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Including Jumps in the Stochastic Valuation of Freight Derivatives

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Abstract: The spot freight rate processes considered in the literature for pricing forward freight agreements (FFA) and freight options usually have a particular dynamics in order to obtain the prices. In those cases, the FFA prices are explicitly obtained. However, for jump-diffusion models, an exact solution is not known for the freight options (Asian-type), in part due to the absence of a suitable valuation framework. In this paper, we consider a general jump-diffusion process to describe the spot freight dynamics and we obtain exact solutions of FFA prices for two parametric models. Moreover, we develop a partial integro-differential equation (PIDE), for pricing freight options for a general unifactorial jump-diffusion model. When we consider that the spot freight follows a geometric process with jumps, we obtain a solution of the freight option price in a part of its domain. Finally, we show the effect of the jumps in the FFA prices by means of numerical simulations.

Keywords: spot freight rates; freight options; stochastic jump-diffusion process; stochastic delay differential equation; risk-neutral measure; arbitrage arguments; partial integro-differential equations

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1. Introduction

From its modest origins, the freight transportation has progressed immeasurably in terms of size and complexity. Over the years, companies devoted to freight transportation have grown exponentially, see [1]. In fact, nowadays, the transportation of goods all over the world has become very important. Moreover, maritime trade is the main source of international trade and transportation. One of the main reasons is the relatively low maritime cost, compared to that associated with other modes of transportation, such as land or air, see [2].

In the international shipping markets, freight derivatives are very useful instruments to deal with risk, but also of interest to market practitioners as well as to academics. On the one hand, institutional investors, like hedge funds or investment banks, have an interest in shipping derivatives as alternative investments. As Grelck et al [3] show, shipping has a very low correlation with stocks. Then, incorporating shipping assets to their portfolios means an important source of diversification, although they are also very sensitive to the global economy, see [4]. In fact, the freight derivative market is relatively new, and it is an emerging market which can still improve considerably. On the other hand, the academic interest in shipping finance is very recent, and the volume of literature on this topic, although increasing, is still considerably lower than the volume of general finance literature. Then, practitioners are interested in this market to hedge their risk and/or get higher returns, and academics to develop high-impact research works to support practitioners.

In freight rate modeling, initially, the considered stochastic processes were very simple models borrowed from financial economics. In early stages, it was assumed that the freight rate followed a geometric Brownian motion, see [5–7]. However, the notion of mean reversion has been always dominant in the maritime economic literature (see, for instance, [8]). An Ornstein–Uhlenbeck process was taken into account by [9,10]. It is very well-known that this process has some deficiencies which as a result can provide negative freight rates in some cases. Tvedt (1998) [11] established that it could be improved by a geometric mean-reverting process and, for example, Prokopczuk [12] assumed that the logarithm of the spot price followed an Ornstein–Uhlenbeck process. More recently, for example, jumps have been considered to model the spot freight rate as they usually exhibit relatively large and abrupt upside and downside movements, which can be represented by jump-diffusion processes. For example, Nomikos et al. [13] added jumps of normal size to the geometric Brownian motion, and Kyriakou et al. [14] established a generalized stochastic freight rate in the form of exponential mean reverting process overlaid with jumps. There is also a recent trend to consider nonparametric techniques in order to avoid the problems surrounding the identification and estimation of parametric stochastic models, see [8,15].

Freight markets are highly volatile. Then, it does not seem strange that market participants rely on FFA contracts and options to manage their freight rate risk exposure. As shown by Cox and Ross [16] and Harrison and Kreps [17], the no-arbitrage price of a derivative is given by the risk-neutral expectation of its cash-flows. For the most financial derivatives, it is possible to obtain a partial differential equation whose solution is the derivative price, see [18]. However, if the spot freight rate process includes jumps, a PIDE for pricing freight options is not known. Then, the Monte Carlo method must be applied to price these options, although it is very expensive and not very accurate from a computational point of view.

The main contribution of this paper is to offer a new framework for pricing both FFA contracts and freight options, when the spot freight rate follows a jump-diffusion stochastic process. We consider a one-factor model which includes the spot processes considered in freight and commodity models as in [6,19,20]. We provide a PIDE which allows us to use other different methods to price these derivatives. The original idea of this approach dates back to [21], but they leave unattainable various modeling processes including, for example, price discontinuities which are important in freight markets.

Considering that FFA prices can be obtained as the average of the futures prices whose maturities are the different time levels of the settlement period, we obtain the price for some of the jump-diffusion processes mostly used in the literature, such as the geometric Brownian motion and the geometric mean reverting process with jumps. Moreover, if a closed-form solution for the future price could not be obtained, it would be possible to apply a numerical method to the futures PIDE pricing equation as a previous step to obtain an approximation to the FFA price.

As far as freight options are concerned, we provide a novel PIDE which verifies the freight option price. This equation depends on three independent variables: the spot freight, its delay and the average of the spot freight over the settlement period (its continuous version). Therefore, this methodology offers a new approach to price this special kind of options. In some cases, this PIDE could provide an explicit solution for the freight option valuation problem and, in other cases, numerical methods could be applied to approximate the solution. In this paper, we obtain a closed-form solution in a part of its domain for the geometric Brownian motion with jumps to be added as a boundary condition when numerical discretization schemes are designed to approximate the price.

Therefore, with this framework, we contribute with a new, fast, and accurate analytical method for pricing freight derivatives when jumps are considered. Furthermore, the obtained FFA prices are useful to obtain lower bounds of the freight options which can provide valuable approximations to the option prices. Numerical simulations illustrate the results.

The paper is structured as follows. In Section 2, a one-factor jump-diffusion model is introduced for pricing FFA contracts and freight options. In Section 3, we provide closed-form solutions for FFA prices when some very well-known assumptions are made for modeling the spot freight rates: the geometric and the geometric mean reversion with jumps. In Section 4, we obtain a novel PIDE for pricing FFA options and a partial closed-form solution for a particular case. In Section 5, we make some numerical experiments to show how to implement our approach and analyze the impact of the mean reversion and jumps in a FFA and option pricing model. Finally, Section 6 presents conclusions.

2. Model Setup

In this section, we introduce the models which we will consider later to price some freight derivatives: FFA contracts and options. In particular, we consider a general jump-diffusion model which allows us to take into account extreme movements and discontinuities in the freight rates.

Define (Ω, $\mathcal{F}$, and $\mathcal{P}$) as a probability space equipped with a filtration $\{\mathcal{F}\}_{t\geq 0}$ which satisfies the usual conditions, see [22,23].

We consider that the spot freight rate, under the risk-neutral measure $\mathcal{Q}$, follows the jump-diffusion process (S is right-continuous (cadlag, see [22]) and we denote the limit $S(t-) = \lim_{z\uparrow t} S(z)$. This notation will be added only when necessary to avoid confusion; otherwise, it will be assumed implied):

$$S(t) = S(0) + \int_0^t \mu(S(z))dz + \int_0^t \sigma(S(z))dW(z) + \int_0^t \gamma(S(z-))dJ(z), \tag{1}$$

where $\mu(S)$ and $\sigma(S)$ are the drift and volatility of the process, respectively, and W is a Wiener process. The jump term is given by the function γ and the compound Poisson process, $J(t) = \sum_{i=1}^{N^{\mathcal{Q}}(t)} Y_i$, with jump times $(\tau_i)_{i\geq 1}$, where $N^{\mathcal{Q}}(t)$ represents a Poisson process with intensity $\lambda(S)$ and $Y_1, Y_2, \ldots$ is a sequence of independent and identically distributed random variables with distribution function Π. We suppose that the functions μ, σ, γ and λ satisfy suitable regularity conditions as follows (see [24,25]). Functions μ, σ, γ, and λ are twice continuously differentiable. Moreover, they satisfy local Lipschitz and grow conditions:

Assumption 1. *For every compact subset $D \subset \mathbb{R}$, there exists a constant C_1^D such that, for all $x, z \in \mathbb{R}$,*

$$|\mu(x) - \mu(z)| + |\sigma(x) - \sigma(z)| + \lambda(x) \int_{-\infty}^{\infty} |(\gamma(x) - \gamma(z))y|\Pi(y)dy \leq C_1^D |x - z|,$$

Assumption 2. *There exists a constant C_2 such that, for all $x \in \mathbb{R}$,*

$$|\mu(x)| + |\sigma(x)| + \lambda(x) \int_{-\infty}^{\infty} |\gamma(x)y|\Pi(y)dy \leq C_2(1 + |x|).$$

Assumption 3. *For a given $\alpha > 2$, there exists a constant C_3 such that, for any $x \in \mathbb{R}$,*

$$\lambda(x) \int_{-\infty}^{\infty} |\gamma(x)y|^{\alpha}\Pi(y)dy \leq C_3(1 + |x|^{\alpha}).$$

Assumption 4. *$\lambda \geq 0$ on $\mathbb{R}$.*

FFAs are forward contracts which trade, at a future date, the arithmetic average of the spot freight rate during a settlement period $[T_1, T_N]$ with $T_1 < \cdots < T_N$. Then, it verifies at time $t \leq T_N$

$$F(t, S; T_1, \ldots, T_N) = E^{\mathcal{Q}}\left[\frac{1}{N}\sum_{i=1}^{N} S(T_i)\Big| S(t) = S\right], \tag{2}$$

where $E^{\mathcal{Q}}$ represents the conditional expectation under $\mathcal{Q}$-measure, see [6].

We express the freight call option price at time t, with strike price K and maturity T_N, as $C(t, S; K, T_1, \ldots, T_N)$. At maturity, its value is

$$C(T_N, S; K, T_1, \ldots, T_N) = \left(\frac{1}{N} \sum_{i=1}^{N} S(T_i) - K \right)^{+}. \tag{3}$$

We assume the discount factor $D(t) = \mathrm{e}^{-rt}$, where the riskless interest rate r is constant. If we use the fundamental theorem of asset pricing (see [18]), the price of a freight call option, at time t, strike price K, and maturity time T_N is given by

$$C(t, S; K, T_1, \ldots, T_N) = \mathrm{e}^{-r(T_N - t)} \, E^{\mathcal{Q}} \left[\left(\frac{1}{N} \sum_{i=1}^{N} S(T_i) - K \right)^{+} \Big| S(t) = S \right]. \tag{4}$$

This Asian-style option can also be expressed as a European call option on a FFA as follows (see [6]):

$$C(t, S; K, T_1, \ldots, T_N) = \mathrm{e}^{-r(T_N - t)} \, E^{\mathcal{Q}} \left[(F(T_N, S; T_1, \ldots, T_N) - K)^{+} \Big| S(t) = S \right],$$

where the FFA price at maturity verifies

$$F(T_N, S; T_1, \ldots, T_N) = E^{\mathcal{Q}} \left[\frac{1}{N} \sum_{i=1}^{N} S(T_i) \Big| S(T_N) = S \right] = \frac{1}{N} \sum_{i=1}^{N} S(T_i).$$

3. Forward Freight Agreement Pricing

In this section, we describe several jump-diffusion models for pricing FFA contracts and calculate their exact solution.

The FFA price can be expressed as the average of futures prices whose maturities are the times of the settlement period. We show this result in the following proposition.

Proposition 1. *Let $F(t, S; T_1, \cdots, T_N)$ be the FFA price defined in (2) and $\widetilde{F}(t, S; T_i)$ the futures prices whose maturities are the times T_i ($i = 1, \ldots, N$) of the settlement period. Then, the FFA price verifies $F(t, S; T_1, \cdots, T_N) = \frac{1}{N} \sum_{i=1}^{N} \widetilde{F}(t, S; T_i)$.*

Proof of Proposition 1. From the definition of the FFA price (2),

$$\begin{aligned} F(t, S; T_1, \cdots, T_N) &= E^{\mathcal{Q}} \left[\frac{1}{N} \sum_{i=1}^{N} S(T_i) \Big| S(t) = S \right] \\ &= \frac{1}{N} \sum_{i=1}^{N} E^{\mathcal{Q}} [S(T_i) | S(t) = S] \\ &= \frac{1}{N} \sum_{i=1}^{N} \widetilde{F}(t, S; T_i). \end{aligned} \tag{5}$$

□

Bellow, we consider several processes already used to model the dynamics of different commodities (see [20,26,27]):

- **GJ model**: In the process (1), we assume that the freight spot rate follows a geometric process where we introduce a jump term proportional to the spot. In this case, we assume that the logarithm of the spot process $v(t) = \log S(t)$ follows the stochastic process

$$v(t) = v(0) + \int_0^t \mu_v dz + \int_0^t \sigma dz + \sum_{i=1}^{N^{\mathcal{Q}}(t)} Y_i, \tag{6}$$

where the distribution of the jump size is Normal ($Y_i \to \mathcal{N}(\mu_J, \sigma_J)$) and the jump intensity λ is constant. Then, the functions in (1) are $\mu(S) = \mu S$, $\sigma(S) = \sigma S$ and $\gamma(S) = S$, where μ and σ are constants and $\mu = \mu_v + \frac{\sigma^2}{2}$. The jump term for the spot freight rate process is $\sum_{i=1}^{N^{\mathcal{Q}}(t)} S(\tau_i)(\mathrm{e}^{Y_i} - 1)$, with τ_i the jump times. Now, the distribution of the jump size becomes Lognormal.

If we express the FFA price as the average of futures prices as in (5), we must solve the following PIDE (see [18,28]) with final conditions $\widetilde{F}(T_i, S; T_i) = S, i = 1, \cdots, N$,

$$\widetilde{F}_t + \mu S \widetilde{F}_S + \frac{1}{2}\sigma^2 S^2 \widetilde{F}_{SS} + \lambda \int_{-\infty}^{\infty} (\widetilde{F}(t, S + S(\mathrm{e}^y - 1); T_i) - \widetilde{F}(t, S; T_i))\Pi(y) dy = 0,$$

where Π is the Normal density function. As a solution of this PIDE, we consider a function, proportional to S, and we obtain that the futures prices can be expressed as

$$\widetilde{F}(t, S; T_i) = S\mathrm{e}^{\Gamma(T_i - t)},$$

where

$$\Gamma = \mu + \lambda(\mu_J^* - 1), \tag{7}$$

with $\mu_J^* - 1 = \mathrm{e}^{\mu_J + \sigma_J^2/2} - 1$, the first moment of the Lognormal distribution. Then, we obtain the FFA price as in [6] (by means of a geometric progression sum, where we assume that the times on the settlement period are equidistant with range $\Delta = T_{i+1} - T_i$)

$$F(t, S; T_1, \cdots, T_N) = \frac{S}{N}\sum_{i=1}^{N} \mathrm{e}^{\Gamma(T_i - t)} = \frac{S\mathrm{e}^{\Gamma(T_N - t)}}{N} \frac{1 - \mathrm{e}^{-\Gamma N \Delta}}{1 - \mathrm{e}^{-\Gamma N}}. \tag{8}$$

Remark 1. *Note that this FFA price only depends on the first moment, $\mu_J^* - 1$, of the jump size distribution of S. Moreover, if we consider that $\lambda = 0$, we obtain a geometric model without jumps (**G model**), and the FFA price is also obtained by (8), with $\Gamma = \mu$.*

- **LogJ model:** In the process (1), we assume that the logarithm of the spot freight rate follows an Ornstein–Uhlenbeck process with a jump term as follows:

$$v(t) = v(0) + \int_0^t k(\mu_v - v) dz + \int_0^t \sigma dz + \sum_{i=1}^{N^{\mathcal{Q}}(t)} Y_i, \tag{9}$$

where the distribution of the jump size is Normal ($Y_i \to \mathcal{N}(\mu_J, \sigma_J)$). Then, the functions in (1) are $\mu(S) = k(\mu - \log S)S$, $\sigma(S) = \sigma S$, $\gamma(S) = S$ and $\mu = \mu_v + \frac{\sigma^2}{2k}$, with k, μ, σ and the jump intensity λ constants. The jump term for the spot process is $\sum_{i=1}^{N^{\mathcal{Q}}(t)} S(\tau_i)(\mathrm{e}^{Y_i} - 1)$, where τ_i are the jump times. Again, the distribution of the jump of the process S becomes Lognormal. We will refer to this dynamics of the spot freight rate as geometric mean reverting.

For this model, the futures price with maturity T_i, $\widetilde{F}(t, S; T_i)$, must verify the following PIDE:

$$\widetilde{F}_t + k(\mu - \log S)S\widetilde{F}_S + \frac{1}{2}\sigma^2 S^2 \widetilde{F}_{SS} + \lambda\int_{-\infty}^{\infty}(\widetilde{F}(t, S + S(e^y - 1); T_i) - \widetilde{F}(t, S; T_i))\Pi(y)dy = 0, \tag{10}$$

(where Π is the Normal density function) and $\widetilde{F}(T_i, S; T_i) = S$.
Thus, as to solve this PIDE, first, we make a change of the time variable $\tau_i = T_i - t$ in (10). If we assume that its solution has the form $\widetilde{F}(\tau_i, S) = e^{B_1(\tau_i)\log S + B_2(\tau_i)}$, with B_1 and B_2 functions of time, we obtain the following system of ordinary differential equations:

$$B_1' = -kB_1,$$
$$B_2' = (k\mu - \frac{1}{2}\sigma^2)B_1 + \frac{1}{2}\sigma^2 B_1^2 + \lambda(e^{B_1\mu_J + \frac{B_1^2\sigma_J^2}{2}} - 1),$$

with the initial conditions $B_1(0) = 1$ and $B_2(0) = 0$. The function B_1 can be obtained explicitly $B_1(\tau_i) = e^{-k\tau_i}$, but the expression for $B_2(\tau_i)$ is:

$$B_2(\tau_i) = \frac{k\mu - \frac{1}{2}\sigma^2}{k}(1 - e^{-k\tau_i}) + \frac{\sigma^2}{4k}(1 - e^{-2k\tau_i}) + \lambda\int_0^{\tau_i}\left(\exp\left(e^{-kz}\mu_J + \frac{e^{-2kz}\sigma_J^2}{2}\right) - 1\right)dz.$$

In order to calculate $B_2(\tau)$, we could apply, for example, numerical quadrature methods to the integral term. Finally, we get the FFA price by means of the relation (5), where we express the FFA as the average of the futures prices:

$$\widetilde{F}(t, S; T_i) = \exp\left(e^{-k(T_i - t)}\log S + \frac{k\mu - \frac{1}{2}\sigma^2}{k}(1 - e^{-k(T_i - t)}) + \frac{\sigma^2}{4k}(1 - e^{-2k(T_i - t)}) + \lambda\int_0^{T_i - t}\left(\exp\left(e^{-kz}\mu_J + \frac{e^{-2kz}\sigma_J^2}{2}\right) - 1\right)dz\right). \tag{11}$$

4. Partial Integro-Differential Equation for Pricing Freight Options

Freight options are a special kind of Asian-options. A large volume of literature is devoted to Asian/Bermudan options where different approaches are considered to approximate its value, see a detailed literature survey on [29] or more recent approaches such as [27,30] or [31].

In the previous section, we have shown several jump-diffusion models which have a closed-form solution for the FFA price. However, this fact does not happen when freight options are priced with those dynamics. In the freight literature, the expectation in (4) is usually used to approximate these Asian-style options. In particular, the Monte Carlo method can be applied to price them.

When the freight rate follows a diffusion process, Gómez-Valle et al. [21] provide a partial differential equation and numerical methods could be developed to discretize it. However, if we consider a jump-diffusion process to reflect possible abrupt changes of the spot rate, a valuation equation to obtain freight option prices is not known in the literature.

In order to fill this gap, in this section, we develop a more general PIDE for pricing freight options assuming that the spot freight rate is given by a jump-diffusion stochastic process. We make a similar reasoning as in [21] for a diffusion process, which is based on pricing standard Asian-style options as in [32], and we extend it to jump-diffusion processes. Moreover, we obtain a partial solution to this PIDE for the GJ model introduced in Section 3.

Thus, as to provide a framework for pricing freight options, we consider a settlement period $[T_1, T_N]$ with a fixed time span $d = T_N - T_1$, which is usually one month. We establish a stochastic process $A(t)$ as an approximation to the average of the spot rate with a continuous version of $\frac{1}{N}\sum_{i=1}^{n} S(T_i)$, as in [21]:

$$A(t) = \begin{cases} \int_0^t S(z)\,dz, & \text{if} \quad 0 \le t \le d, \\ \int_{t-d}^t S(z)\,dz, & \text{if} \quad t > d. \end{cases} \tag{12}$$

This new stochastic process verifies that $dA(t) = (S(t) - S(t-d))dt$ for $t > d$. Then, we consider an additional variable which is the delay of the spot freight rate:

$$X(t) = \begin{cases} S(0), & \text{if} \quad 0 \le t \le d, \\ S(t-d), & \text{if} \quad t > d. \end{cases}$$

This new variable X takes a constant value for $0 \le t \le d$ and follows the jump-diffusion process for $t \ge d$

$$X(t) = X(d) + \int_d^t \mu(S(z))dz + \int_d^t \sigma(S(z))dW(z-d) + \sum_{i=1}^{N^{\mathcal{Q}}(t-d)} \gamma(S(\tau_i-))Y_i,$$

Then, we write $dA(t)$ as

$$dA(t) = \begin{cases} S(t)\,dt, & \text{if} \quad 0 \le t \le d, \\ (S(t) - X(t))\,dt, & \text{if} \quad t > d. \end{cases}$$

The average value of the spot freight rate in (3) is approximated by (12) in continuous-time as:

$$C(T_N, S, X, A; K, T_1, \ldots, T_N) = \left(\frac{1}{d}A(T_N) - K\right)^+, \tag{13}$$

and the freight call option in (4) as

$$C(t, S, X, A; K, T_1, \ldots, T_N) = \tag{14}$$

$$e^{-r(T_N - t)}\, E^{\mathcal{Q}}\left[\left(\frac{1}{d}A(T_N) - K\right)^+ \middle| S(t) = S, X(t) = X, A(t) = A\right]. \tag{15}$$

The following results allow us to deal with the freight option valuation problem in a jump-diffusion setting. We develop a PIDE, which is verified by the freight call option price (analogously a PIDE for the freight put option can be obtained). In some cases, we will be able to obtain an exact solution of the valuation PIDE, but, in other cases, we will have to apply numerical methods in order to approximate it.

Theorem 1. *The freight call option price $C(t, S, X, A; K, T_1, \ldots, T_N)$ in Equation (14) verifies the following PIDE for $d < t < T_N$,*

$$
\begin{aligned}
C_t \quad &+ \quad \mu(S)C_S + \mu(X)C_X + (S-X)C_A \\
&+ \quad \frac{1}{2}\sigma^2(S)C_{SS} + \frac{1}{2}\sigma^2(X)C_{XX} - rC \\
&+ \quad \lambda \int_{-\infty}^{\infty} \Big(C(t, S+\gamma(S)y, X, A; T_1, \cdots, T_N) \\
&\quad\quad -C(t, S, X, A; T_1, \cdots, T_N) \Big) \Pi(y)dy \\
&+ \quad \lambda \int_{-\infty}^{\infty} \Big(C(t, S, X+\gamma(X)y, A; T_1, \cdots, T_N) \\
&\quad\quad -C(t, S, X, A; T_1, \cdots, T_N) \Big) \Pi(y)dy = 0, \\
&\quad\quad S > 0, \quad X > 0, \quad A > 0.
\end{aligned} \tag{16}
$$

Moreover, when $0 < t < d$*, the function* C *in Equation (14) satisfies the PIDE*

$$
\begin{aligned}
C_t \quad &+ \quad \mu(S)C_S + SC_A + \frac{1}{2}\sigma^2(S)C_{SS} - rC \\
&+ \quad \lambda \int_{-\infty}^{\infty} \Big(C(t, S+\gamma(S)y, X, A; T_1, \cdots, T_N) \\
&\quad\quad -C(t, S, X, A; T_1, \cdots, T_N) \Big) \Pi(y)dy = 0, \\
&\quad\quad S > 0, \quad X > 0, \quad A > 0.
\end{aligned} \tag{17}
$$

Proof of Theorem 1. By means of no-arbitrage arguments, the discounted freight option price, under the risk-neutral measure $\mathcal{Q}$, is a martingale, see [18]. Then, the option price must verify

$$
\begin{aligned}
&E^{\mathcal{Q}}[D(T_N)C(T_N, S(T_N), X(T_N), A; K, T_1, \ldots, T_N) | S(t) = S, X(t) = X, A(t) = A] \\
&\quad = D(t)C(t, S, X, A; K, T_1, \ldots, T_N).
\end{aligned}
$$

In this case, if we develop the differential $d(D(t)C(t, S, X, A; K, T_1, \ldots, T_N))$, then the dt term must be zero.

From now, S^c and X^c will be denoted as the continuous part (that is, without the jump term) of the processes S and X, respectively. Then, we note that $dS^c dS^c = \sigma^2(S)dt$ and

$$
dX^c dX^c = \begin{cases} 0, & \text{if} \quad 0 < t < d, \\ \sigma^2(X)\, dt, & \text{if} \quad t > d. \end{cases}
$$

Moreover, $dS^c dX^c = 0$ because $dW(t)dW(t-d) = 0$, and $dAdA = dSdA = dXdA = 0$. Applying the Ito Lemma, for $d < t < T_N$, we get

$$
\begin{aligned}
d(\mathrm{e}^{-rt}C) &= \mathrm{e}^{-rt}\Big(-rC + C_t + \mu(S)C_S + \mu(X)C_X + (S-X)C_A \\
&\quad +\frac{1}{2}\sigma^2(S)C_{SS} + \frac{1}{2}\sigma^2(X)C_{XX} \\
&\quad +\lambda E_y\Big[C(t,S+\gamma(S)y,X,A;T_1,\cdots,T_N) \\
&\quad -C(t,S,X,A;T_1,\cdots,T_N)\Big] \\
&\quad +\lambda E_y\Big[C(t,S,X+\gamma(X)y,A;T_1,\cdots,T_N) \\
&\quad -C(t,S,X,A;T_1,\cdots,T_N)\Big]\Big)dt \\
&+ \mathrm{e}^{-rt}\Big(C_S\sigma(S)dW(t) + C_X\sigma(X)dW(t-d) \\
&\quad +\big(C(t,S+\gamma(S)y,X,A;T_1,\cdots,T_N) \\
&\quad -C(t,S,X,A;T_1,\cdots,T_N)\big)d\widetilde{N}^{Q}(t) \\
&\quad +\big(C(t,S,X+\gamma(X)y,A;T_1,\cdots,T_N) \\
&\quad -C(t,S,X,A;T_1,\cdots,T_N)\big)d\widetilde{N}^{Q}(t-d)\Big),
\end{aligned} \tag{18}
$$

where E_y represents the expectation with respect to the jump size, and $\widetilde{N}^{Q}$ is the compensated Poisson process.

For $0 < t < d$,

$$
\begin{aligned}
d(\mathrm{e}^{-rt}C) &= \mathrm{e}^{-rt}\Big(-rC + C_t + \mu(S)C_S + SC_A + \frac{1}{2}\sigma^2(S)C_{SS} \\
&\quad +\lambda E_y\Big[C(t,S+\gamma(S)y,X,A;T_1,\cdots,T_N) \\
&\quad -C(t,S,X,A;T_1,\cdots,T_N)\Big]\Big)dt \\
&+ \mathrm{e}^{-rt}\Big(C_S\sigma(S)dW(t) \\
&\quad +\big(C(t,S+\gamma(S)y,X,A;T_1,\cdots,T_N) \\
&\quad -C(t,S,X,A;T_1,\cdots,T_N)\big)d\widetilde{N}^{Q}(t)\Big).
\end{aligned} \tag{19}
$$

Finally, we vanish the dt terms in (18) and (19), and this fact leads to (16) and (17), respectively. □

This result provides a final value problem (PIDEs (16) and (17) with the final condition (13)). These equations can be very complex to solve explicitly. Nevertheless, suitable numerical methods can be applied to approximate the solution.

In order to illustrate how to implement this approach, we consider a simple stochastic process commonly used in the freight literature to model the spot freight rate: the GJ model. In the literature, there exist several techniques to obtain the freight option prices with this model, such as [33]. In a similar way to [21], but considering jumps, we calculate a solution to the PIDEs (16)–(17) in Theorem 1 when the average of the spot freight verifies $A \geq dK$.

Proposition 2. *Let $\mu(S) = \mu S$, $\sigma(S) = \sigma S$, and $\gamma(S) = S$ be the drift, volatility, and jump size factor, respectively, of the process (1), λ the jump intensity, and $\mu_J^* - 1$ the first moment of the jump size distribution (Normal distribution), with μ, σ, λ, and μ_J^* constants, as in the GJ model. Then, the following function is a solution to the PIDEs (16)–(17) and verifies the final condition (13) when $A \geq dK$*

$$\widetilde{C}(t,S,X,A;K,T_1,\dots,T_N) =$$

$$\begin{cases} e^{-r(T_N-t)}\left(\left(\frac{1}{d}A-K\right)+\frac{S(e^{\Gamma(T_N-t)}-1)-X(e^{\Gamma(T_N-d)}-1)}{d\Gamma}\right), & 0\le t\le d, \\ e^{-r(T_N-t)}\left(\left(\frac{1}{d}A-K\right)+\frac{(S-X)(e^{\Gamma(T_N-t)}-1)}{d\Gamma}\right), & d\le t\le T_N, \end{cases} \tag{20}$$

where Γ *is given by (7).*

Proof of Proposition 2. We get a new initial value problem if we change the time variable $\tau = T_N - t$ and use (13) and (16):

$$\begin{aligned} C_\tau &= \mu S C_S + \mu X C_X + (S-X)C_A \\ &+ \frac{1}{2}\sigma^2 S^2 C_{SS} + \frac{1}{2}\sigma^2 X^2 C_{XX} - rC \\ &+ \lambda \int_{\infty}^{\infty} \Big(C(\tau, S+Sy, X, A; K) \\ &\quad -C(\tau,S,X,A;K)\Big)\Pi(y)dy \\ &+ \lambda \int_{\infty}^{\infty} \Big(C(\tau, S, X+Xy, A; K) \\ &\quad -C(\tau,S,X,A;K)\Big)\Pi(y)dy, \quad 0<\tau<T_N-d, \end{aligned} \tag{21}$$

$$C(0,S,X,A;K) = \left(\frac{1}{d}A-K\right)^+. \tag{22}$$

As in [21], for $A \ge dK$, we consider this linear solution to this problem

$$C(\tau,S,X,A;K) = \left(\frac{1}{d}A-K\right)B_1(\tau) + (S-X)B_2(\tau), \quad 0\le\tau\le T_N-d, \tag{23}$$

where B_1 and B_2 depend only on the new time variable τ. Replacing (23) in the PIDE (21), and taking into account that the first integral term in (21) can be expressed as

$$\lambda\int_{-\infty}^{\infty} S(\mathrm{e}^y-1)B_2(\tau)\Pi(y)dy = \lambda(\mu_J^*-1)SB_2(\tau),$$

and analogously for the second integral term, we obtain that B_1 and B_2 verify the following system of ordinary differential equations

$$\begin{aligned} B_1'(\tau) &= -rB_1(\tau), \\ B_2'(\tau) &= (\Gamma - r)B_2(\tau) + \frac{1}{d}B_1(\tau). \end{aligned}$$

We get, from (22), the initial conditions: $B_1(0)=1$ and $B_2(0)=0$. Solving this system, we obtain the solution to (21)–(22)

$$C(\tau,S,X,A;K) = \left(\frac{1}{d}A-K\right)\mathrm{e}^{-r\tau} + (S-X)\frac{\mathrm{e}^{-r\tau}}{d\Gamma}(\mathrm{e}^{\Gamma\tau}-1), \quad 0\le\tau\le T_N-d. \tag{24}$$

We also make the change of the time variable in (17). Moreover, we use the value of (24) in T_N-d as an initial condition. Then, we obtain the initial value problem

$$C_\tau = \mu S C_S + S C_A + \frac{1}{2}\sigma^2 S^2 - rC$$
$$+\lambda \int_{-\infty}^{\infty} \Big(C(\tau, S+Sy, X, A; K) - C(\tau, S, X, A; K) \Big) \Pi(y) dy, \quad (25)$$
$$T_N - d < \tau < T_N,$$

$$C(T_N - d, S, X, A; K) = \mathrm{e}^{-r(T_N-d)} \left(\left(\frac{1}{d}A - K \right) + (S - X) \frac{\mathrm{e}^{\Gamma(T_N-d)} - 1}{d\Gamma} \right). \quad (26)$$

Again, in this case, we consider this linear solution

$$C(\tau, S, X, A; K) = \left(\frac{1}{d}A - K \right) A_1(\tau) + S A_2(\tau) + X A_3(\tau), \quad T_N - d \leq \tau \leq T_N, \quad (27)$$

where A_1, A_2, and A_3 are functions of time.

We replace (27) into the PIDE (25) and obtain that A_1, A_2 and A_3 satisfy the following system of ordinary differential equations:

$$A_1'(\tau) = -r A_1(\tau),$$
$$A_2'(\tau) = (\Gamma - r) A_2(\tau) + \frac{1}{d} A_1(\tau), \quad (28)$$
$$A_3'(\tau) = -r A_3(\tau),$$

and, from (26), we get the initial conditions in $\tau = T_N - d$

$$A_1(T_N - d) = \mathrm{e}^{-r(T_N-d)},$$
$$A_2(T_N - d) = \frac{\mathrm{e}^{-r(T_N-d)}}{d\Gamma} (\mathrm{e}^{\Gamma(T_N-d)} - 1), \quad (29)$$
$$A_3(T_N - d) = -\frac{\mathrm{e}^{-r(T_N-d)}}{d\Gamma} (\mathrm{e}^{\Gamma(T_N-d)} - 1).$$

We solve the problems (28) and (29) and obtain the solution

$$C(\tau, S, X, A; K) = \left(\frac{1}{d}A - K \right) \mathrm{e}^{-r\tau} + S \frac{\mathrm{e}^{-r\tau}}{d\Gamma} (\mathrm{e}^{\Gamma\tau} - 1) - X \frac{\mathrm{e}^{-r\tau}}{d\Gamma} (\mathrm{e}^{\Gamma(T_N-d)} - 1),$$
$$T_N - d \leq \tau \leq T_N.$$

Finally, if we unmake the change of variable $t = T_N - \tau$, we get the expression in (20) for $\widetilde{C}$, that is, the call freight option price when $A \geq dK$. □

Note that the solution in (20) is only valid for $A \geq dK$. However, (20) could be used to obtain some boundary conditions for the Equations (16) and (17), in order to approximate the complete solution to these PIDEs with numerical methods (in a similar way to [21,32]).

5. Computational Aspects

Through this section, and because of a lack of FFA and freight option observations, we make some numerical experiments in order to show how to implement our framework for FFA and freight option low bound valuation. At the same time, we analyze the effect of considering mean reverting characteristics and jumps in the freight rates, on the FFA and freight options.

By means of a test problem (an academic model with closed-form solution), we make some Monte Carlo simulations of the spot freight rates in order to generate observations. Then, we calculate FFA prices and low bounds for the freight option prices and analyze the effect of the mean reversion and jumps.

It is very well known that mean reversion is an interesting characteristic in the maritime economic theory but also the price discontinuities. Then, as a test problem, we consider that the spot freight rate follows a geometric mean reverting process with jumps as in (9). This process is highly supported in the literature for modeling commodities in general, and freights in particular because of its important properties, as we detailed in Section 1.

Following [34], we generate 5000 simulated paths of the geometric mean reverting model with jumps, see (9), using parameter values (we have estimated these parameters using the Baltic Dry Index (BDI) from January 2013 to January 2019 and a log process (without jumps). Then, we have added reasonable values for the jumps parameters to make the comparisons. We have also used other parameter values and have obtained similar conclusions) in Table 1 (first row) and a starting value S_0 = 2000. We use these freight rate observations to obtain FFA prices (for simplicity, we assume that the parameters are equal under $\mathcal{P}$ and $\mathcal{Q}$ measures) using (11) for a settlement period of one month and maturities of 1, 2 , 3, 6, and 12 months. All of these data are considered our "market" observations along this section.

Table 1. Parameter values for models LogJ, Log, and GJ.

Model	k	μ	σ	λ	μ_J	σ_J
LogJ	0.4041	6.8805	0.3740	1.25	0.5	0.9
Log	0.4044	11.3885	1.9218	-	-	-
GJ	$\Gamma = 0.4592$					

In order to analyze the impact of the jumps, we compare some FFA prices obtained with the LogJ model with those obtained with a geometric mean reversion model without jumps (**Log model**):

$$v(t) = v(0) + \int_0^t k(\mu_v - v)dz + \int_0^t \sigma dz,$$

where $v(t) = log(S(t))$ and then the functions in (1) are $\mu(S) = k(\mu - \log S)S$, $\sigma(S) = \sigma S$, $\gamma(S) = 0$ and $\mu = \mu_v + \frac{\sigma^2}{2k}$, with k, μ, and σ constants. A closed-form solution for the FFA price can be easily obtained replacing $\lambda = 0$ in (11).

Thus, in order to calibrate this model, we rely on standard practices for extracting parameters, see [14,33]. We minimize the distance between the FFA observations and the theoretical FFA prices using a standard nonlinear least-squares solver in MATLAB (MathWorks R2018a). We consider the FFA prices obtained with parameters in Table 1 (first row) and maturities of 1, 2, 3 , 6, and 12 months and spot freight values $S \in [2000, 10{,}000]$ as observations. The resulting estimated parameters for the Log model are in Table 1 (second row). As you can see, the value of the speed of mean reversion nearly changes, but the volatility and the level of mean reversion do. In fact, we have also changed the parameter values of the jump term in the simulations with the LogJ model, and we have observed the following. Whenever we increase one of these parameters (keeping the rest of them fixed), the estimated speed of mean-reversion of the Log model nearly changes, but its level of mean-reversion and volatility increase in a similar way. For example, if we double the mean of the jump size distribution, both increase by about 50%.

In Figure 1, we plot the prices with both models and maturities of 1 and 3 months, and we see that they have the same behavior. That is, FFA prices increase with the spot freight rate and with the maturity, and the differences between both models are nearly undistinguishable for both maturities. Therefore, taking into account the jumps is interesting, although, for very short maturities, it is not so important. However, we think that this is because all the parameters are estimated jointly and a highest volatility and mean reversion level in the Log model try to compensate the changes of the prices due to the jumps in the LogJ model.

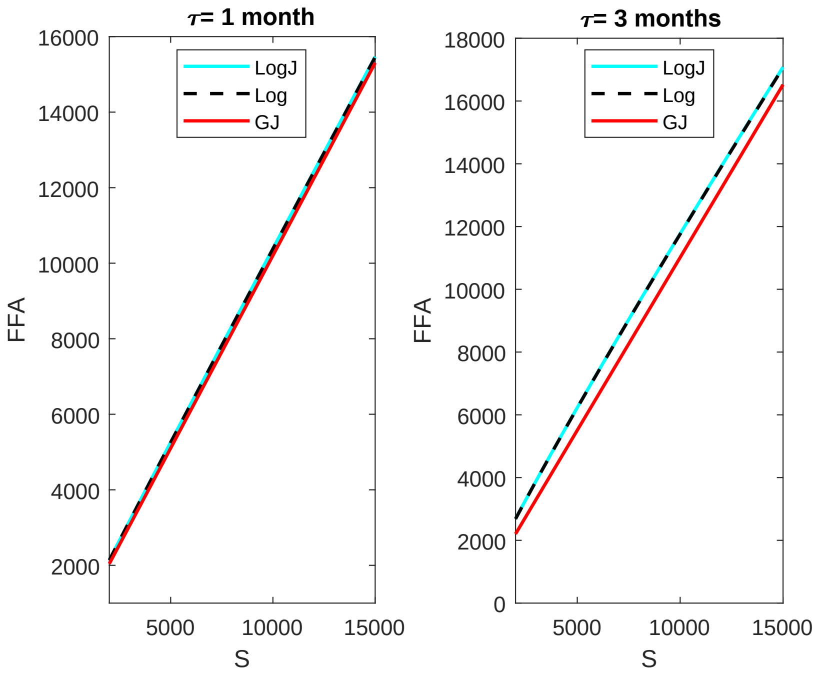

Figure 1. FFA prices with the LogJ, Log, and GJ models for maturities: 1 and 3 months.

In order to analyze the impact of the mean reversion, we estimate the parameters when the spot freight rate follows the GJ model, see (6). We calibrate this model with the FFA prices of the LogJ model (our "market"), and we use the standard nonlinear least-squares solver in Matlab and the FFA price expression (8). The values of the estimated parameters are in Table 1 (third row). Then, we also price FFA contracts with the GJ model, $S \in [2000, 10{,}000]$ and maturities: 1 and 3 months and plot them in Figure 1. As you can see, the behavior of the GJ FFA prices is quite similar to the other models, but the GJ model underprices FFA contracts with respect to the Log and LogJ models.

For a deeper comparison, we use the following measures of error: the root mean square error (RMSE), the root mean square percentage error (RMSPE), the mean absolute error (MAE), and the mean absolute percentage error (MAPE):

$$\begin{aligned}
RMSE_\tau &= \sqrt{\frac{1}{M}\sum_{i=1}^{M}(FFA_{i\tau}^{LogJ} - FFA_{i\tau}^{\theta})^2}, \\
RMSPE_\tau &= \sqrt{\frac{1}{M}\sum_{i=1}^{M}\left(1 - \frac{FFA_{i\tau}^{\theta}}{FFA_{i\tau}^{LogJ}}\right)^2}, \\
MAE\tau &= \frac{1}{M}\sum_{i=1}^{M}\left|FFA_{i\tau}^{LogJ} - FFA_{i\tau}^{\theta}\right|, \\
MAPE_\tau &= \frac{1}{M}\sum_{i=1}^{M}\left|1 - \frac{FFA_{i\tau}^{\theta}}{FFA_{i\tau}^{LogJ}}\right|,
\end{aligned}$$

where $FFA_{i\tau}^{LogJ}$ is the price with the LogJ model, $FFA_{i\tau}^{\theta}$ the price with the corresponding estimated model (Log or GJ), τ the maturity and M the number of observations.

We obtain these errors comparing the Log and the GJ model with the LogJ model, which we assume is the true model as it takes into account the desirable properties of mean reversion and jumps. We show these errors in Table 2, where we will suppress the maturity to show the errors with the all of the considered maturities. Notice that, in all the cases, the errors are the highest when we use the GJ model, i.e., this process does not look very suitable to model the behavior of the freight rates. We think that this fact is mainly due to two reasons. Firstly, the GJ model does not take into account mean reversion, which is a characteristic of the freight rates highly documented in the literature; see, for instance [8]. Hence, it would be very adequate to use it to model other financial variables, such as stocks, but not freight rates. Finally, although this geometric Brownian motion has also a compound poisson process to collect the possible jumps of the spot freight rate, the FFA price is not affected by the jumps. If we compare this price with the one obtained with a geometric Brownian motion without the jumps (G model), we see that, in both cases, the solution has the same expression, and it depends only on a single parameter Γ. Therefore, if we calibrated both models with FFA prices, we would obtain the same value for the parameter and, consequently, for the FFA prices. Therefore, it is very important to use a suitable model that takes into account the properties of the spot freight rate in order to accurately price FFA contracts.

Table 2. Root mean square, mean absolute, root mean square percentage, and mean absolute percentage errors for Log and GJ models.

Maturity	1 Month	3 Months	6 Months	12 Months	24 Months	Total
RMSE						
Log	5.9921	11.7458	10.6176	13.5907	2.4494	9.7705
GJ	1.6597×10^{2}	4.5397×10^{2}	6.985×10^{2}	1.1781×10^{3}	1.6365×10^{3}	9.7852×10^{2}
MAE						
Log	5.4710	10.7474	9.7113	−12.8888	1.6455	2.9375
GJ	1.6450×10^{2}	4.510×10^{2}	6.9426×10^{2}	1.1397×10^{3}	1.4561×10^{3}	7.8110×10^{2}
RMSPE						
Log	6.1634×10^{-4}	1.1269×10^{-3}	9.5168×10^{-4}	1.1360×10^{-3}	1.4368×10^{-4}	8.7939×10^{-4}
GJ	2.4376×10^{-2}	6.3327×10^{-2}	9.3780×10^{-2}	1.4847×10^{-1}	1.7540×10^{-1}	1.1507×10^{-1}
MAPE						
Log	6.1630×10^{-4}	1.1267×10^{-3}	9.5100×10^{-4}	1.1338×10^{-3}	1.2656×10^{-4}	7.9086×10^{-4}
GJ	2.2558×10^{-2}	5.8132×10^{-2}	8.5078×10^{-2}	1.2691×10^{-1}	1.3553×10^{-1}	8.5640×10^{-2}

Obtaining a closed form-solution for the FFA prices also has some other additional applications. It is very well known that a closed-form solution for a freight option cannot be obtained. Then, some authors, such as [14], obtain an analytical approximation in the form of the lower bound. Gómez-Valle et al. [21] showed that it is possible to get a lower bound using the FFA price:

$$e^{-r(T_N-t)}(F(t,S;T_1,\ldots,T_N)-K)^{+} \leq C(t,S;K,T_1,\ldots,T_N).$$

Therefore, we can use the FFA prices obtained in Section 3 to calculate a lower bound that can give us an approximation for the freight option price.

In Figure 2, we show the lower bounds for a freight option with different strikes from 70% to 130%, maturities of 1 and 3 months, and $S = 2000$. In our computations, we use then a proxy for the risk free interest rate $r = 0.005$, which is a representative value of the interest rates around the world nowadays. For both maturities, the GJ lower bounds are always lower than the Log and LogJ lower bounds, but the differences increase with the maturity. These differences are even more evident than those between the FFA prices with these models, see Figure 1. Therefore, as the GJ model shows the lowest FFA prices and bounds, it is expected that it will also underprice freight options. However, if we consider the Log model, its lower bounds cannot be nearly distinguished from the LogJ lower bounds.

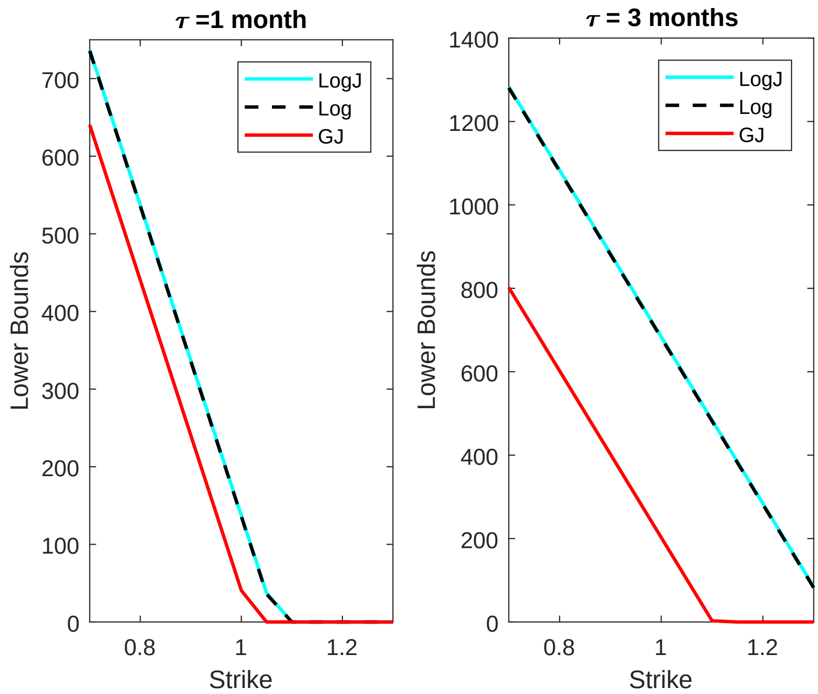

Figure 2. Lower bounds of freight options for $S = 2000$ and maturities of 1 and 3 months with the LogJ, Log, and GJ models.

As far as freight option prices are concerned, in order to avoid applying the inaccurate Monte Carlo method, numerical discretization schemes could be designed to approximate the solutions to the PIDEs (16) and (17). Notice that, from (13), the freight option price verifies a pure final value problem which must be solved. Therefore, a bounded domain for the state variables should be defined by truncation, and then suitable boundary conditions should be established. For example, some boundary conditions have been proposed to price European options and bonds like in [35–38], but these conditions are not suitable for freight options. In this case, the partial solution obtained in Proposition 2 can be used to design appropriate boundary conditions for these kinds of problems. However, freight option pricing by means of numerical methods to PIDE problems is beyond the scope of this study.

6. Discussion and Conclusions

The freight market is very new; however, its importance is increasing considerably nowadays. Therefore, in this area, additional scientific research is necessary. As in this market large participants can manipulate prices, the setting is made against the average of some freight index values. This fact results in derivatives with average-style payoffs, which are more difficult to value.

Recently, there has been a growing amount of literature on freight derivative valuation, which, inspired by the general finance literature, uses diffusion processes. However, considering jump-diffusion processes provides a more realistic behavior of the spot freight rate, and we try to fill this gap with this paper.

The main contributions of this paper are as follows. Firstly, we obtain a closed-form solution for the FFA prices when a jump term is added to some very well-known diffusion spot freight rate processes. More precisely, we incorporate a jump term to the geometric Brownian motion and the geometric mean reversion model. Additionally, we show how these prices can also be used to provide lower bounds for freight option prices, which, in some cases, are used as approximations for these prices.

Secondly, we provide a general framework to obtain a PIDE to price freight options under the dynamics of a jump-diffusion spot freight rate process. This PIDE has three variables: the spot freight rate, its delay, and the continuous version of the average of the spot rate in the settlement period.The main benefits of this approach are as follows. It opens the door to developing numerical methods for the PIDE to obtain an accurate approximation. Furthermore, it allows us to obtain in some cases, such as the geometric Brownian motion with jumps, a partial closed form solution. This solution can be used for attaining suitable boundary conditions in order to design a numerical method for the PIDE in a bounded domain.

Finally, by means of a test problem, we show different facts. For example, if we do not consider mean reversion in the spot freight rate, the FFAs can be underpriced. However, the effect of the jump could be compensated by a higher volatility of the process, as the parameters are globally estimated. We also show that, when the mean reversion and the jumps are taken into account, the differences between the lower bounds of the freight options are more remarkable.

As future research, on the one hand, we could improve the model by considering a more general process for the spot freight rate or adding a new variable such as the stochastic volatility. In the latest case, the PIDE will depend on another additional independent variable, and, therefore, its resolution will be considerably more complicated. On the other hand, inspired by the results of the Feynman–Kac problems shown in [22], a study of the uniqueness of the solution to the PIDE obtained in this work and its representation as the conditional expectation of the freight option price at the maturity could be very interesting.

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Abbreviations

The following abbreviations are used in this manuscript:

PIDE	Partial Integro-Differential Equation
FFA	Forward Freight Agreement
BDI	Baltic Dry Index

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