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On the Generalization of Factoriangular Numbers

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

A factoriangular number is a sum of a factorial and its corresponding triangular number. This paper presents some forms of the generalization of factoriangular numbers. One generalization is the $n^{(m)}$-factoriangular number which is of the form $(n!)^m + S_m(n)$, where $(n!)^m$ is the mth power of the factorial of n and $S_m(n)$ is the sum of the m-powers of n. This generalized form is explored for the different values of the natural number m. The investigation results to some interesting proofs of theorems related thereto. Two important formulas were generated for $n^{(m)}$-factoriangular number: $Ft_{n^{(m)}} = Ft_{n^{(2k)}} = (n!)^{2k} + \frac{2n+1}{2k+1}\left[n^{2k-2} + P(n^{2k-3})\right]T_n$ for even $m = 2k$, and $Ft_{n^{(m)}} = Ft_{n^{(2k+1)}} = (n!)^{2k+1} + \frac{n(n+1)}{k+1}\left[n^{2k-2} + P(n^{2k-3})\right]T_n$ for odd $m = 2k + 1$.

Keywords: Factoriangular numbers; generalized factoriangular numbers; factorial; triangular numbers; sums of powers; Faulhaber's sums; integer sequences; number theory.

1 Introduction

Studies on factorials, triangular numbers, and other numbers associated with them abound the literature and have a long history of research in number theory. A survey on factorials, triangular numbers, and factoriangular numbers is provided in a recent article [1]. While factorials and triangular numbers have long been studied,

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factoriangular numbers have been studied only in the past few years. In 2004, a sequence of numbers of the form *a(n) = n! + Sum_{i=1..n} i* was introduced in Sloane's The Online Encyclopedia of Integer Sequences (OEIS) [2]. In 2015, this number was named *factoriangular*, a contraction of the terms *factorial* and *triangular* [3]. A factoriangular number is a sum of a factorial and its corresponding triangular number and some recent studies were conducted on this relatively new sequence of numbers [3-12].

The first study on factoriangular numbers presents its characteristics as regards parity, compositeness, number and sum of its positive divisors, abundancy and deficiency, Zeckendorf's decomposition, end digits, and digital roots [3]. The second study presents the runsum representations of factoriangular numbers and as difference of two triangular numbers, as well as its trapezoidal arrangements and politeness [4]. A study presents factoriangular numbers that are sums of two triangular numbers or sums of two squares [5] while another study gives some recurrence relations and exponential generating functions of the sequence of factoriangular numbers [6].

Ruiz and Luca [7] prove that 2, 5, and 34 are the only Fibonacci factoriangular numbers, which confirms the conjecture of Castillo [3]. Luca, Odjoumani and Togbe [8] show that the only Pell factoriangular numbers are 2, 5, and 12 while Kafle, Luca and Togbe [9] show that the only Lucas factoriangular numbers are 1 and 2. In addition, Rayaguru, Odjoumani and Panda [10] prove that there is no factoriangular number in the sequence of balancing numbers, as well as in the sequence of Lucas-balancing numbers.

Two recent studies present generalizations of factoriangular numbers. In one study, a factoriangular number (as being a sum of corresponding factorial and triangular number) is generalized as a sum of any factorial and any triangular number [11]. In another study, the factoriangular number is generalized as sum of a power of factorial and its corresponding power sum, which is dubbed multiple factoriangular numbers [12]. This paper aims to explore this second generalization, named here as $n^{(m)}$-factoriangular numbers. In particular, the generalized factoriangular numbers in the form of $(n!)^m + S_m(n)$, where $(n!)^m$ is the mth power of the factorial of n and $S_m(n)$ is the sum of the m-powers of n, are examined for the different values of $m \geq 1$.

The different forms of the generalized factoriangular numbers are presented in the next section. The third section presents the different cases for the $n^{(m)}$-factoriangular numbers and the proofs of some theorems related thereto. Conclusion is given in the last section.

2 Generalization of Factoriangular Numbers

A factoriangular number is a sum of the corresponding factorial and triangular number. The definitions of factorial and triangular number are given as follows:

Definition 2.1. *For natural number n, the factorial of n is given by*

$$n! = 1 \cdot 2 \cdot 3 \cdots n\,.$$

Definition 2.2. *For natural number n, the nth triangular number is given by*

$$T_n = 1 + 2 + 3 + ... + n = n(n+1)/2\,.$$

The first few factoriangular numbers, denoted by Ft_n [3], are {2, 5, 12, 34, 135, 741, 5068, 40356, 362925, 3628855, …}. This sequence is included in OEIS as sequence A101292 in 2004, but in 2016, 1 was added as the first term [2].

Definition 2.3. *The nth factoriangular number is given by the formula*

$$Ft_n = n! \ + \ T_n\,.$$

Here, numbers of this form will be called n-factoriangular numbers.

In the sequence of factoriangular numbers, $\{Ft_n\}$, each entry is given by

$$Ft_n = (1 \cdot 2 \cdot 3 \cdots n) + (1 + 2 + 3 + ... + n)$$

for natural number $n \geq 1$. This sequence can be generalized in several ways. One generalization [11] is the sequence $\{Ft_{n,k}\}$ where each entry is given by

$$Ft_{n,k} = (1 \cdot 2 \cdot 3 \cdots n) + (1 + 2 + 3 + ... + k)$$

for natural numbers $n, k \geq 1$. Clearly, when $n = k$, $Ft_{n,k} = Ft_n$. Numbers of this form will be called *(n, k)*-factoriangular numbers.

Definition 2.4. *The* (n, k) *-factoriangular number is defined by the formula*

$$Ft_{n,k} = n! \ + \ T_k .$$

Another generalization is the sequence $\{Ft_{n^{(m)}}\}$, where, for natural numbers $n, m \geq 1$,

$$Ft_{n^{(m)}} = (1^m \cdot 2^m \cdot 3^m \cdots n^m) + (1^m + 2^m + 3^m + ... + n^m) .$$

It is also clear that when $m = 1$, $Ft_{n^{(m)}} = Ft_n$. Here, numbers of this form will be called $n^{(m)}$-factoriangular numbers.

Definition 2.5. *The* $n^{(m)}$ *-factoriangular number is defined by the formula*

$$Ft_{n^{(m)}} = (n!)^m + S_m(n)$$

where $(n!)^m = 1^m \cdot 2^m \cdot 3^m \cdots n^m$ *and* $S_m(n) = 1^m + 2^m + 3^m + ... + n^m$ *for natural numbers* $n, m \geq 1$.

These $n^{(m)}$-factoriangular numbers are similar to the multiple factoriangular numbers [12] given as

$$F_t(n,k) = (n!)^k \ + \ \sum n^k$$

where $\sum n^k = T_n(k)$.

Further generalization of factoriangular numbers is the sequence $\{Ft_{n^{(m)},k^{(m)}}\}$, where, for natural numbers $n, k, m \geq 1$,

$$Ft_{n^{(m)},k^{(m)}} = (1^m \cdot 2^m \cdot 3^m \cdots n^m) + (1^m + 2^m + 3^m + ... + k^m) .$$

Notice also that when $n = k$ and $m = 1$, $Ft_{n^{(m)},k^{(m)}} = Ft_n$. Numbers of this form will be called $(n^{(m)},k^{(m)})$-factoriangular numbers.

Definition 2.6. *The* $(n^{(m)}, k^{(m)})$ *-factoriangular number is defined by the formula*

$$Ft_{n^{(m)},k^{(m)}} = (n!)^m + S_m(k)$$

where $(n!)^m = 1^m \cdot 2^m \cdot 3^m \cdots n^m$ and $S_m(k) = 1^m + 2^m + 3^m + ... + k^m$ for natural numbers $n, k, m \geq 1$.

Still another generalization is the sequence $\left\{Ft_{n^{(a)},k^{(b)}}\right\}$, where, for natural numbers $n, k, a, b \geq 1$,

$$Ft_{n^{(a)},k^{(b)}} = (1^a \cdot 2^a \cdot 3^a \cdots n^a) + (1^b + 2^b + 3^b + ... + k^b) .$$

If $n = k$ and $a = b = 1$, then $Ft_{n^{(a)},k^{(b)}} = Ft_n$. Numbers of this form will be called *($n^{(a)}$,$k^{(b)}$)*-factoriangular numbers.

Definition 2.7. *The* $(n^{(a)}, k^{(b)})$ *-factoriangular number is defined by the formula*

$$Ft_{n^{(a)},k^{(b)}} = (n!)^a + S_b(k)$$

where $(n!)^a = 1^a \cdot 2^a \cdot 3^a \cdots n^a$ and $S_b(k) = 1^b + 2^b + 3^b + ... + k^b$ for natural numbers $n, k, a, b \geq 1$.

As given above, the *n*-factoriangular numbers can be generalized in several ways. This paper, however, focuses only on the $n^{(m)}$ -factoriangular numbers as presented in the next section.

3 The $n^{(m)}$-Factoriangular Numbers

The $n^{(m)}$ -factoriangular number is a generalization of the *n*-factoriangular number. The symbol $Ft_{n^{(m)}}$ is used here to denote $n^{(m)}$ -factoriangular number and it should be cautiously noted that this is different from Ft_n^m (the *m*th power of the *n*-factoriangular number) and from Ft_{n^m} (the *n*-factoriangular of the *m*th power of *n*). Recall that the $n^{(m)}$ -factoriangular number is given by the formula

$$Ft_{n^{(m)}} = (n!)^m + S_m(n)$$

where $(n!)^m = 1^m \cdot 2^m \cdot 3^m \cdots n^m$ and $S_m(n) = 1^m + 2^m + 3^m + ... + n^m$ for natural numbers $n, m \geq 1$. Here, several cases in connection with the value of *m* are analyzed.

3.1 The First Four Cases

Case 1. When $m = 1$,

$$Ft_{n^{(1)}} = n! + S_1(n) = (1 \cdot 2 \cdot 3 \cdots n) + (1 + 2 + 3 + ... + n) = Ft_n ,$$

which is the *n*-factoriangular number.

Case 2. When $m = 2$,

$$Ft_{n^{(2)}} = (n!)^2 + S_2(n)$$
$$\Leftrightarrow \quad Ft_{n^{(2)}} = (1^2 \cdot 2^2 \cdot 3^2 \cdots n^2) + (1^2 + 2^2 + 3^2 + ... + n^2) .$$

Theorem 3.1. *For natural number* $n \geq 1$*, the* $n^{(2)}$*-factoriangular number is given by the formula*

$$Ft_{n^{(2)}} = (n!)^2 + \frac{1}{3}(2n+1)T_n .$$

Proof. It is a well-known fact that

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} ,$$

which can be proven easily through mathematical induction. However, instead of proving by induction, the following discussion shows how this identity was derived.

For ease of writing, let the sum of powers of natural numbers be denoted by S_m instead of $S_m(n)$. If $m = 0$ is included then,

$$S_0 = 1^0 + 2^0 + 3^0 + ... + n^0 = 1+1+1+...+1 = 1 \cdot n = n .$$

If $m = 1$,

$$S_1 = 1+2+3+...+n = n(n+1)/2 = T_n ,$$

the triangular number. This formula for S_1 or T_n can also be proven easily through induction but the derivation is more interesting here. An easy derivation is through the addition of S_1 to itself. In particular,

$$\begin{aligned} S_1 &= 1 + 2 + 3 + ... + n \\ + S_1 &= n + (n-1) + (n-2) + ... + 1, \end{aligned}$$

which results to

$$2S_1 = (n+1)+(n+1)+(n+1)+...+(n+1) = n(n+1)$$
$$\Leftrightarrow \qquad S_1 = \frac{n(n+1)}{2} .$$

A more complex derivation is also presented below and this will be more useful later in the derivation of formulas for sum of higher powers of natural numbers. For any positive integer r,

$$(r+1)^2 - r^2 = r^2 + 2r + 1 - r^2 = 2r + 1.$$

Then, for $r = 1, 2, 3, ..., n$,

$$\begin{aligned} 2^2 - 1^2 &= 2(1)+1 \\ 3^2 - 2^2 &= 2(2)+1 \\ 4^2 - 3^2 &= 2(3)+1 \\ &\vdots \\ (n+1)^2 - n^2 &= 2(n)+1. \end{aligned}$$

Adding the corresponding members of the above equations through a technique called telescoping and simplifying, results to

$$(n+1)^2 - 1 = 2(1+2+3+...+n) + (1+1+1+...+1)$$

$$\Leftrightarrow \qquad n^2 + 2n + 1 - 1 = 2S_1 + n$$

$$\Leftrightarrow \qquad S_1 = \frac{n(n+1)}{2}.$$

This technique was actually already known to Pascal who had shown how to use the binomial coefficients to find the sum of the *k*th powers of the first (*n*-1) positive integers if the formulas for the sums of the powers less than *k* are known [13].

Similarly, for any positive integer r,

$$(r+1)^3 - r^3 = r^3 + 3r^2 + 3r + 1 - r^3 = 3r^2 + 3r + 1.$$

Then, for $r = 1, 2, 3, ..., n$,

$$\begin{aligned} 2^3 - 1^3 &= 3(1^2) + 3(1) + 1 \\ 3^3 - 2^3 &= 3(2^2) + 3(2) + 1 \\ 4^3 - 3^3 &= 3(3^2) + 3(3) + 1 \\ &\vdots \\ (n+1)^3 - n^3 &= 3(n^2) + 3(n) + 1. \end{aligned}$$

Adding the corresponding members of these equations and simplifying, results to

$$(n+1)^3 - 1 = 3(1^2 + 2^2 + 3^2 + ... + n^2) + 3(1+2+3+...+n) + (1+1+1+...+1)$$

$$\Leftrightarrow \qquad n^3 + 3n^2 + 3n + 1 - 1 = 3S_2 + 3S_1 + n.$$

Hence,

$$3S_2 = n^3 + 3n^2 + 3n - 3\left[\frac{n(n+1)}{2}\right] - n$$

$$\Leftrightarrow \qquad 6S_2 = 2n^3 + 3n^2 + n$$

$$\Leftrightarrow \qquad S_2 = \frac{n(n+1)(2n+1)}{6}.$$

Thus,

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6},$$

which can also be written as

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{1}{3}(2n+1)T_n .$$

Hence,

$$Ft_{n^{(2)}} = (n!)^2 + \frac{1}{3}(2n+1)T_n$$

and the theorem was proven.

With this formula, the entries in the sequence of $n^{(2)}$-factoriangular numbers can be computed and the sequence is given as follows:

$$\left\{Ft_{n^{(2)}}\right\} = \{2, 9, 50, 606, 14455, ...\} .$$

Note again that $Ft_{n^{(2)}}$ is different from Ft_n^2 and from Ft_{n^2}. For instance, if $n = 2$,

$$Ft_{2^{(2)}} = (2!)^2 + \frac{1}{3}[2(2)+1](3) = 9 \text{ or } Ft_{2^{(2)}} = (1^2 \cdot 2^2) + (1^2 + 2^2) = 9$$

while $Ft_2^2 = [(1 \cdot 2) + (1+2)]^2 = 25$ and $Ft_{2^2} = Ft_4 = 4! + 4(4+1)/2 = 34$.

Case 3. When $m = 3$,

$$Ft_{n^{(3)}} = (n!)^3 + S_3(n)$$

$$\Leftrightarrow \qquad Ft_{n^{(3)}} = (1^3 \cdot 2^3 \cdot 3^3 \cdots n^3) + (1^3 + 2^3 + 3^3 + ... + n^3)$$

Theorem 3.2. *For natural number $n \geq 1$, the $n^{(3)}$-factoriangular number is given by the formula*

$$Ft_{n^{(3)}} = (n!)^3 + T_n^2 .$$

Proof. For any positive integer r,

$$(r+1)^4 - r^4 = r^4 + 4r^3 + 6r^2 + 4r + 1 - r^4 = 4r^3 + 6r^2 + 4r + 1 .$$

Then, for $r = 1, 2, 3, ..., n$,

$$\begin{aligned} 2^4 - 1^4 &= 4(1^3) + 6(1^2) + 4(1) + 1 \\ 3^4 - 2^4 &= 4(2^3) + 6(2^2) + 4(2) + 1 \\ 4^4 - 3^4 &= 4(3^3) + 6(3^2) + 4(3) + 1 \\ &\vdots \\ (n+1)^4 - n^4 &= 4(n^3) + 6(n^2) + 4(n) + 1. \end{aligned}$$

Adding the corresponding members of these equations and simplifying, results to

$$(n+1)^4 - 1 = 4(1^3 + 2^3 + 3^3 + ... + n^3) + 6(1^2 + 2^2 + 3^2 + ... + n^2)$$
$$+ 4(1 + 2 + 3 + ... + n) + (1 + 1 + 1 + ... + 1)$$

$$\Leftrightarrow \qquad n^4 + 4n^3 + 6n^2 + 4n + 1 - 1 = 4S_3 + 6S_2 + 4S_1 + n.$$

It follows that,

$$4S_3 = n^4 + 4n^3 + 6n^2 + 4n - 6\left[\frac{n(n+1)(2n+1)}{6}\right] - 4\left[\frac{n(n+1)}{2}\right] - n$$

$$\Leftrightarrow \qquad 4S_3 = n^4 + 2n^3 + n^2$$

$$\Leftrightarrow \qquad S_3 = \frac{n^2(n+1)^2}{4}.$$

With this,

$$1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4},$$

which can also be written as

$$1^3 + 2^3 + 3^3 + ... + n^3 = T_n^2.$$

Hence,

$$Ft_{n^{(3)}} = (n!)^3 + T_n^2$$

and the theorem was proven.

With this formula, the sequence of $n^{(3)}$-factoriangular numbers is as follows:

$$\left\{Ft_{n^{(3)}}\right\} = \left\{2, 17, 252, 13924, 1728225, ...\right\}.$$

Case 4. When $m = 4$,

$$Ft_{n^{(4)}} = (n!)^4 + S_4(n)$$

$$\Leftrightarrow \qquad Ft_{n^{(4)}} = (1^4 \cdot 2^4 \cdot 3^4 \cdots n^4) + (1^4 + 2^4 + 3^4 + ... + n^4)$$

Theorem 3.3. *For natural number $n \geq 1$, the $n^{(4)}$-factoriangular number is given by the formula*

$$Ft_{n^{(4)}} = (n!)^4 + \frac{1}{15}(6n^3 + 9n^2 + n - 1)T_n.$$

Proof. For any positive integer r,

$$(r+1)^5 - r^5 = r^5 + 5r^4 + 10r^3 + 10r^2 + 5r + 1 - r^5 = 5r^4 + 10r^3 + 10r^2 + 5r + 1.$$

Then, similar to the previous cases, it can be shown that

$$(n+1)^5 - 1 = 5S_4 + 10S_3 + 10S_2 + 5S_1 + n$$

and solving for S_4, results to

$$S_4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30} = \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}.$$

Thus,

$$1^4 + 2^4 + 3^4 + ... + n^4 = \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30},$$

which can also be written as

$$1^4 + 2^4 + 3^4 + ... + n^4 = \frac{1}{15}(6n^3 + 9n^2 + n - 1)T_n.$$

Hence,

$$Ft_{n^{(4)}} = (n!)^4 + \frac{1}{15}(6n^3 + 9n^2 + n - 1)T_n$$

and the theorem was proven.

The sequence of $n^{(4)}$-factoriangular numbers is now given as follows:

$$\left\{Ft_{n^{(4)}}\right\} = \{2, 33, 1394, 332130, 207360979, ...\}.$$

Summarizing the results for the first four cases regarding the sums of powers of positive integers, S_m, for $m \geq 1$:

$$S_1 = \frac{n^2 + n}{2} = \frac{n(n+1)}{2} = T_n;$$

$$S_2 = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}(2n+1)T_n;$$

$$S_3 = \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^2(n+1)^2}{4} = T_n^2; \text{ and}$$

$$S_4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30} = \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30} = \frac{1}{15}(6n^3 + 9n^2 + n - 1)T_n.$$

Notice that the triangular number, T_n, is a factor in S_m. This shows the recursive nature of the sums of powers of positive integers. Similar to these results are the Faulhaber's formulas for sums of odd powers expressing each sum of odd powers from 1 to 17 as a polynomial in N where $N=(n^2+n)/2$ [14]. The first five cases of Faulhaber's results are as follows:

$$1^1+2^1+...+n^1=N;$$
$$1^3+2^3+...+n^3=N^2;$$
$$1^5+2^5+...+n^5=(4N^3-N^2)/3;$$
$$1^7+2^7+...+n^7=(12N^4-8N^3+2N^2)/6; \text{ and}$$
$$1^9+2^9+...+n^9=(16N^5-20N^4+12N^3-3N^2)/5.$$

3.2 The Next Few Higher Cases

When $m=5,6,7,8,9$,

$$Ft_{n^{(5)}}=(n!)^5+S_5(n)=(1^5\cdot 2^5\cdot 3^5\cdots n^5)+(1^5+2^5+3^5+...+n^5);$$
$$Ft_{n^{(6)}}=(n!)^6+S_6(n)=(1^6\cdot 2^6\cdot 3^6\cdots n^6)+(1^6+2^6+3^6+...+n^6);$$
$$Ft_{n^{(7)}}=(n!)^7+S_7(n)=(1^7\cdot 2^7\cdot 3^7\cdots n^7)+(1^7+2^7+3^7+...+n^7);$$
$$Ft_{n^{(8)}}=(n!)^8+S_8(n)=(1^8\cdot 2^8\cdot 3^8\cdots n^8)+(1^5+2^8+3^8+...+n^8); \text{ and}$$
$$Ft_{n^{(9)}}=(n!)^9+S_9(n)=(1^9\cdot 2^9\cdot 3^9\cdots n^9)+(1^9+2^9+3^9+...+n^9),$$

respectively.

Theorem 3.4. *For natural number $n\geq 1$, then $n^{(5)}$, $n^{(6)}$, $n^{(7)}$, $n^{(8)}$ and $n^{(9)}$-factoriangular numbers are, respectively, given by the formulas*

$$Ft_{n^{(5)}}=(n!)^5+\frac{1}{3}(2n^2+2n-1)T_n^2;$$
$$Ft_{n^{(6)}}=(n!)^6+\frac{1}{21}(6n^5+15n^4+6n^3-6n^2-n+1)T_n;$$
$$Ft_{n^{(7)}}=(n!)^7+\frac{1}{6}(3n^4+6n^3-n^2-4n+2)T_n^2;$$
$$Ft_{n^{(8)}}=(n!)^8+\frac{1}{45}(10n^7+35n^6+25n^5-25n^4-17n^3+17n^2+3n-3)T_n; \text{ and}$$
$$Ft_{n^{(9)}}=(n!)^9+\frac{1}{5}(2n^6+6n^5+n^4-8n^3+n^2+6n-3)T_n^2.$$

Proof. As in the first four lower cases for sums of powers of positive integers, it can be shown that for any positive integer r,

$$(r+1)^6-r^6=6r^5+15r^4+20r^3+15r^2+6r+1 \ \Rightarrow$$
$$(n+1)^6-1=6S_5+15S_4+20S_3+15S_2+6S_1+n;$$

$$(r+1)^7 - r^7 = 7r^6 + 21r^5 + 35r^4 + 35r^3 + 21r^2 + 7r + 1 \Rightarrow$$
$$(n+1)^7 - 1 = 7S_6 + 21S_5 + 35S_4 + 35S_3 + 21S_2 + 7S_1 + n;$$
$$(r+1)^8 - r^8 = 8r^7 + 28r^6 + 56r^5 + 70r^4 + 56r^3 + 28r^2 + 8r + 1 \Rightarrow$$
$$(n+1)^8 - 1 = 8S_7 + 28S_6 + 56S_5 + 70S_4 + 56S_3 + 28S_2 + 8S_1 + n;$$
$$(r+1)^9 - r^9 = 9r^8 + 36r^7 + 84r^6 + 126r^5 + 126r^4 + 85r^3 + 36r^2 + 9r + 1 \Rightarrow$$
$$(n+1)^9 - 1 = 9S_8 + 36S_7 + 84S_6 + 126S_5 + 126S_4 + 84S_3 + 36S_2 + 9S_1 + n; \text{ and}$$
$$(r+1)^{10} - r^{10} = 10r^9 + 45r^8 + 120r^7 + 210r^6 + 252r^5 + 210r^4 + 120r^3 + 45r^2 + 10r + 1 \Rightarrow$$
$$(n+1)^{10} - 1 = 10S_9 + 45S_8 + 120S_7 + 210S_6 + 252S_5 + 210S_4 + 120S_3 + 45S_2 + 10S_1 + n.$$

Solving for S_5, S_6, S_7, S_8 and S_9 successively, using previously derived formulas for sums of lower powers, results to

$$S_5 = \frac{1}{3}(2n^2 + 2n - 1)T_n^2;$$
$$S_6 = \frac{1}{21}(6n^5 + 15n^4 + 6n^3 - 6n^2 - n + 1)T_n;$$
$$S_7 = \frac{1}{6}(3n^4 + 6n^3 - n^2 - 4n + 2)T_n^2;$$
$$S_8 = \frac{1}{45}(10n^7 + 35n^6 + 25n^5 - 25n^4 - 17n^3 + 17n^2 + 3n - 3)T_n; \text{ and}$$
$$S_9 = \frac{1}{5}(2n^6 + 6n^5 + n^4 - 8n^3 + n^2 + 6n - 3)T_n^2.$$

Hence,

$$Ft_{n^{(5)}} = (n!)^5 + \frac{1}{3}(2n^2 + 2n - 1)T_n^2;$$
$$Ft_{n^{(6)}} = (n!)^6 + \frac{1}{21}(6n^5 + 15n^4 + 6n^3 - 6n^2 - n + 1)T_n;$$
$$Ft_{n^{(7)}} = (n!)^7 + \frac{1}{6}(3n^4 + 6n^3 - n^2 - 4n + 2)T_n^2;$$
$$Ft_{n^{(8)}} = (n!)^8 + \frac{1}{45}(10n^7 + 35n^6 + 25n^5 - 25n^4 - 17n^3 + 17n^2 + 3n - 3)T_n; \text{ and}$$
$$Ft_{n^{(9)}} = (n!)^9 + \frac{1}{5}(2n^6 + 6n^5 + n^4 - 8n^3 + n^2 + 6n - 3)T_n^2;$$

and the theorem was proven.

Notice that for odd $m > 3$, T_n^2 is a factor of S_m. Further explorations on these resulted to the following:
For $m = 5$,

$$S_5 = \frac{1}{3}T_n^2(2n^2 + 2n - 1)$$

$$=\frac{1}{3}T_n^2[2(n^2+n)-1]$$
$$=\frac{1}{3}T_n^2\left[\frac{4(n^2+n)}{2}-1\right]$$
$$=\frac{1}{3}T_n^2(4T_n-1)$$
$$=\frac{1}{3}(4T_n^3-T_n^2);$$

for $m=7$,

$$S_7=\frac{1}{6}T_n^2(3n^4+6n^3-n^2-4n+2)$$
$$=\frac{1}{6}T_n^2[(3n^4+6n^3+3n^2)-(4n^2+4n)+2]$$
$$=\frac{1}{6}T_n^2[3(n^2+n)^2-4(n^2+n)+2]$$
$$=\frac{1}{6}T_n^2\left[\frac{12(n^2+n)^2}{4}-\frac{8(n^2+n)}{2}+2\right]$$
$$=\frac{1}{6}T_n^2(12T_n^2-8T_n+2)$$
$$=\frac{1}{6}(12T_n^4-8T_n^3+2T_n^2);$$

and for $m=9$,

$$S_9=\frac{1}{5}T_n^2(2n^6+6n^5+n^4-8n^3+n^2+6n-3)$$
$$=\frac{1}{5}T_n^2[(2n^6+6n^5+6n^4+2n^3)-(5n^4+10n^3+5n^2)+(6n^2+6n)-3]$$
$$=\frac{1}{5}T_n^2[2(n^2+n)^3-5(n^2+n)^2+6(n^2+n)-3]$$
$$=\frac{1}{5}T_n^2\left[\frac{16(n^2+n)^3}{8}-\frac{20(n^2+n)^2}{4}+\frac{12(n^2+n)}{2}-3\right]$$
$$=\frac{1}{5}T_n^2(16T_n^3-20T_n^2+12T_n-3)$$
$$=\frac{1}{5}(16T_n^5-20T_n^4+12T_n^3-3T_n^2);$$

which are clearly seen now as exactly the same as Faulhaber's sums of odd powers mentioned earlier.

Similarly, the sums of even powers for $m>2$ were further explored and the results are as follows:
For $m=4$,

$$S_4 = \frac{1}{15}(6n^3 + 9n^2 + n - 1)T_n$$

$$= \frac{1}{15}(2n+1)(3n^2 + 3n - 1)T_n$$

$$= \frac{1}{15}(2n+1)\left[\frac{6(n^2+n)}{2} - 1\right]T_n$$

$$= \frac{1}{15}(2n+1)(6T_n^2 - T_n)\,;$$

for $m = 6$

$$S_6 = \frac{1}{21}(6n^5 + 15n^4 + 6n^3 - 6n^2 - n + 1)T_n$$

$$= \frac{1}{21}(2n+1)(3n^4 + 6n^3 - 3n + 1)T_n$$

$$= \frac{1}{21}(2n+1)[(3n^4 + 6n^3 + 3n^2) - (3n^2 + 3n) + 1)]T_n$$

$$= \frac{1}{21}(2n+1)[3(n^2+n)^2 - 3(n^2+n) + 1)]T_n$$

$$= \frac{1}{21}(2n+1)\left[\frac{12(n^2+n)^2}{4} - \frac{6(n^2+n)}{2} + 1\right]T_n$$

$$= \frac{1}{21}(2n+1)(12T_n^2 - 6T_n + 1)T_n$$

$$= \frac{1}{21}(2n+1)(12T_n^3 - 6T_n^2 + T_n)\,;$$

and for $m = 8$,

$$S_8 = \frac{1}{45}(10n^7 + 35n^6 + 25n^5 - 25n^4 - 17n^3 + 17n^2 + 3n - 3)T_n$$

$$= \frac{1}{45}(2n+1)(5n^6 + 15n^5 + 5n^4 - 15n^3 - n^2 + 9n - 3)T_n$$

$$= \frac{1}{45}(2n+1)[(5n^6 + 15n^5 + 15n^4 + 5n^3) - (10n^4 + 20n^3 + 10n^2) + (9n^2 + 9n) - 3)]T_n$$

$$= \frac{1}{45}(2n+1)[5(n^2+n)^3 - 10(n^2+n)^2 + 9(n^2+n) - 3)]T_n$$

$$= \frac{1}{45}(2n+1)\left[\frac{40(n^2+n)^3}{8} - \frac{40(n^2+n)^2}{4} + \frac{18(n^2+n)}{2} - 3\right]T_n$$

$$= \frac{1}{45}(2n+1)(40T_n^3 - 40T_n^2 + 18T_n - 3)T_n$$

$$= \frac{1}{45}(2n+1)(40T_n^4 - 40T_n^3 + 18T_n^2 - 3T_n).$$

Since

$$S_2 = \frac{1}{3}(2n+1)T_n,$$

it follows that

$$S_4 = \frac{1}{15}(2n+1)(6T_n - 1)T_n = \frac{1}{5}(6T_n - 1)S_2;$$
$$S_6 = \frac{1}{21}(2n+1)(12T_n^2 - 6T_n + 1)T_n = \frac{1}{7}(12T_n^2 - 6T_n + 1)S_2; \text{ and}$$
$$S_8 = \frac{1}{45}(2n+1)(40T_n^3 - 40T_n^2 + 18T_n - 3)T_n = \frac{1}{15}(40T_n^3 - 40T_n^2 + 18T_n - 3)S_2;$$

which shows that for even $m > 2$, S_2 divides S_m and the quotient is a polynomial in T_n as also noted and proven in a previous study [15].

3.3 The General Case

When m is any positive integer,

$$Ft_{n^{(m)}} = (n!)^m + S_m(n) = (1^m \cdot 2^m \cdot 3^m \cdots n^m) + (1^m + 2^m + 3^m + ... + n^m)$$

of which the formula for the second term (the sum of *m*th power of positive integers) is to be determined in a way similar to the previous specific cases.

Theorem 3.5. *For natural numbers* $n, m \geq 1$, *the* $n^{(m)}$ *-factoriangular numbers can be determined by the formula*

$$Ft_{n^{(m)}} = (n!)^m + \frac{1}{m+1}\left[(n+1)[(n+1)^m - 1] - \sum_{i=1}^{m-1}\binom{m+1}{i}S_i(n)\right]$$

where n! is the factorial of n and $S_i(n) = 1^i + 2^i + 3^i + ... + n^i$.

Proof. From the previous cases, it is not difficult to see that

$$(n+1)^{m+1} - 1 = \binom{m+1}{m}S_m + \binom{m+1}{m-1}S_{m-1} + \binom{m+1}{m-2}S_{m-2} + ... + \binom{m+1}{1}S_1 + n$$

or

$$(n+1)^{m+1} - (n+1) = (m+1)S_m + \binom{m+1}{1}S_1 + \binom{m+1}{2}S_2 + ... + \binom{m+1}{m-1}S_{m-1}$$

and then,

$$S_m = \frac{1}{m+1}\left[(n+1)[(n+1)^m - 1] - \sum_{i=1}^{m-1}\binom{m+1}{i}S_i\right].$$

Hence,

$$Ft_{n^{(m)}} = (n!)^m + \frac{1}{m+1}\left[(n+1)[(n+1)^m - 1] - \sum_{i=1}^{m-1}\binom{m+1}{i}S_i(n)\right]$$

and the theorem was proven.

What remains to be shown is the general expression of $n^{(m)}$-factoriangular numbers in terms of the factorial and the triangular number as in the previous specific cases for $m = 2,3,...,9$.

Theorem 3.6. *The* $n^{(m)}$ *-factoriangular number, for even* $m = 2k$ *, is given by the formula*

$$Ft_{n^{(m)}} = Ft_{n^{(2k)}} = (n!)^{2k} + \frac{2n+1}{2k+1}\left[n^{2k-2} + P(n^{2k-3})\right]T_n$$

and for odd $m = 2k+1$*, is given by the formula*

$$Ft_{n^{(m)}} = Ft_{n^{(2k+1)}} = (n!)^{2k+1} + \frac{n(n+1)}{k+1}\left[n^{2k-2} + P(n^{2k-3})\right]T_n$$

where $n!$ *is the factorial of n,* T_n *is the nth triangular number, and* $P(n^{2k-3})$ *is a polynomial in* n *of degree* $2k-3$*, for natural numbers* $n, k \geq 1$.

Proof. Further explorations on sums of powers of positive integers are needed and this time, consider separating the sums of even powers from the sums of odd powers.

Consider first the sum of even powers of integers $n \geq 1$, that is for natural numbers k, consider $S_{2k}(n)$. Again, for ease of writing, S_m is used in lieu of $S_m(n)$. Using the previous results for some specific cases, it can be shown that:

For $k = 1$,

$$S_{2(1)} \quad = \quad S_2 \quad = \frac{2n+1}{3}T_n;$$

for $k = 2$,

$$S_{2(2)} \quad = \quad S_4 \quad = \frac{2n+1}{15}(3n^2 + 3n - 1)T_n$$

$$= \frac{2n+1}{15}[3(n^2 + n - \tfrac{1}{3})]T_n$$

$$= \frac{2n+1}{5}[n^2 + (n - \tfrac{1}{3})]T_n;$$

for $k = 3$,

$$S_{2(3)} \quad = \quad S_6 \quad = \frac{2n+1}{21}(3n^4 + 6n^3 - 3n + 1)T_n$$
$$= \frac{2n+1}{21}[3(n^4 + 2n^3 - n + \tfrac{1}{3})]T_n$$
$$= \frac{2n+1}{7}[n^4 + (2n^3 - n + \tfrac{1}{3})]T_n;$$

and for $k = 4$,

$$S_{2(4)} \quad = \quad S_8 \quad = \frac{2n+1}{45}(5n^6 + 15n^5 + 5n^4 - 15n^3 - n^2 + 9n - 3)T_n$$
$$= \frac{2n+1}{45}[5(n^6 + 3n^5 + n^4 - 3n^3 - \tfrac{1}{5}n^2 + \tfrac{9}{5}n - \tfrac{3}{5})]T_n$$
$$= \frac{2n+1}{9}[n^6 + (3n^5 + n^4 - 3n^3 - \tfrac{1}{5}n^2 + \tfrac{9}{5}n - \tfrac{3}{5})]T_n.$$

These results can be generalized into:

$$S_{2k} = \frac{2n+1}{2k+1}\left[n^{2k-2} + P(n^{2k-3})\right]T_n$$

where the second term in the bracket is a polynomial in n of degree $2k-3$.
Hence, for $m = 2k$

$$Ft_{n^{(m)}} = Ft_{n^{(2k)}} = (n!)^{2k} + \frac{2n+1}{2k+1}\left[n^{2k-2} + P(n^{2k-3})\right]T_n$$

which proves the first part of the theorem.

Consider next the sum of odd powers of integers $n \geq 1$, that is for natural numbers k, consider $S_{2k+1}(n)$. From the previous results, but in this case using the factor T_n instead of T_n^2, it can be further shown that:

For $k = 1$,

$$S_{2(1)+1} \quad = \quad S_3 \quad = \frac{n(n+1)}{2}T_n;$$

for $k = 2$,

$$S_{2(2)+1} \quad = \quad S_5 \quad = \frac{n(n+1)}{6}(2n^2 + 2n - 1)T_n$$

$$= \frac{n(n+1)}{6}[2(n^2 + n - \tfrac{1}{2})]T_n$$
$$= \frac{n(n+1)}{3}[n^2 + (n - \tfrac{1}{2})]T_n;$$

for $k = 3$,

$$S_{2(3)+1} = S_7 = \frac{n(n+1)}{12}(3n^4 + 6n^3 - n^2 - 4n + 2)T_n$$
$$= \frac{n(n+1)}{12}[3(n^4 + 2n^3 - \tfrac{1}{3}n^2 - \tfrac{4}{3}n + \tfrac{2}{3})]T_n$$
$$= \frac{n(n+1)}{4}[n^4 + (2n^3 - \tfrac{1}{3}n^2 - \tfrac{4}{3}n + \tfrac{2}{3})]T_n;$$

and for $k = 4$,

$$S_{2(4)+1} = S_9 = \frac{n(n+1)}{10}(2n^6 + 6n^5 + n^4 - 8n^3 + n^2 + 6n - 3)T_n$$
$$= \frac{n(n+1)}{10}[2(n^6 + 3n^5 + \tfrac{1}{2}n^4 - 4n^3 - \tfrac{1}{2}n^2 + 3n - \tfrac{3}{2})]T_n$$
$$= \frac{n(n+1)}{5}[n^6 + (3n^5 + \tfrac{1}{2}n^4 - 4n^3 + \tfrac{1}{2}n^2 + 3n - \tfrac{3}{2})]T_n.$$

These results can be generalized into:

$$S_{2k+1} = \frac{n(n+1)}{k+1}\left[n^{2k-2} + P(n^{2k-3})\right]T_n$$

where the second term in the bracket is a polynomial in n of degree $2k-3$.
Hence, for $m = 2k+1$

$$Ft_{n^{(m)}} = Ft_{n^{(2k+1)}} = (n!)^{2k+1} + \frac{n(n+1)}{k+1}\left[n^{2k-2} + P(n^{2k-3})\right]T_n$$

which proves the second part and completes the proof of the theorem.

A previous study presented similar formulas as the above but without any explanation, proof or derivation [16].

Another set of formulas is stated in the following theorem.

Theorem 3.7. *The $n^{(m)}$-factoriangular number, for even $m = 2k$, is given by the formula*

$$Ft_{n^{(m)}} = Ft_{n^{(2k)}} = (n!)^{2k} + \frac{3}{2k+1}\left[n^{2k-2} + P(n^{2k-3})\right]S_2(n)$$

and for odd $m = 2k+1$, is given by the formula

$$Ft_{n^{(m)}} = Ft_{n^{(2k+1)}} = (n!)^{2k+1} + \frac{2}{k+1}\left[n^{2k-2} + P(n^{2k-3})\right]S_3(n)$$

where $n!$ *is the factorial of n,* $S_2(n) = n(n+1)(2n+1)/6$, $S_3(n) = n^2(n+1)^2/4$, *and* $P(n^{2k-3})$ *is a polynomial in* n *of degree* $2k-3$, *for natural numbers* $n \ge 1$ *and* $k > 1$.

Proof. With the identities

$$S_2 = \frac{2n+1}{3}T_n \text{ and } S_3 = T_n^2$$

and using the previous results that the sums of even powers of positive integers can be expressed in terms of S_2 while the sums of odd powers of positive integers can be expressed in terms of T_n^2 or S_3 , it can be further shown that for sums of even powers:

For $k = 2$,

$$S_{2(2)} \quad = \quad S_4 \quad = \frac{1}{5}(3n^2 + 3n - 1)S_2$$
$$= \frac{1}{5}[3(n^2 + n - \tfrac{1}{3})]S_2$$
$$= \frac{3}{5}[n^2 + (n - \tfrac{1}{3})]S_2 ;$$

for $k = 3$,

$$S_{2(3)} \quad = \quad S_6 \quad = \frac{1}{7}(3n^4 + 6n^3 - 3n + 1)S_2$$
$$= \frac{1}{7}[3(n^4 + 2n^3 - n + \tfrac{1}{3})]S_2$$
$$= \frac{3}{7}[n^4 + (2n^3 - n + \tfrac{1}{3})]S_2 ;$$

and for $k = 4$,

$$S_{2(4)} \quad = \quad S_8 \quad = \frac{1}{15}(5n^6 + 15n^5 + 5n^4 - 15n^3 - n^2 + 9n - 3)S_2$$
$$= \frac{1}{15}[5(n^6 + 3n^5 + n^4 - 3n^3 - \tfrac{1}{5}n^2 + \tfrac{9}{5}n - \tfrac{3}{5})]S_2$$
$$= \frac{3}{9}[n^6 + (3n^5 + n^4 - 3n^3 - \tfrac{1}{5}n^2 + \tfrac{9}{5}n - \tfrac{3}{5})]S_2 .$$

These results can be generalized into:

$$S_{2k} = \frac{3}{2k+1}\left[n^{2k-2} + P(n^{2k-3})\right]S_2$$

where the second term in the bracket is a polynomial in n of degree $2k-3$.

Hence, for $m=2k$

$$Ft_{n^{(m)}} = Ft_{n^{(2k)}} = (n!)^{2k} + \frac{3}{2k+1}\left[n^{2k-2} + P(n^{2k-3})\right]S_2(n)$$

which proves the first part of the theorem.

Similarly for sums of odd powers:

For $k=2$,

$$\begin{aligned} S_{2(2)+1} \;=\; S_5 &= \frac{1}{3}(2n^2+2n-1)S_3 \\ &= \frac{1}{3}[2(n^2+n-\tfrac{1}{2})]S_3 \\ &= \frac{2}{3}[n^2+(n-\tfrac{1}{2})]S_3; \end{aligned}$$

for $k=3$,

$$\begin{aligned} S_{2(3)+1} \;=\; S_7 &= \frac{1}{6}(3n^4+6n^3-n^2-4n+2)S_3 \\ &= \frac{1}{6}[3(n^4+2n^3-\tfrac{1}{3}n^2-\tfrac{4}{3}n+\tfrac{2}{3})]S_3 \\ &= \frac{2}{4}[n^4+(2n^3-\tfrac{1}{3}n^2-\tfrac{4}{3}n+\tfrac{2}{3})]S_3; \end{aligned}$$

and for $k=4$,

$$\begin{aligned} S_{2(4)+1} \;=\; S_9 &= \frac{1}{5}(2n^6+6n^5+n^4-8n^3+n^2+6n-3)S_3 \\ &= \frac{1}{5}[2(n^6+3n^5+\tfrac{1}{2}n^4-4n^3-\tfrac{1}{2}n^2+3n-\tfrac{3}{2})]S_3 \\ &= \frac{2}{5}[n^6+(3n^5+\tfrac{1}{2}n^4-4n^3+\tfrac{1}{2}n^2+3n-\tfrac{3}{2})]S_3. \end{aligned}$$

These results can be generalized into:

$$S_{2k+1} = \frac{2}{k+1}\left[n^{2k-2} + P(n^{2k-3})\right]S_3$$

where the second term in the bracket is a polynomial in n of degree $2k-3$.

Hence, for $m=2k+1$

$$Ft_{n^{(m)}} = Ft_{n^{(2k+1)}} = (n!)^{2k+1} + \frac{2}{k+1}\left[n^{2k-2} + P(n^{2k-3})\right]S_3(n)$$

which proves the second part and completes the proof of the theorem.

The next question is how to determine the polynomial $P(n^{2k-3})$ given in Theorems 3.6 and 3.7. The following is for further investigation:

Open Question. *Is there an explicit formula or method to determine the polynomial* $P(n^{2k-3})$ *as stated in Theorems 3.6 and 3.7?*

4 Conclusion

The sequence of factoriangular numbers is a relatively new topic of research in number theory. A factoriangular number can be simply defined as a sum of a factorial and its corresponding triangular number. In this study some forms of the generalization of factoriangular numbers are presented. One particular generalization of factoriangular numbers of the form $(n!)^m + S_m(n)$ is explored for the different values of the natural number m and this results to some interesting proofs of theorems related thereto.

Competing Interests

Author has declared that no competing interests exist.

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