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# The Solutions of Generalized Euler Function Equation $\varphi_2(n - \varphi_2(n)) = 2^{\omega(n)}$

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***Authors' contributions***

*This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.*

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## Abstract

By using the properties of Euler function, an upper bound of solutions of Euler function equation $\varphi(x) = 2^s$ is given, where s is a positive integer. By using the classification discussion and the upper bound we obtained, all positive integer solutions of the generalized Euler function equation $\varphi_2\left(n - \varphi_2(n)\right) = 2^{\omega(n)}$ are given, where $\omega(n)$ is the number of distinct prime factors of n.

*Keywords: Euler functions; generalized Euler functions; solutions.*

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# 1 Introduction

Euler function $\varphi(n)$ is a relatively important content in elementary number theory, and Euler function is the number of positive integers not greater than $n$ and prime to $n$. According to the definition, $\varphi(1) = 1$. If $n > 1$, let canonical form of $n$ be $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \cdots {p_k}^{\alpha_k}$, where $p_i$ is a prime and $\alpha_i$ is a positive integer, then

$$\varphi(n) = \sum_{i=1}^{k} {p_i}^{\alpha_{i-1}}(p_i - 1).$$

Lv Zhihong [1,2] get the solutions of $\varphi(n) = 2^{\omega(n)}$ and $\varphi(\varphi(n)) = 2^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors of $n$. Li Yijun [3] calculated the solutions of $\varphi\left(\varphi(\varphi(n))\right) = 2^{\omega(n)}$, and studied the properties of $\left(\varphi(\varphi(n))\right)$ .

For positive integer $n$, the generalized Euler function [4] $\varphi_e(n)$ is defined as the number of positive integers not exceeding $\left[\frac{n}{e}\right]$ and prime to $n$ , namely

$$\varphi_e(n) = \sum_{i=1,\gcd(i,n)=1}^{\left[\frac{n}{e}\right]} 1,$$

where $[x]$ is the greatest integer not greater than $x$.

In particular, when $e = 2$, the generalized Euler function $\varphi_2(1) = 0$, $\varphi_2(2) = 1$. When $n \geq 3$, then $\varphi_2(n) = \frac{1}{2}\varphi(n)$.

Ding [5] introduced the generalized Euler function $\varphi_e(n)$ and its properties in detail. Cai [6,7] et. studied the parity of $\varphi_e(n)$ for $e = 2, 3, 4$ and 6. Zhang [8] solved the generalized Euler function $\varphi_2(x - \varphi_2(x)) = 2$ and $\varphi_2(\varphi_2(x - \varphi_2(x))) = 2$. At the same time, when $e = 6$ was selected in Reference [9], the equation $\varphi_6(n) = 2^{\omega(n)}$ was solved.

Yu and Shen [10] extended $\varphi(\varphi(n)) = 2^{\omega(n)}$ to the generalized Euler function, and obtained the positive integer solutions $\varphi_2(n) = 2^{\omega(n)}$ and $\varphi_2(\varphi_2(n)) = 2^{\omega(n)}$. Jin and Shen [11] changed $\omega(n)$ into $\Omega(n)$ on the basis of [10], and calculated the positive integer solutions of $\varphi_2(n) = 2^{\Omega(n)}$ and $\varphi_2(\varphi_2(n)) = 2^{\Omega(n)}$ , where $\Omega(n)$ is the number of prime factors of $n$, counting repetitions.

In this paper, we consider the equation $\varphi_2(n - \varphi_2(n)) = 2^{\omega(n)}$, and obtain the following theorem.

**Theorem 1.1** All positive integer solutions of the equation

$$\varphi_2(n - \varphi_2(n)) = 2^{\omega(n)}$$

are $n = 18, 19, 20, 21, 23, 24, 26, 36$.

# 2 Lemmas

In order to prove Theorem 1.1, we need to use the following lemmas.

**Lemma 2.1** Let $t_i$ $(1 \leq i \leq k)$ be distinct positive integers, then

$$\prod_{i=1}^{k}\left(2^{2^{t_i}} + 1\right) \leq 2^{\sum_{i=1}^{k} 2^{t_i}+1}.$$

**Proof** Set $n \geq max\{t_1, t_2, \cdots, t_k\}$, then

$$\prod_{i=0}^{n}\left(2^{2^i}+1\right) = \left(2^{2^0}+1\right)\left(2^{2^1}+1\right)\cdots\left(2^{2^n}+1\right)$$

$$= \left(2^{2^0}-1\right)\left(2^{2^0}+1\right)\left(2^{2^1}+1\right)\cdots\left(2^{2^n}+1\right)$$

$$= 2^{2^{n+1}} - 1$$

$$= 2^{\sum_{i=0}^{n} 2^i + 1} - 1$$

$$\leq 2^{\sum_{i=0}^{n} 2^i + 1}.$$

Hence

$$\prod_{i=1}^{k}\left(2^{2^{t_i}}+1\right) = \frac{\prod_{i=0}^{n}\left(2^{2^i}+1\right)}{\prod_{i=0, i\neq t_i}^{n}\left(2^{2^i}+1\right)} \leq \frac{2^{\sum_{i=0}^{n} 2^i+1}}{\prod_{i=0, i\neq t_i}^{n} 2^{2^i}} = 2^{\sum_{i=1}^{k} 2^{t_i}+1},$$

This completes the proof of Lemma 2.1.

**Lemma 2.2** Let $s$ be a positive integer, then the equation $\varphi(x) = 2^s$ has at most one odd solution.

**Proof** Let $x = \prod_{i=1}^{k} p_i^{\alpha_i}$ be an odd solution of $\varphi(x) = 2^s$, then

$$\prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1) = 2^s,$$

Hence, $\alpha_i = 1 (1 \leq i \leq k)$,$p_i - 1 = 2^{s_i}$,$\sum_{i=1}^{k} s_i = s$. If $p_i = 2^{s_i} + 1$ is a prime, then there is $t_i$ such that $s_i = 2^{t_i}$. According to binary representation, there are unique $t_i$ $(1 \leq i \leq k)$ such that

$$s = \sum_{i=1}^{k} s_i = \sum_{i=1}^{k} 2^{t_i}.$$

Therefore if there is an odd solution $x$ such that $\varphi(x) = 2^s$, then the expression is unique. If $2^{2^{t_i}} + 1$ is not a prime, then the equation $\varphi(x) = 2^s$ has no solution. Hence, the equation $\varphi(x) = 2^s$ has at most one odd solution.

This completes the proof of Lemma 2.2.

**Lemma 2.3** Let $s$ be a positive integer, and $x$ be a solution to equation

$$\varphi(x) = 2^s.$$

Then

$$x \leq 2^{s+2}.$$

**Proof** For $s = 1$, then $x = 3,4 \; or \; 6 \leq 2^{1+2}$, the proposition holds.

Assume that the proposition is true for $s$, and we attempt to prove the validity of the proposition for $s + 1$.

(1) If $x$ is an odd solution of $\varphi(x) = 2^{s+1}$, it can be seen from the proof of Lemma 2.2 that there are unique positive integers $t_i$ $(1 \leq i \leq k)$which distinct, such that $\sum_{i=1}^{k} 2^{t_i}$=$s$+1, and $x = \prod_{i=1}^{k}\left(2^{2^{t_i}}+1\right)$. By Lemma 2.1 we have

$$x = \prod_{i=1}^{k}\left(2^{2^{t_i}} + 1\right) \leq 2^{\sum_{i=1}^{k} 2^{t_i+1}} = 2^{s+2} \leq 2^{s+3}.$$

(2) If $x(2||x)$ is an even solution to equation $\varphi(x) = 2^{s+1}$, then $\frac{x}{2}$ is exactly the unique odd solution of $\varphi(x) = 2^{s+1}$. We can obtain $\frac{x}{2} \leq 2^{s+2}$ by case (1), therefore

$$x \leq 2^{s+3} = 2^{(s+1)+2}.$$

(3) If $x(4|x)$ is an even solution of $\varphi(x) = 2^{s+1}$, then $\frac{x}{2}$ is also an even solution of $\varphi(x) = 2^{s}$, then we know $\frac{x}{2} \leq 2^{s+2}$by assumption, then we have

$$x \leq 2^{s+3} = 2^{(s+1)+2}.$$

This completes the proof of Lemma 2.3.

**Lemma 2.4** When $\omega(n) = 2$, all positive integer solutions of $\varphi_2(n - \varphi_2(n)) = 2^{\omega(n)}$ are

$$n = 18,20,21,24,26,36.$$

**Proof** When $\omega(n) = 2$, the equation

$$\varphi_2(n - \varphi_2(n)) = 2^{\omega(n)}$$

is

$$\varphi_2(n - \varphi_2(n)) = 4.$$

By Lemma 4 of [10], we obtain $n - \varphi_2(n) = 15,16,20,24 \ or \ 30$, i.e.,

$$2n - \varphi(n) = 30,32,40,48 \ or \ 60. \qquad (1)$$

Set the standard decomposition formula of $n$ as $n = 2^{\beta}{p_1}^{\alpha_1}{p_2}^{\alpha_2} \cdots {p_k}^{\alpha_k}$, where $p_i$ is prime and $3 \leq p_1 < p_2 < \cdots < p_k$.

Since $\omega(n) = 2$, if $\beta = 0$, then $k = 2$, (1) is equivalent to

$$2{p_1}^{\alpha_1}{p_2}^{\alpha_2} - {p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}(p_1 - 1)(p_2 - 1) = 30,32,40,48 \ or \ 60.$$

We can obtain $2{p_1}^{\alpha_1}{p_2}^{\alpha_2} - {p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}(p_1 - 1)(p_2 - 1) \equiv 2 \pmod 4$, thus we only need to solve the equation $2{p_1}^{\alpha_1}{p_2}^{\alpha_2} - {p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}(p_1 - 1)(p_2 - 1) = 30$.That is ${p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}[(p_1 + 1)(p_2 + 1) - 2] = 30$, we have $\alpha_1 = \alpha_2 = 1, p_1 = 3, p_2 = 7, n = 21$.

If $\beta \geq 1$, then $k = 1$, (1) can be simplified to

$$2^{\beta+1}{p_1}^{\alpha_1} - 2^{\beta-1}{p_1}^{\alpha_1-1}(p_1 - 1) = 30,32,40,48 \ or \ 60.$$

After simplification, we get

$$2^{\beta-1}{p_1}^{\alpha_1-1}(3p_1 + 1) = 30,32,40,48 \ or \ 60.$$

If $2^{\beta-1}{p_1}^{\alpha_1-1}(3p_1 + 1) = 30$, then $3p_1 + 1 \leq 30$ and $(3p_1 + 1)|30$, we can get $p_1 = 3, \alpha_1 = 2, \beta = 1$, then

$$n = 2 \times 3^2 = 18.$$

If $2^{\beta-1}{p_1}^{\alpha_1-1}(3p_1+1)=32$, then $3p_1+1\le 32$ and $(3p_1+1)|32$, we can get $p_1=5, \alpha_1=1, \beta=2$, then

$$n=2^2\times 5=20.$$

If $2^{\beta-1}{p_1}^{\alpha_1-1}(3p_1+1)=40$, then $3p_1+1\le 40$ and $(3p_1+1)|40$, we can get $p_1=3, \alpha_1=1, \beta=3$,

$$n=2^3\times 3=24,$$

or

$$p_1=13, \alpha_1=1, \beta=1,$$

$$n=2\times 13=26.$$

If $2^{\beta-1}{p_1}^{\alpha_1-1}(3p_1+1)=48$, then $3p_1+1\le 48$ and $(3p_1+1)|48$, we know there is no solution in this case.

If $2^{\beta-1}{p_1}^{\alpha_1-1}(3p_1+1)=60$, then $3p_1+1\le 60$ and $(3p_1+1)|60$, we can get $p_1=3, \alpha_1=2, \beta=2$,

$$n=2^2\times 3^2=36.$$

In summary, when $\omega(n)=2$, the solutions of equation $\varphi_2\big(n-\varphi_2(n)\big)=2^{\omega(n)}$ are $n=18,20,21,24,26,36$.

This completes the proof of Lemma 2.4.

**Lemma 2.5** When $\omega(n)=3$, the equation $\varphi_2\big(n-\varphi_2(n)\big)=2^{\omega(n)}$ has no solution.

**Proof** When $\omega(n)=3$, equation $\varphi_2\big(n-\varphi_2(n)\big)=2^{\omega(n)}$ is

$$\varphi_2\big(n-\varphi_2(n)\big)=8.$$

By Lemma 5 of [10], we can get $n-\varphi_2(n)=17,32,34,40,48,60$, then

$$2n-\varphi(n)=34,64,68,80,96 \ or\ 120. \quad (2)$$

Set the standard decomposition formula of $n$ as $n=2^\beta {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots {p_k}^{\alpha_k}$, where $p_i$ is prime and $3\le p_1<p_2<\cdots<p_k$.

Since $\omega(n)=3$, if $\beta=0$, then $k=3$, we can get (2) to

$$2{p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}-{p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}{p_3}^{\alpha_3-1}(p_1-1)(p_2-1)(p_3-1)=34,64,68,80,96 \ or\ 120,$$

Since

$$2{p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}-{p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}{p_3}^{\alpha_3-1}(p_1-1)(p_2-1)(p_3-1)\equiv 2(mod 4),$$

thus we only need to solve the equation

$$2{p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}-{p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}{p_3}^{\alpha_3-1}(p_1-1)(p_2-1)(p_3-1)=34.$$

By calculation, there is no solution.

When $\beta\ge 1$, $k=2$, (2) is equivalent to

$$2^{\beta+1}{p_1}^{\alpha_1}{p_2}^{\alpha_2}-2^{\beta-1}{p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}(p_1-1)(p_2-1)=34,64,68,80,96 \ or\ 120.$$

Since $2^{\beta+1}{p_1}^{\alpha_1}{p_2}^{\alpha_2}-2^{\beta-1}{p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}(p_1-1)(p_2-1)\equiv 0 \pmod 4$, we only need to consider the equations

$$2^{\beta+1}{p_1}^{\alpha_1}{p_2}^{\alpha_2} - 2^{\beta-1}{p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}(p_1-1)(p_2-1) = 64,68,80,96\ or\ 120.$$

After simplification we get

$$2^{\beta-1}{p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}(3p_1p_2+p_1+p_2-1) = 64,68,80,96,120.$$

By calculation, there is no corresponding odd primes $p_1, p_2$ such that

$$(3p_1p_2+p_1+p_2-1)|64,68,80,96\ or\ 120.$$

Therefore there is no solution.

This completes the proof of Lemma 2.5.

**Lemma 2.6** When $\omega(n)=4$, equation $\varphi_2(n-\varphi_2(n)) = 2^{\omega(n)}$ has no solution.

**Proof** When $\omega(n)=4$, equation $\varphi_2(n-\varphi_2(n)) = 2^{\omega(n)}$ is

$$\varphi_2(n-\varphi_2(n)) = 16.$$

By Lemma 6 of [10], we can get $n-\varphi_2(n) = 51,64,68,80,96,102,120$, then

$$2n-\varphi(n) = 102,128,136,160,192,204\ or\ 240. \quad (3)$$

Set the standard decomposition formula of $n$ as $n = 2^{\beta}{p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, where $p_i$ is prime and $3 \le p_1 < p_2 < \cdots < p_k$.

Since $\omega(n)=4$, if $\beta=0$, then $k=4$, (3) is equivalent to

$$2\prod_{i=1}^{4}{p_i}^{\alpha_i} - \prod_{i=1}^{4}{p_i}^{\alpha_i-1}(p_i-1) = 102,128,136,160,192,204\ or\ 240.$$

However,

$$2\prod_{i=1}^{4}{p_i}^{\alpha_i} - \prod_{i=1}^{4}{p_i}^{\alpha_i-1}(p_i-1) > \prod_{i=1}^{4}{p_i}^{\alpha_i} > 3\times5\times7\times11 = 1155,$$

therefore, when $\beta = 0, k = 4$, equations (3) have no solution.

If $\beta \ge 1$, then $k=3$, (3) is equivalent to

$$2^{\beta+1}{p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3} - 2^{\beta-1}{p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}{p_3}^{\alpha_3-1}(p_1-1)(p_2-1)(p_3-1) = 102,128,136,160,192,204,240.$$

However,

$$2^{\beta+1}{p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3} - 2^{\beta-1}{p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}{p_3}^{\alpha_3-1}(p_1-1)(p_2-1)(p_3-1)$$

$$= 2^{\beta-1}{p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}{p_3}^{\alpha_3-1}[4p_1p_2p_3-(p_1-1)(p_2-1)(p_3-1)]$$

$$\ge 2^{\beta-1}\times 3{p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3} \ge 3\times3\times5\times7 = 315.$$

Hence when $\beta \ge 1$, $k=3$, (3) has no solution.

This completes the proof of Lemma 2.6.

# 3 Proof of Theorems

**Proof of Theorem1.1** When $n=1$, we can know that the left side of the original formula $\varphi_2(1-\varphi_2(1)) = \varphi_2(1-0) = \varphi_2(1) = 0$, and the right side of the original formula $2^0 = 1$. The original equation has no solution.

When $\omega(n) = 1$, the equation can be reduced to $\varphi_2(n-\varphi_2(n)) = 2$. In [9], five solutions of this formula have been obtained $n = 6,10,12,19,23$ to $\varphi_2(n-\varphi_2(n)) = 2$, of which $n = 19,23$ satisfies the condition of $\omega(n) = 1$.

When $\omega(n) \geq 5$, equation $\varphi_2(n-\varphi_2(n)) = 2^{\omega(n)}$ can be transformed to $\varphi(n-\varphi_2(n)) = 2^{\omega(n)+1}$.

By lemma 2.3, we can know $n-\varphi_2(n) \leq 2^{\omega(n)+3}$, then

$$n \leq 2n-\varphi(n) \leq 2^{\omega(n)+4}. \quad (4)$$

However, when $\omega(n) \geq 5$,

$$n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}{p_5}^{\alpha_5}\cdots{p_{\omega(n)}}^{\alpha_{\omega(n)}}$$

$$\geq 2\times 3\times 5\times 7\times 11\times\cdots\times p_{\omega(n)}$$

$$= 2310\times\cdots\times p_{\omega(n)} > 2^9\times 2^{\omega(n)-5} = 2^{\omega(n)+4}.$$

This contradicts (4), therefore when $\omega(n) \geq 5$, equation $\varphi_2(n-\varphi_2(n)) = 2^{\omega(n)}$ has no solution.

Together with Lemma 2.4 – 2.6, all positive integer solutions of equation $\varphi_2(n-\varphi_2(n)) = 2^{\omega(n)}$ are

$$n = 18,19,20,21,23,24,26,36.$$

This completes the proof of Theorem 1.1.

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## Competing Interests

Authors have declared that no competing interests exist.

## References

[1] Lv Zhihong. An equation involving the Euler function [J](Chinese). Journal of Northwest University (Natural Science Edition). 2006;(01):17-20.

[2] Lv Zhihong. Two functions of number theory and the equations [J](Chinese). Pure and Applied Mathematices. 2006;(03):303-306.

[3] Li Yijun. Properties of some kind of number theoretic function [J](Chinese). Journal of Shangqiu Teachers College. 2007;(09):20-22.

[4] Cai TX, Fu XD, Zhou X. A congruence involving the quotients of Euler and its application ( II ), Acta Arithmetic. 2007;130(3):203-214.

[5] Ding Yu. Generalized Euler function and its properties [D]. Zhejiang University; 2008.

[6] Tianxin Cai, Zhongyan Shen, Mengjun Hu. On the Parity of the Generalized Euler Function [J]. Advances in Mathematics. 2013;42(04):505-510.

[7] Shen Zhongyan, CAI Tianxin, HU Mengjun. On the Parity of the Generalized Euler Function (II), Advances in Mathematics (China). 2016;45(4):509-519.

[8] Zhang Sibao, Adim • Yoldax. Two Equation of Generalized Euler Function $\varphi_2(n)$[J] (Chinese). Journal of Northeast Normal University (Natural Science Edition). 2019;51(02):7-12.

[9] Zhang Sibao. Solutions of Equation$\varphi_6(n) = 2^{\omega(n)}$on Generalized Euler function[J] (Chinese). Journal of Southwest China Normal University (Natural Science Edition). 2018;43(02):36-41.

[10] Yu Honglin, Shen Zhongyan. The Equations Related with Generalized Euler Function [J](Chinese). Journal of Zhejiang International Studies University. 2012;(03):91-97.

[11] Jin Mingyan, Shen Zhongyan. Solvability of Equations $\varphi_2(n) = 2^{\Omega(n)}$ and $\varphi_2\left(\varphi_2(n)\right) = 2^{\Omega(n)}$ [J] (Chinese). Journal of Zhejiang International Studies University. 2013;(04):47-52.

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