n\}$ such that $b_n = ha_n = ia_{n+1}$, where $n \in \mathbb{N}\cup\{0\}$. If $b_p = b_{p+1}$ for some $p \in \mathbb{N}\cup\{0\}$, then $ib_{p+1} = b_p = hb_{p+1}$, and h and i have a unique point of coincidence. Accordingly, suppose that $b_n \neq b_{n+1}$ for every $n \in \mathbb{N}\cup\{0\}$. By replacing $a = a_n$ and $b = a_{n+1}$ into (9), we get

$$\varpi\left(\int_0^{w(ia_n,ia_{n+1})} \tilde{\zeta}(t)dt\right) + F\left(\int_0^{w(ha_n,ha_{n+1})} \tilde{\zeta}(t)dt\right) \leq F\left(\int_0^{w(ia_n,ia_{n+1})} \tilde{\zeta}(t)dt\right). \tag{12}$$

Guided by the properties of the $\varpi, \tilde{\zeta}$ and F, we get that $w(b_n, b_{n+1}) < w(b_{n-1}, b_n)$, for all $n \in \mathbb{N}$. Therefore, there exists $\bar{\delta} \geq 0$ so that $\lim_{n\to+\infty} w(b_n, b_{n+1}) = \bar{\delta}$. Suppose that $\bar{\delta} > 0$. Based on the condition of the function ϖ we know that there exist $\varpi_0 > 0$ and $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have

$$\varpi_0 + F\left(\int_0^{w(b_n,b_{n+1})} \tilde{\zeta}(t)dt\right) <$$

$$\varpi\left(\int_0^{w(b_{n-1},b_n)}\tilde{\zeta}(t)dt\right)+F\left(\int_0^{w(b_n,b_{n+1})}\tilde{\zeta}(t)dt\right)\le F\left(\int_0^{w(b_{n-1},b_n)}\tilde{\zeta}(t)dt\right), \tag{13}$$

that is,

$$\varpi_0+F\left(\int_0^{w(b_n,b_{n+1})}\tilde{\zeta}(t)dt\right)<F\left(\int_0^{w(b_{n-1},b_n)}\tilde{\zeta}(t)dt\right), \tag{14}$$

for all $n \ge n_1$. Based on the conditions (F1), the last relation yields

$$\varpi_0+F\left(\int_0^{\bar{\delta}+0}\tilde{\zeta}(t)dt\right)\le F\left(\int_0^{\bar{\delta}+0}\tilde{\zeta}(t)dt\right),$$

and it is a contradiction. Hence, $\lim_{n\to+\infty} w(b_n, b_{n+1}) = 0$.

Moreover, it remains to be shown $b_n \ne b_m$ whenever $n \ne m$. We will assume the opposite, i.e., $b_n = b_m$ for some $n > m$. Based on the definition of the Picard–Jungck sequence $\{b_n\}$ we can choose $b_{n+1} = b_{m+1}$. Using the previous arguments, we have

$$w(b_n,b_{n+1})=w(b_m,b_{m+1})<w(b_{m-1},b_m)<\ldots<w(b_{n+1},b_{n+2})<w(b_n,b_{n+1})$$

which is a contradiction.

Further we need to show that the sequence $\{b_n\}$ is a Cauchy sequence. We will show this by the method of contradiction. Including $a = a_{n_k+1}$ and $b = a_{m_k+1}$ in (9), we obtain

$$\varpi\left(\int_0^{w\left(ia_{n_k+1},ia_{m_k+1}\right)}\tilde{\zeta}(t)dt\right)+F\left(\int_0^{w\left(ha_{n_k+1},ha_{m_k+1}\right)}\tilde{\zeta}(t)dt\right)\le F\left(\int_0^{w\left(ia_{n_k+1},ia_{m_k+1}\right)}\tilde{\zeta}(t)dt\right),$$

i.e.,

$$\varpi\left(\int_0^{w\left(b_{n_k},b_{m_k}\right)}\tilde{\zeta}(t)dt\right)+F\left(\int_0^{w\left(b_{n_k+1},b_{m_k+1}\right)}\tilde{\zeta}(t)dt\right)\le F\left(\int_0^{w\left(b_{n_k},b_{m_k}\right)}\tilde{\zeta}(t)dt\right). \tag{15}$$

Using Lemma 1, $w\left(b_{n_k+1}, b_{m_k+1}\right)$ and $w\left(b_{n_k}, b_{m_k}\right)$ tend to ε^+ as $k \to +\infty$, and accordingly we obtain

$$\lim_{w\left(b_{n_k},b_{m_k}\right)\to\varepsilon^+}\inf\varpi\left(\int_0^{w\left(b_{n_k},b_{m_k}\right)}\tilde{\zeta}(t)dt\right)+\lim_{w\left(b_{n_k},b_{m_k}\right)\to\varepsilon^+}\inf F\left(\int_0^{w\left(b_{n_k+1},b_{m_k+1}\right)}\tilde{\zeta}(t)dt\right)$$

$$\le\lim_{w\left(b_{n_k},b_{m_k}\right)\to\varepsilon^+}\inf F\left(\int_0^{w\left(b_{n_k},b_{m_k}\right)}\tilde{\zeta}(t)dt\right),$$

that is,

$$\lim_{w\left(b_{n_k},b_{m_k}\right)\to\varepsilon^+}\inf\varpi\left(\int_0^{w\left(b_{n_k},b_{m_k}\right)}\tilde{\zeta}(t)dt\right)+F\left(\int_0^{\varepsilon^++0}\tilde{\zeta}(t)dt\right)\le F\left(\int_0^{\varepsilon^++0}\tilde{\zeta}(t)dt\right), \tag{16}$$

which is a contradiction with

$$\lim \inf_{w(b_{n_k}, b_{m_k}) \to \varepsilon^+} \varpi \left(\int_0^{w(b_{n_k}, b_{m_k})} \tilde{\zeta}(t)dt \right) > 0.$$

So, we showed that the sequence $\{b_n\}$ is a Cauchy sequence.

Now let (i) hold. Then, there exists $z \in X$ so that $b_n = ia_n \to iz$ as $n \to +\infty$. We shall prove that $hz = iz$. Since $b_n \neq b_m$ whenever $n \neq m$, we can suppose that $hz, iz \notin \{b_n : n \in \mathbb{N} \cup \{0\}\}$. Therefore, by (9) we have

$$F\left(\int_0^{w(b_n,hz)} \tilde{\zeta}(t)dt \right) < \varpi \left(\int_0^{w(b_{n-1},iz)} \tilde{\zeta}(t)dt \right) + F\left(\int_0^{w(b_n,hz)} \tilde{\zeta}(t)dt \right) \leq F\left(\int_0^{w(b_{n-1},iz)} \tilde{\zeta}(t)dt \right). \quad (17)$$

Based on the properties of the function F, we get that $w(b_n, hz) < w(b_{n-1}, iz) \to 0$ as $n \to +\infty$. Hence $hz = iz$ and z is unique.

At the end, let (ii) hold. From completeness of (Ω, ω) it follows that there exists $v \in X$ such that $ha_n \to v$, when $n \to +\infty$. As i is continuous, $iha_n \to iv$ when $n \to +\infty$. By (9) and the continuity of i we conclude that h must also be continuous. Therefore, $hia_n \to hv$ as $n \to +\infty$. As h and i are compatible, we have

$$w(hv, iv) \leq w(hv, hia_n) + w(hia_n, iha_n) + w(iha_n, iv) \to 0 + 0 + 0 = 0. \quad (18)$$

Thus, our result is proved in both cases, and we realize that the mappings h and i have a unique point of coincidence. □

Remark 4. *(1) If (i) is satisfied and (h, i) are weakly compatible, using Proposition 1, we conclude that h and i have a common fixed point. Moreover, the common fixed point is unique.*

(2) Assuming that (ii) holds, h and i also have a unique common fixed point using Proposition 1. We conclude this based on the fact that every compatible pair (h, i) is weakly compatible.

In the following corollary the mapping $F : (0, +\infty) \to (-\infty, +\infty)$ is only strictly increasing one. Therefore our new Theorem 6 generalizes, improves, complements, unifies and enriches several results from $F-$contraction type in existing literature.

Corollary 2. *Putting in Theorem 6 condition $\tilde{\zeta}(t) \equiv 1$ for all $t \in [0, +\infty)$ we get a Jungck–Wardowski type result, i.e., Theorem 8 from [21]. Further if $\tilde{\zeta}(t) \equiv 1$ for all $t \in [0, +\infty)$ and $i = I_\Omega$ the identity mapping on Ω then we obtain Theorem 2.1 from Wardowski [34]. If $\tilde{\zeta}(t) \equiv 1$ for all $t \in [0, +\infty), \varpi(t) = \varpi =$constant from $(0, +\infty)$, and $i = I_\Omega$ the identity mapping on X we have Wardowski's Theorem 2.1. from [3]. Putting in Theorem 6 $i = I_X$ the identity mapping on Ω we get a Branciari–Wardowski type fixed point result in the sense of [34]. While for $i = I_\Omega$ the identity mapping on Ω and $\varpi(t) = \varpi =$constant from $(0, +\infty)$ our Theorem 6 gives a Branciari–Wardowski type fixed point result in the sense of [3].*

The direct consequences of the Theorem 6 are new contraction conditions that complement results from [18,28].

Corollary 3. *Let (Ω, w) be a metric space, $h, i : \Omega \to \Omega$ be a self-mapping and h be an $(\varpi_i, \tilde{\zeta}, F, i)-$ contraction, where $C_i > 0, i = \overline{1,6}$ such that for all $a, b \in \Omega$ with $w(ha, hb) > 0$ and $w(ia, ib) > 0$ any of the following inequalities hold true*

$$C_1 + \int_0^{w(ha,hb)} \tilde{\zeta}(t)dt \leq \int_0^{w(ia,ib)} \tilde{\zeta}(t)dt,$$

$$C_2 + \exp\left(\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt\right) \leq \exp\left(\int_0^{w(ia,ib)} \tilde{\zeta}(t)dt\right),$$

$$C_3 - \frac{1}{\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt} \leq -\frac{1}{\int_0^{w(ia,ib)} \tilde{\zeta}(t)dt},$$

$$C_4 - \frac{1}{\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt} + \int_0^{w(ha,hb)} \tilde{\zeta}(t)dt \leq -\frac{1}{\int_0^{w(ia,ib)} \tilde{\zeta}(t)dt} + \int_0^{w(ia,ib)} \tilde{\zeta}(t)dt,$$

$$C_5 + \frac{1}{1-\exp\left(\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt\right)} \leq \frac{1}{1-\exp\left(\int_0^{w(ia,ib)} \tilde{\zeta}(t)dt\right)},$$

$$C_6 + \frac{1}{\exp\left(-\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt\right) - \exp\left(\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt\right)} \leq \frac{1}{\exp\left(-\int_0^{w(ia,ib)} \tilde{\zeta}(t)dt\right) - \exp\left(\int_0^{w(ia,ib)} \tilde{\zeta}(t)dt\right)},$$

$$C_7 + (\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt)^k \leq (\int_0^{w(ia,ib)} \tilde{\zeta}(t)dt)^k,\ k > 0$$

$$C_8 + \int_0^{w(ha,hb)} \tilde{\zeta}(t)dt \cdot \exp(\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt) \leq \int_0^{w(ia,ib)} \tilde{\zeta}(t)dt \cdot \exp(\int_0^{w(ia,ib)} \tilde{\zeta}(t)dt),$$

$$C_9 + \exp(\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt) \cdot \ln(\int_0^{w(ha,hb)} \tilde{\zeta}(t)dt) \leq \exp(\int_0^{w(ia,ib)} \tilde{\zeta}(t)dt) \cdot \ln(\int_0^{w(ia,ib)} \tilde{\zeta}(t)dt).$$

Suppose that there exists a Picard–Jungck sequence $\{b_n\}_{n\in\mathbb{N}\cup\{0\}}$ *of* (h,i) *and assume that at least one of the following two conditions holds true:*

(i) $(i\Omega, w)$ *is a complete metric space;*
(ii) (Ω, w) *is complete metric space,* i *is continuous and* (h,i) *is compatible pair of self-mappings on* (X,w).

Then, in each of these cases, h *and* i *have a unique point of coincidence.*

Proof. First of all, put $\varpi_i(t) = C_i \in (0,+\infty), i = \overline{1,9}$ for all $t \in (0,+\infty)$ and $F(l) = l, F(l) = \exp(l), F(l) = -\frac{1}{l}, F(l) = -\frac{1}{l} + l, F(l) = \frac{1}{1-\exp(l)}, F(l) = \frac{1}{\exp(-l)-\exp(l)}, F(l) = l^k,\ k > 0, F(l) = l \cdot \exp(l)$ and $F(l) = \exp(l) \cdot \ln(l)$, respectively. Because every of the functions $l \mapsto F(l)$ is strictly increasing on $(0,+\infty)$ then the result follows by Theorem 6. □

Example 2. *Let* Ω, ω, h, $\tilde{\zeta}$ *and* F *be the same as in Example 1. Let* $i(a) = \frac{2}{3}a$. *Then all conditions of Corollary 3 are satisfied for* $C_3 \in (0, \frac{7}{4}]$ *and obviously* 0 *is a unique point of coincidence for the mappings* h *and* i.

3. Conclusions

In this paper, the new term of $F-$integral contraction is introduced. Fixed point and common fixed point theorems are established, and as a consequence of the main results we obtain Jungck–Wardowski, Branciari–Wardowski and Jungck–Branciari type results. The results presented in the article enhance and complement some of known results in literature.

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