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Bayesian Prediction for Exponentiated Generalized Xgamma Distribution Based on Dual Generalized Order Statistics with Application to Poverty and COVID-19 Mortality Rates

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

Statistical prediction is one of the most important problems in life testing; it has been applied in medicine, engineering, business and other areas as well. In this paper, the exponentiated generalized xgamma distribution is introduced as an application on the exponentiated generalized general class of distributions. Bayesian point and interval prediction of exponentiated generalized xgamma distribution based on dual generalized order statistics are considered. All results are specialized to lower records. The results are verified using simulation study as well as applications to real data sets to demonstrate the flexibility and potential applications of the distribution.

Keywords: *Exponentiated generalized distributions; bayesian prediction; dual generalized order statistics; exponentiated generalized xgamma distribution.*

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1 Introduction

Prediction for observations in a future sample has received much attention in recent years. Bayesian prediction based on two samples of an unknown observable is to provide some estimators for future observations based on the current available sample, known as an informative sample. Many researchers studied the prediction and its applications; for example, Aitchison and Dunsmore [1], AL-Hussaini and Jaheen [2], Geisser [3], Kim et al. [4] and Vidovi´c [5].

Cordeiro et al. [6] proposed a class of distributions as an extension of the exponentiated type distribution which has greater flexibility of its tails and can be widely applied in many areas of biology and engineering. Given a non-negative continuous random variable X, the *cumulative distribution function* (cdf) for the *exponentiated generalized* (EG) of distributions is defined by

$$F(x;\alpha,\beta)=[1-(1-G(x))^{\alpha}]^{\beta}; \qquad \alpha,\beta>0, \tag{1}$$

where α and β are additional shape parameters, the corresponding *probability density function* (pdf) for (1) is given by

$$f(x;\alpha,\beta)=\alpha\beta g(x)\big(1-G(x)\big)^{\alpha-1}[1-(1-G(x))^{\alpha}]^{\beta-1}; \qquad \alpha,\beta>0, \tag{2}$$

where $g(x)$ is the first derivative of $G(x)$ with respect to x.

By setting $\alpha=1$ in (1) the exponentiated type distributions are derived by Gupta et al. [7]; further the exponentiated exponential and exponentiated gamma distributions can be obtained if $G(x)$ is the exponential or gamma cdfs, respectively. For $\beta=1$ in (1) and if $G(x)$ is the Gumbel or Fre$\acute{c}$het cdfs, then, one can get the exponentiated Gumbel and exponentiated Fre$\acute{c}$het distributions, respectively, as defined by Nadarajah and Kotz [8].

Many authors focused on the EG distributions and its applications; for example, Oguntunde et al. [9], Yousof et al. [10], De Andrade et al. [11], Mustafa et al. [12], Sindi et al. [13]], Nasiru et al. [14], Abbas et al. [15] and Oluyede et al. [16].

Abd AL-Fattah et al. [17] introduced the *EG general class* (EGGC) of distributions, Bayesian estimation for the unknown parameters, *reliability function* (rf), *hazard rate function* (hrf) of the EGGC of distributions based on *dual generalized order statistics* (dgos) are introduced, the cdf and pdf are given, respectively, by

$$F(x;\alpha,\beta)=\big[1-exp[-\alpha\lambda(x)]\big]^{\beta}; \quad x>0,\ \alpha,\beta>0, \tag{3}$$

and

$$f(x;\alpha,\beta)=\alpha\beta\grave{\lambda}(x)exp[-\alpha\lambda(x)]\big[1-exp[-\alpha\lambda(x)]\big]^{\beta-1}; \quad x>0,\ \alpha,\beta>0, \tag{4}$$

where $\lambda(x)\equiv\lambda\big(x,\ \underline{\theta}\big)$ is a non-negative continuous function of x such that $\lambda\big(x,\ \underline{\theta}\big)\to 0\ as$

$x\to 0^{+}$ and $\lambda\big(x,\ \underline{\theta}\big)\to\infty\ as\ x\to\infty,\ \underline{\theta}=(\theta_1,\theta_2,\dots,\theta_s)$ are known parameters and $\grave{\lambda}(x)$ is the first derivative of $\lambda(x)$ with respect to x.

Burkschat et al. [18] studied the dgos that enables a common approach to descending ordered random variables as reversed ordered order statistics, lower record models and lower Pfeifer records.

Let $X_{(1,n,m,k)},X_{(2,n,m,k)},\dots\ X_{(n,n,m,k)}$ be n dgos from an absolutely cdf with corresponding pdf. Then, the joint pdf has the form

$$f_{X_{(1,n,m,k)},X_{(2,n,m,k)},\dots X_{(n,n,m,k)}}\left(x_{(1)},\dots,x_{(n)}\right) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[\prod_{i=1}^{n-1}\left(F\left(x_{(i)}\right)\right)^m f\left(x_{(i)}\right)\right]\left(F\left(x_{(n)}\right)\right)^{k-1} f\left(x_{(n)}\right); \tag{5}$$

where

$$F^{-1}(1) \geq x_{(1)} \dots \geq x_{(n)} \geq F^{-1}(0), n \in N,\ k \geq 1,\ m_1,\dots,m_{n-1} = m,$$
$m \in \mathbb{R}$ be the parameters such that $\gamma_r = k + (n-r)(m+1) \geq 1$, $\quad for\ all\ 1 \leq r \leq n.$

The marginal pdf of the r^{th} dgos $X\ (r,n,m,k), 1 \leq r \leq n$ is given by; [See Khan and Khan [19]

$$f^{(r,n,m,k)}\left(x_{(r)}\right) = \frac{\zeta_{r-1}}{(r-1)!}\left[F\left(x_{(r)}\right)\right]^{\gamma_r-1} f\left(x_{(r)}\right) g_m^{r-1}\left(F\left(x_{(r)}\right)\right), \tag{6}$$

where

$$\zeta_{r-1} = \prod_{j=1}^{r}\gamma_j, \qquad g_m(x) = h_m(x) - h_m(1), \qquad x \in [0,1),$$
$$h_m(x) = \begin{cases} -\frac{1}{m+1}x^{m+1}, & m \neq -1 \\ -\ln x, & m = -1 \end{cases} \quad \text{and} \quad h_m(1) = \begin{cases} -\frac{1}{m+1}, & m \neq -1 \\ 0, & m = -1 \end{cases}. \tag{7}$$

This paper is organized as follows: In Section 2, Bayesian prediction (point and interval) for a future observation of the EGGC of distributions based on dgos is obtained. The results of Bayesian prediction of EGGC are specialized to the *exponentiated generalized xgamma* (EG-Xg) distribution and studied based on dgos in Section 3. A numerical study is presented in Section 4 to illustrate the application procedures of the various results developed in this paper.

2 Bayesian Prediction for Exponentiated Generalized General Class of Distributions

This section developed Bayesian prediction for future observations from the EGGC of distributions based on dgos under *squared error* (SE) and *linear exponential* (LINEX) loss functions. Suppose that $X_{(1,n,m,k)}, X_{(2,n,m,k)}, \dots, X_{(n,n,m,k)}$ are n dgos from EGGC distribution, the likelihood function can be derived by substituting (3) and (4) in (5) as follows:

$$L\left(\alpha,\beta|\underline{x}\right) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\alpha^n\beta^n\prod_{i=1}^{n}\grave{\lambda}(x_i)\ exp\left[-\alpha\sum_{i=1}^{n}\lambda(x_i)\right]\prod_{i=1}^{n-1}\left[1-exp[-\alpha\lambda(x_i)]\right]^{\beta(m+1)-1} \times \left[1-exp[-\alpha\lambda(x_n)]\right]^{\beta k-1}. \tag{8}$$

Let α and β are independent random variables with gamma prior distribution with the pdf as follows:

$$\pi(\alpha,\beta) \propto \alpha^{c_1-1}\beta^{c_2-1}e^{-[d_1\alpha+d_2\beta]}, \tag{9}$$

where c_1, c_2, d_1 and d_2 are known hyper parameters.

The joint posterior of α and β can be derived by using (8) and (9) as follows:

$$\pi\left(\alpha,\beta|\underline{x}\right) \propto \alpha^{n+c_1-1}\beta^{n+c_2-1}e^{-\alpha\left[d_1+\sum_{i=1}^{n}\lambda(x_i)\right]}e^{-\beta\left[d_2-(m+1)\sum_{i=1}^{n-1}\ln u_i-k\ln u_n\right]}\prod_{i=1}^{n}(u_i)^{-1}, \tag{10}$$

where

$$u_i = \left[1-exp[-\alpha\lambda(x_i)]\right] \text{ and } u_n = \left[1-exp[-\alpha\lambda(x_n)]\right],$$

hence, the joint posterior distribution of α and β is given by;

$$\pi\big(\alpha,\beta|\underline{x}\big)=\frac{\alpha^{n+c_1-1}\beta^{n+c_2-1}e^{-\alpha[d_1+\sum_{i=1}^{n}\lambda(x_i)]}e^{-\beta[d_2-(m+1)\sum_{i=1}^{n-1}\ln u_i-k\ln u_n]}\prod_{i=1}^{n}(u_i)^{-1}}{\varphi\,\Gamma(n+c_2)}\ , \tag{11}$$

where

$$\varphi=\int_0^{\infty}\frac{\alpha^{n+c_1-1}e^{-\alpha[d_1+\sum_{i=1}^{n}\lambda(x_i)]}\prod_{i=1}^{n}(u_i)^{-1}}{[d_2-(m+1)\sum_{i=1}^{n-1}\ln u_i-k\ln u_n]^{n+c_2}}d\alpha. \tag{12}$$

Let $X(1,n,m,k),\dots,X(r,n,m,k)$ be a dgos of size n with the pdf $f(x;\theta)$ and suppose $Y\big(1,n_y,m_y,k_y\big),\dots,Y\big(r_y,n_y,m_y,k_y\big),k_y>0,m_y\in\mathbb{R}$ is a second unobserved dgos of size n_y. Using (3), (4), (6) and (7), the pdf of the dgos $Y_{(s)}$ can be obtained and just replacing $x_{(r)}$ by $y_{(s)}$ as follows:

$$f\big(y_{(s)}|\alpha,\beta\big)=\frac{\zeta_{s-1}}{(s-1)!}\Big[1-exp\big[-\alpha\lambda\big(y_{(s)}\big)\big]\Big]^{\beta\gamma_s-1}\alpha\beta\acute{\lambda}\big(y_{(s)}\big)exp\big[-\alpha\lambda\big(y_{(s)}\big)\big]g_{m_y}^{s-1}\Big(F\big(y_{(s)}\big)\Big), \tag{13}$$

where

$$\zeta_{s-1}=\prod_{j=1}^{s}\gamma_j\ ,\ g_M(y_s)=h_M(y_s)-h_M(1),\ \gamma_r^*=k_y+\big(n_y-r\big)\big(m_y+1\big)\,,$$

$$g_{m_y}^{s-1}\Big(F\big(y_{(s)}\big)\Big)=\begin{cases}\frac{1}{(m_y+1)^{s-1}}\Big[1-\big(1-exp\big[-\alpha\lambda(y_{(s)})\big]\big)^{\beta(m_y+1)}\Big]^{s-1}, & m_y\neq-1\,,\\ \Big[-\ln\big(1-exp\big[-\alpha\lambda(y_{(s)})\big]\big)^{\beta}\Big]^{s-1}, & m_y=-1\,.\end{cases} \tag{14}$$

For the future sample of size n_y, let $Y_{(s)}$ denotes the s^{th} ordered life time, $1\le s\le n_y$, The pdf of the dgos $y_{(s)}$; $y_{(s)}>0$, from EGGC distribution is obtained by substituting (14) in (13) .

Case one: for $m_y\neq-1$

$$f_1(y_{(s)}|\alpha,\beta)=\frac{\alpha\beta\zeta_{s-1}}{(m_y+1)^{s-1}(s-1)!}\acute{\lambda}\big(y_{(s)}\big)exp\big[-\alpha\lambda(y_{(s)})\big]\Big[1-exp\big[-\alpha\lambda(y_{(s)})\big]\Big]^{\beta\gamma_s-1}$$
$$\times\Big[1-\big(1-exp\big[-\alpha\lambda(y_{(s)})\big]\big)^{\beta(m_y+1)}\Big]^{s-1}.$$

Using the binomial expansion, one obtains

$$f_1(y_{(s)}|\alpha,\beta)=\frac{\alpha\beta\zeta_{s-1}}{(m_y+1)^{s-1}(s-1)!}\sum_{i_1=0}^{s-1}(-1)^{i_1}\binom{s-1}{i_1}\big(1+y_{(s)}\big)^{-(\alpha+1)}\big[\big(1-exp\big[-\alpha\lambda(y_{(s)})\big]\big)\big]^{\beta(\gamma_s+i_1(m_y+1))-1},$$

let

$\xi=\frac{\zeta_{s-1}}{(m_y+1)^{s-1}(s-1)!}$, $\eta_{i_1}=(-1)^{i_1}\binom{s-1}{i_1}$, and $\varpi_{i_1}=\big(\gamma_s+i_1(m_y+1)\big)$.

Then

$$f_1\big(y_{(s)}|\alpha,\beta\big)=\xi\alpha\beta\acute{\lambda}\big(y_{(s)}\big)exp\big[-\alpha\lambda\big(y_{(s)}\big)\big]\sum_{i_1=0}^{s-1}\eta_{i_1}\big[\big(1-exp\big[-\alpha\lambda\big(y_{(s)}\big)\big]\big)\big]^{\beta\varpi_{i_1}-1};\ y_{(s)>0}\,,\alpha,\beta>0, \tag{15}$$

which can be rewritten as

$$f_1(y_{(s)}|\alpha,\beta)=\frac{\xi\alpha\beta\lambda(y_{(s)})exp[-\alpha\lambda(y_{(s)})]}{1-exp[-\alpha\lambda(y_{(s)})]}e^{\beta\sum_{i_1=0}^{s-1}\varpi_{i_1}\ln[\,\eta_{i_1}(1-exp[-\alpha\lambda(y_{(s)})])]}\ .$$

Case two: for $m_y=-1$

$$f_{1^*}(y_{(s)}|\alpha,\beta) = \frac{k_y^s \alpha\beta^s \acute{\lambda}(y_{(s)}) exp[-\alpha\lambda(y_{(s)})]}{(s-1)!}\left[\left(1-exp[-\alpha\lambda(y_{(s)})]\right)\right]^{\beta k_y - 1} \times \left[-\ln\left(1-exp[-\alpha\lambda(y_{(s)})]\right)\right]^{s-1}; \quad y_{(s)} > 0,\ \alpha,\beta > 0. \quad (16)$$

The *Bayesian predictive density* (BPD) for the future observation $Y_{(s)}$, $1 \le s \le n_y$, based on dgos can be derived as follows:

$$f\left(y_{(s)}|\underline{x}\right) = \int_0^\infty \int_0^\infty f\left(y_{(s)}|\alpha,\beta\right)\pi\left(\alpha,\beta|\underline{x}\right)d\alpha d\beta\ , \quad (17)$$

where $\pi\left(\alpha,\beta|\underline{x}\right)$ is the posterior density function of α,β and $f\left(y_{(s)}|\alpha,\beta\right)$ is the pdf of $y_{(s)}$.

Case one: for $m_y \neq -1$

The BPD of $y_{(s)}$ given $\underline{x}$ is obtained by substituting (11) and (15) in (17) as given below

$$f_1\left(y_{(s)}|\underline{x}\right) = \int_0^\infty \int_0^\infty \frac{\xi\alpha\beta\acute{\lambda}(y_{(s)})exp[-\alpha\lambda(y_{(s)})]}{1-exp[-\alpha\lambda(y_{(s)})]} e^{\beta\sum_{i_1=0}^{s-1}\varpi_{i_1}\ln[\eta_{i_1}(1-exp[-\alpha\lambda(y_{(s)})])]} \times \left[\frac{\alpha^{n+c_1-1}\beta^{n+c_2-1}e^{-\alpha[d_1+\sum_{i=1}^n\lambda(x_i)]}e^{-\beta[d_2-(m+1)\sum_{i=1}^{n-1}\ln u_i - k\ln u_n]}\prod_{i=1}^n(u_i)^{-1}}{\varphi\Gamma(n+c_2)}\right]d\beta d\alpha,$$

hence

$$f_1\left(y_{(s)}|\underline{x}\right) = \int_0^\infty \frac{\xi\alpha^{n+c_1}\acute{\lambda}(y_{(s)})exp[-\alpha\lambda(y_{(s)})]e^{-\alpha[d_1+\sum_{i=1}^n\lambda(x_i)]}}{\varphi\left(1-exp[-\alpha\lambda(y_{(s)})]\right)\prod_{i=1}^n u_i\,\Gamma(n+c_2)} \times \left[\int_0^\infty \beta^{n+c_2}e^{-\beta[(d_2-(m+1)\sum_{i=1}^{n-1}\ln u_i - k\ln u_n)-\sum_{i_1=0}^{s-1}\varpi_{i_1}\ln[\eta_{i_1}(1-exp[-\alpha\lambda(y_{(s)})])]]}d\beta\right]d\alpha$$
$$= \int_0^\infty \frac{\xi(n+c_2)\alpha^{n+c_1}\acute{\lambda}(y_{(s)})exp[-\alpha\lambda(y_{(s)})]e^{-\alpha[d_1+\sum_{i=1}^n\lambda(x_i)]}}{\varphi\left(1-exp[-\alpha\lambda(y_{(s)})]\right)\prod_{i=1}^n u_i\ \tau_1^{(n+c_2+1)}}d\alpha\ , \quad (18)$$

where

$$\tau_1 = (d_2-(m+1)\sum_{i=1}^{n-1}\ln u_i - k\ln u_n) - \sum_{i_1=0}^{s-1}\varpi_{i_1}\ln[\eta_{i_1}\left(1-exp[-\alpha\lambda(y_{(s)})]\right)].$$

Case two: for $m_y = -1$

Substituting (11) and (16) in (17), the BPD of $y_{(s)}$ given $\underline{x}$ is

$$f_{1^*}\left(y_{(s)}|\underline{x}\right) = \int_0^\infty\int_0^\infty \frac{k_y^s\alpha\beta^s\acute{\lambda}(y_{(s)})exp[-\alpha\lambda(y_{(s)})]}{(s-1)!}\left[\left(1-exp[-\alpha\lambda(y_{(s)})]\right)\right]^{\beta k_y-1}\left[-\ln\left(1-exp[-\alpha\lambda(y_{(s)})]\right)\right]^{s-1} \times \left[\frac{\alpha^{n+c_1-1}\beta^{n+c_2-1}e^{-\alpha[d_1+\sum_{i=1}^n\lambda(x_i)]}e^{-\beta[d_2-(m+1)\sum_{i=1}^{n-1}\ln u_i-k\ln u_n]}\prod_{i=1}^n(u_i)^{-1}}{\varphi\Gamma(n+c_2)}\right]d\beta d\alpha$$
$$= \int_0^\infty \frac{k_y^s\alpha^{n+c_1}\acute{\lambda}(y_{(s)})exp[-\alpha\lambda(y_{(s)})]\left[-\ln\left(1-exp[-\alpha\lambda(y_{(s)})]\right)\right]^{s-1}e^{-\alpha[d_1+\sum_{i=1}^n\lambda(x_i)]}}{\varphi(s-1)!\left(1-exp[-\alpha\lambda(y_{(s)})]\right)\prod_{i=1}^n u_i\Gamma(n+c_2)} \times \left[\int_0^\infty \beta^{n+c_2+s-1}e^{-\beta[(d_2-(m+1)\sum_{i=1}^{n-1}\ln u_i-k\ln u_n)-k\,ln(1-exp[-\alpha\lambda(y_{(s)})])]}d\beta\right]d\alpha$$
$$= \int_0^\infty \frac{k_y^s\alpha^{n+c_1}\acute{\lambda}(y_{(s)})exp[-\alpha\lambda(y_{(s)})]\left[-\ln\left(1-exp[-\alpha\lambda(y_{(s)})]\right)\right]^{s-1}e^{-\alpha[d_1+\sum_{i=1}^n\lambda(x_i)]}\Gamma(n+c_2+s)}{\varphi(s-1)!\left(1-exp[-\alpha\lambda(y_{(s)})]\right)\prod_{i=1}^n u_i\Gamma(n+c_2)\ \tau_2^{(n+c_2+s)}}d\alpha, \quad (19)$$

where

$\tau_2 = (d_2 - (m+1)\sum_{i=1}^{n-1} \ln u_i - k \ln u_n) - k\, ln\left(1 - exp\left[-\alpha\lambda(y_{(s)})\right]\right).$

2.1 Point prediction

The *Bayesian Predictor* (BP) of the future dgos $Y_{(s)}$ can be derived under SE and LINEX loss function as given below

Case one: for $m_y \neq -1$

The BP of the future dgos $Y_{(s)}$ can be obtained under SE loss function using (18), then

$$\tilde{y}_{(s)1SE} = E\left(y_{(s)}|\underline{x}\right) = \int_0^\infty y_{(s)}\, f_1\left(y_{(s)}|\underline{x}\right)\, dy_{(s)} ,$$

$$= \int_0^\infty \int_0^\infty \frac{\xi y_{(s)}(n+c_2)\alpha^{n+c_1}\acute{\lambda}(y_{(s)}) exp[-\alpha\lambda(y_{(s)})] e^{-\alpha[d_1+\sum_{i=1}^{n}\lambda(x_i)]}}{\varphi(1-exp[-\alpha\lambda(y_{(s)})])\prod_{i=1}^{n} u_i\, \tau_1^{(n+c_2+1)}}\, d\alpha\, dy_{(s)}. \tag{20}$$

The BP of the future dgos $Y_{(s)}$ under LINEX loss function using (18) is given by

$$\tilde{y}_{(s)1LNX} = \frac{-1}{\vartheta} \ln E\left(e^{-\vartheta y_{(s)}}|\underline{x}\right),$$

where $E\left(e^{-\vartheta y_{(s)}}|\underline{x}\right) = \int_0^\infty e^{-\vartheta y_{(s)}}\, f_1\left(y_{(s)}|\underline{x}\right)\, dy_{(s)}$

$$= \int_0^\infty \int_0^\infty \frac{e^{-\vartheta y_{(s)}}\xi(n+c_2)\alpha^{n+c_1}\acute{\lambda}\left(y_{(s)}\right) exp\left[-\alpha\lambda\left(y_{(s)}\right)\right] e^{-\alpha[d_1+\sum_{i=1}^{n}\lambda(x_i)]}}{\varphi\left(1-exp\left[-\alpha\lambda\left(y_{(s)}\right)\right]\right)\prod_{i=1}^{n} u_i\;\; \tau_1^{(n+c_2+1)}}\, d\alpha\, dy_{(s)}. \tag{21}$$

Case two: for $m_y = -1$

From (19), the BP of dgos $Y_{(s)}$ under SE and LINEX loss functions are, respectively, given by

$$\tilde{y}_{(s)1^*SE} = E\left(y_{(s)}|\underline{x}\right) = \int_0^\infty y_{(s)}\, f_{1^*}\left(y_{(s)}|\underline{x}\right)\, dy_{(s)}$$

$$= \int_0^\infty \int_0^\infty \frac{y_{(s)} k_y^s \alpha^{n+c_1}\acute{\lambda}(y_{(s)}) exp[-\alpha\lambda(y_{(s)})]\left[-\ln(1-exp[-\alpha\lambda(y_{(s)})])\right]^{s-1} e^{-\alpha[d_1+\sum_{i=1}^{n}\lambda(x_i)]}\Gamma(n+c_2+s)}{\varphi(s-1)!(1-exp[-\alpha\lambda(y_{(s)})])\prod_{i=1}^{n} u_i \Gamma(n+c_2)\, \tau_2^{(n+c_2+s)}}\, d\alpha\, dy_{(s)} , \tag{22}$$

and

$$\tilde{y}_{(s)1^*LNX} = \frac{-1}{\vartheta} \ln E\left(e^{-\vartheta y_{(s)}}|\underline{x}\right) , \tag{23}$$

where

$$E\left(e^{-\vartheta y_{(s)}}|\underline{x}\right) = \int_0^\infty \int_0^\infty \frac{e^{-\vartheta y_{(s)}} k_y^s \alpha^{n+c_1}\acute{\lambda}\left(y_{(s)}\right) exp\left[-\alpha\lambda\left(y_{(s)}\right)\right]\left[-\ln\left(1-exp\left[-\alpha\lambda\left(y_{(s)}\right)\right]\right)\right]^{s-1} e^{-\alpha[d_1+\sum_{i=1}^{n}\lambda(x_i)]}\Gamma(n+c_2+s)}{\varphi(s-1)!\left(1-exp\left[-\alpha\lambda\left(y_{(s)}\right)\right]\right)\prod_{i=1}^{n} u_i \Gamma(n+c_2)\;\tau_2^{(n+c_2+s)}}\, d\alpha\, dy_{(s)}.$$

2.2 Credible interval prediction

The 100 (1- ω) % credible interval prediction for $Y_{(s)}$ is $\left(L_{(s)}(\underline{x}), U_{(s)}(\underline{x})\right)$,

where

$$P\left[L_{(s)}(\underline{x}) < Y_{(s)} < U_{(s)}(\underline{x})|\underline{x}\right] = \int_{L_{(s)}(\underline{x})}^{U_{(s)}(\underline{x})} f\left(y_{(s)}|\underline{x}\right) dy_{(s)} = 1 - \omega.$$

The *Bayesian predictive bounds* (BPB) of the future dgos $Y_{(s)}$ can be obtained using (18) and (19) as follows:

Case one: for $m_y \neq -1$

$$P\left[Y_{(s)} > L_{(s)1}(\underline{x})|\underline{x}\right] = \int_{L_{(s)1}(\underline{x})}^{\infty}\int_0^{\infty}\frac{\xi(n+c_2)\alpha^{n+c_1}\dot{\lambda}(y_{(s)})exp[-\alpha\lambda(y_{(s)})]e^{-\alpha[d_1+\sum_{i=1}^{n}\lambda(x_i)]}}{\varphi(1-exp[-\alpha\lambda(y_{(s)})])\prod_{i=1}^{n}u_i\,\tau_1^{(n+c_2+1)}}d\alpha\, dy_{(s)} = 1-\frac{\omega}{2}, \quad (24)$$

and

$$P\left[Y_{(s)} > U_{(s)1}(\underline{x})|\underline{x}\right] = \int_{U_{(s)1}(\underline{x})}^{\infty}\int_0^{\infty}\frac{\xi(n+c_2)\alpha^{n+c_1}\dot{\lambda}(y_{(s)})exp[-\alpha\lambda(y_{(s)})]e^{-\alpha[d_1+\sum_{i=1}^{n}\lambda(x_i)]}}{\varphi(1-exp[-\alpha\lambda(y_{(s)})])\prod_{i=1}^{n}u_i\,\tau_1^{(n+c_2+1)}}d\alpha\, dy_{(s)} = \frac{\omega}{2}. \quad (25)$$

Case two: for $m_y = -1$

$$P\left[Y_{(s)} > L_{(s)1^*}(\underline{x})|\underline{x}\right] = \int_{L_{(s)1^*}(\underline{x})}^{\infty}\int_0^{\infty}\frac{k_y^s\alpha^{n+c_1}\dot{\lambda}(y_{(s)})exp[-\alpha\lambda(y_{(s)})]\left[-\ln(1-exp[-\alpha\lambda(y_{(s)})])\right]^{s-1}e^{-\alpha[d_1+\sum_{i=1}^{n}\lambda(x_i)]}\Gamma(n+c_2+s)}{\varphi(s-1)!(1-exp[-\alpha\lambda(y_{(s)})])\prod_{i=1}^{n}u_i\Gamma(n+c_2)\,\tau_2^{(n+c_2+s)}}d\alpha\, dy_{(s)}=1-\frac{\omega}{2}, \quad (26)$$

and $P\left[Y_{(s)} > U_{(s)1^*}(\underline{x})|\underline{x}\right] =$

$$\int_{U_{(s)1^*}(\underline{x})}^{\infty}\int_0^{\infty}\frac{k_y^s\alpha^{n+c_1}\dot{\lambda}(y_{(s)})exp[-\alpha\lambda(y_{(s)})]\left[-\ln(1-exp[-\alpha\lambda(y_{(s)})])\right]^{s-1}e^{-\alpha[d_1+\sum_{i=1}^{n}\lambda(x_i)]}\Gamma(n+c_2+s)}{\varphi(s-1)!(1-exp[-\alpha\lambda(y_{(s)})])\prod_{i=1}^{n}u_i\Gamma(n+c_2)\,\tau_2^{(n+c_2+s)}}d\alpha\, dy_{(s)} = \frac{\omega}{2}. \quad (27)$$

3 Application to Exponentiated Generalized Xgamma Distribution

Sen et al. [20] introduced the xgamma distribution which is generated as a special finite mixture of exponential (θ) and gamma $(3,\theta)$ distributions with mixing proportion $\pi_1 = \frac{\theta}{1+\theta}$ and $\pi_2 = 1-\pi_1 = \frac{1}{1+\theta}$. The cdf and the pdf of the xgamma distribution are, respectively,

$$G(x;\theta) = 1-\frac{1+\theta+\theta x+\frac{\theta^2x^2}{2}}{1+\theta}e^{-\theta x}, \qquad x>0,\ \theta>0, \quad (28)$$

and

$$g(x;\theta) = \frac{\theta^2}{1+\theta}\left(1+\frac{\theta x^2}{2}\right)e^{-\theta x}, \qquad x>0,\ \theta>0. \quad (29)$$

Yadav et al. [21] studied the generalized xgamma distribution by adding power shape parameter to the cdf and some statistical properties of this generalized xgamma are discussed. They used many methods of estimation to estimate the rf and hrf of the generalized xgamma distribution.

Abd AL-Fattah et al. [17] introduced the EG-Xg distribution. Bayesian estimation for the unknown parameters, rf and hrf of EG-Xg distribution based on dgos were obtained and a real data set was used to insure the theoretical results. The cdf of the EG-Xg distribution is given by

$$F(x;\alpha,\beta,\theta) = \left[1-\frac{(1+\theta+\theta x+\frac{\theta^2x^2}{2})^{\alpha}}{(1+\theta)^{\alpha}}e^{-\theta\alpha x}\right]^{\beta}, \qquad x>0,\ \alpha,\beta,\theta>0, \quad (30)$$

and by substituting (28) in (1), one can get the pdf of the EG-Xg distribution as

$$f(x;\alpha,\beta,\theta) = \frac{\alpha\beta\theta^2e^{-\alpha\theta x}}{(1+\theta)^{\alpha}}\left(1+\frac{\theta x^2}{2}\right)\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)^{\alpha-1}\left[1-\frac{\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)^{\alpha}}{(1+\theta)^{\alpha}}e^{-\theta\alpha x}\right]^{\beta-1},$$

$$x>0, \qquad \alpha,\beta,\theta>0\,. \quad (31)$$

The rf and the hrf are given, respectively, by

$$R(x;\alpha,\beta,\theta) = 1 - \left[1 - \frac{\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)^{\alpha}}{(1+\theta)^{\alpha}} e^{-\theta\alpha x}\right]^{\beta}, \qquad x > 0,\ \alpha,\beta,\theta > 0\,. \tag{32}$$

$$h(x;\alpha,\beta,\theta) == \frac{\frac{\alpha\beta\theta^2 e^{-\alpha\theta x}}{(1+\theta)^{\alpha}}\left(1+\frac{\theta x^2}{2}\right)\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)^{\alpha-1}\left[1-\frac{\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)^{\alpha}}{(1+\theta)^{\alpha}} e^{-\theta\alpha x}\right]^{\beta-1}}{1-\left[1-\frac{\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)^{\alpha}}{(1+\theta)^{\alpha}} e^{-\theta\alpha x}\right]^{\beta-1}},$$
$$x > 0,\ \alpha,\beta,\theta > 0\,. \tag{33}$$

Suppose that $X_{(1,n,m,k)}, X_{(2,n,m,k)},, \ldots, X_{(n,n,m,k)}$ are n dgos from EG-Xg distribution, the likelihood function can be derived by substituting (30) and (31) in (5) as follows:

$$L(\alpha,\beta,\theta|\underline{x}) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\frac{\alpha^n\beta^n\theta^{2n}e^{-\theta\alpha\sum_{i=1}^{n}x_i}}{(1+\theta)^{n\alpha}}\prod_{i=1}^{n}\left(1+\frac{\theta x_i^2}{2}\right){\delta_i}^{\alpha-1}$$
$$\times\prod_{i=1}^{n-1}\left[1-\left(\frac{\delta_i}{(1+\theta)}\right)^{\alpha} e^{-\theta\alpha x_i}\right]^{\beta(m+1)-1}\ \left[1-\left(\frac{\delta_n}{(1+\theta)}\right)^{\alpha} e^{-\theta\alpha x_n}\right]^{\beta k-1}, \tag{34}$$

where

$$\delta_i = \left(1+\theta+\theta x_i+\frac{\theta^2 {x_i}^2}{2}\right) \text{ and } \delta_n = \left(1+\theta+\theta x_n+\frac{\theta^2 {x_n}^2}{2}\right). \tag{35}$$

Let $\theta_1 = \alpha, \theta_2 = \beta$ and $\theta_3 = \theta$ are independent random variables with gamma prior distributions with the pdf as given below

$$\pi(\theta_l) = \frac{d_l^{c_l}}{\Gamma(c_l)}{\theta_l}^{c_l-1}e^{-d_l\theta_l}, \qquad \theta_l, d_l, c_l > 0,\ l = 1,2,3, \text{ where } c_l, d_l \text{ are the known hyper parameters.}$$

A joint prior density function of $\underline{\theta} = (\theta_1,\theta_2,\theta_3)'$ is

$$\pi(\underline{\theta}) \propto \prod_{l=1}^{3}{\theta_l}^{c_l-1}e^{-d_l\theta_l} = \alpha^{c_1-1}\beta^{c_2-1}\theta^{c_3-1}e^{-[d_1\alpha+d_2\beta+d_3\theta]}. \tag{36}$$

The likelihood function given by (34) can be rewritten as

$$L(\alpha,\beta,\theta|\underline{x}) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\frac{\alpha^n\beta^n\theta^{2n}e^{-\theta\alpha\sum_{i=1}^{n}x_i}}{(1+\theta)^{n\alpha}}\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}\prod_{i=1}^{n-1}{u_i^*}^{[\beta(m+1)]}\ \ {u_n^*}^{[\beta k]}, \tag{37}$$

where

$$u_i^* = \left[1-\left(\frac{\delta_i}{(1+\theta)}\right)^{\alpha}e^{-\theta\alpha x_i}\right],\ u_n^* = \left[1-\left(\frac{\delta_n}{(1+\theta)}\right)^{\alpha}e^{-\theta\alpha x_n}\right] \text{ and } \rho_i = \left(1+\frac{\theta x_i^2}{2}\right){\delta_i}^{\alpha-1}. \tag{38}$$

The joint posterior density can be derived by using (36) and (37) as follows:

$$\pi(\underline{\theta}|\underline{\mathrm{x}}) = T(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1})e^{-\alpha\left(d_1+\theta\sum_{i=1}^{n}x_i+n\ln(1+\theta)\right)}$$
$$\times e^{-\beta\left[d_2-(m+1)\sum_{i=1}^{n-1}\ln(u_i^*)-k\ln(u_n^*)\right]}\ e^{-\theta d_3}\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1},$$

where T is the normalizing constant and is defined by

$$T^{-1} = \Gamma(c_2+n)\int_0^\infty\int_0^\infty \alpha^{c_1+n-1}\theta^{c_3+2n-1}\, e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}\, e^{-\theta d_3}$$
$$\times\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}\left[\frac{1}{[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^*)-kln(u_n^*)]^{c_2+n}}\right]d\alpha d\theta.$$

Let

$$\varphi^* = \int_0^\infty\int_0^\infty \left(\alpha^{c_1+n-1}\theta^{c_3+2n-1}\, e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}\, e^{-\theta d_3}\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}\right)$$
$$\times\left[\frac{1}{[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^*)-kln(u_n^*)]^{c_2+n}}\right]d\alpha d\theta.$$

Hence, the joint posterior distribution of α,β and θ given $\underline{x}$ can be written as follows;

$$\pi_{EGxg}(\alpha,\beta,\theta|\underline{x}) = \frac{(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1})}{\varphi^*\Gamma(c_2+n)}$$
$$\times e^{-\alpha(d_1+\theta\sum_{i=1}^{n}x_i+nln(1+\theta))}e^{-\theta d_3}\prod_{i=1}^{n}\rho_i\,(u_i^*)^{-1}\, e^{-\beta[d_2-(m+1)\sum_{i=1}^{n-1}ln(u_i^*)-kln(u_n^*)]}. \quad (39)$$

Using (30), (31), (6) and (7), the pdf of the s^{th} dgos $Y_{(s)}$ can be obtained and just replacing $x_{(r)}$ by $y_{(s)}$, one obtains

$$f_{EGxg}(y_{(s)}|\alpha,\beta,\theta) = \frac{\zeta_{s-1}\alpha\beta\theta^2e^{-\alpha\theta y_{(s)}}}{(s-1)!(1+\theta)^\alpha}\left(1+\frac{\theta(y_{(s)})^2}{2}\right)\left(\delta_{y_{(s)}}\right)^{\alpha-1}\left[u^*_{y_{(s)}}\right]^{\beta\gamma_s-1} g_{m_y}^{s-1}\left(F(y_{(s)})\right), \quad (40)$$

where

$$\zeta_{s-1} = \prod_{j=1}^{s}\gamma_j, \qquad g_{m_y}(y_s) = h_{m_y}(y_s) - h_{m_y}(1),$$

$$g_{m_y}^{s-1}\left(F(y_{(s)})\right) = \begin{cases} \frac{1}{(m_y+1)^{s-1}}\left[1-\left(u^*_{y_{(s)}}\right)^{\beta(m_y+1)}\right]^{s-1}, & m_y \neq -1, \\ \left[-\ln\left(u^*_{y_{(s)}}\right)^{\beta}\right]^{s-1}, & m_y = -1, \end{cases} \quad (41)$$

where

$$u^*_{y_{(s)}} = \left[1-\frac{\left(\delta_{y_{(s)}}\right)^\alpha}{(1+\theta)^\alpha}e^{-\theta\alpha y_{(s)}}\right] \text{and } \delta_{y_{(s)}} = \left(1+\theta+\theta y_{(s)}+\frac{\theta^2(y_{(s)})^2}{2}\right). \quad (42)$$

For the future sample of size n_y, let $Y_{(s)}$ denotes the s^{th} ordered life time, $1 \le s \le n_y$. The pdf of the dgos $Y_{(s)}$ from EG-Xg distribution is obtained by substituting (41) in (40).

Case one: for $m_y \neq -1$

$$f_{2EGxg}(y_{(s)}|\alpha,\beta,\theta) = \left(\frac{\zeta_{s-1}}{(s-1)!}\right)\frac{\alpha\beta\theta^2e^{-\alpha\theta y_{(s)}}}{(1+\theta)^\alpha(m_y+1)^{s-1}}\left(1+\frac{\theta(y_{(s)})^2}{2}\right)\left(\delta_{y_{(s)}}\right)^{\alpha-1}\left[u^*_{y_{(s)}}\right]^{\beta\gamma_s-1}\left[1-\left(u^*_{y_{(s)}}\right)^{\beta(m_y+1)}\right]^{s-1}.$$
$$= \xi\,\frac{\alpha\beta\theta^2e^{-\alpha\theta y_{(s)}}}{(1+\theta)^\alpha}\left(1+\frac{\theta(y_{(s)})^2}{2}\right)\left(\delta_{y_{(s)}}\right)^{\alpha-1}\sum_{i=0}^{s-1}\eta_{i_1}\left[u^*_{y_{(s)}}\right]^{\beta\varpi_{i_1}-1},\ y_{(s)}>0,\alpha,\beta,\theta>0, \quad (43)$$

where

$$\xi = \frac{\zeta_{s-1}}{(m_y+1)^{s-1}(s-1)!},\ \eta_{i_1} = (-1)^{i_1}\binom{s-1}{i_1},\ \text{and}\ \varpi_{i_1} = \left(\gamma_s + i_1(m_y+1)\right).$$

Case two: for $m_y = -1$

$$f_{2^*EGxg}\left(y_{(s)}|\alpha,\beta,\theta\right) = \frac{k_y^s \alpha\beta^s\theta^2 e^{-\alpha\theta y_{(s)}}}{(s-1)!\,(1+\theta)^\alpha}\left(1+\frac{\theta\left(y_{(s)}\right)^2}{2}\right)\left(\delta_{y_{(s)}}\right)^{\alpha-1}\left[u^*_{y_{(s)}}\right]^{\beta k_y - 1}\left[-\ln\left(u^*_{y_{(s)}}\right)\right]^{s-1},$$
$$y_{(s)} > 0,\ \alpha,\beta,\theta > 0\,. \qquad (44)$$

In this subsection, Bayesian two-sample prediction based on dgos for the future observation of $Y_{(s)}$, $1 \le s \le n_y$ is obtained. The BPD function can be derived as follows:

$$f_{EGxg}\left(y_{(s)}|\underline{x}\right) = \int_0^\infty\int_0^\infty\int_0^\infty f_{EGxg}\left(y_{(s)}|\alpha,\beta,\theta\right)\pi_{EGxg}\left(\alpha,\beta,\theta|\underline{x}\right)d\alpha d\beta d\theta\,, \qquad (45)$$

where $\pi_{EGxg}\left(\alpha,\beta,\theta|\underline{x}\right)$ is the posterior density function of α,β and θ and $f_{EGxg}\left(y_{(s)}|\alpha,\beta,\theta\right)$ is the pdf of $y_{(s)}$.

3.1 Point prediction

Case one: for $m_y \neq -1$

The BPD of $y_{(s)}$ given $\underline{x}$ is obtained by substituting (39) and (43) in (45) as given below

$$f_2\left(y_{(s)}|\underline{x}\right)$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty\left[\xi\frac{\alpha\beta\theta^2 e^{-\alpha\theta y_{(s)}}}{(1+\theta)^\alpha}\left(1+\frac{\theta\left(y_{(s)}\right)^2}{2}\right)\left(\delta_{y_{(s)}}\right)^{\alpha-1}\sum_{i=0}^{s-1}\eta_{i_1}\left[u^*_{y_{(s)}}\right]^{\beta\varpi_{i_1}-1}\right.$$
$$\times\left[\frac{\left(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1}\right)}{\varphi^*\Gamma(c_2+n)}\right]e^{-\alpha\left(d_1+\theta\sum_{i=1}^n x_i+nln(1+\theta)\right)}e^{-\theta d_3}$$
$$\left.\times\prod_{i=1}^n\rho_i\left(u_i^*\right)^{-1}e^{-\beta\left[d_2-(m+1)\sum_{i=1}^{n-1}ln\left(u_i^*\right)-kln\left(u_n^*\right)\right]}\right]d\beta d\alpha d\theta$$
$$= \int_0^\infty\int_0^\infty\frac{\xi\alpha\theta^{c_3+2n+1}(c_2+n)e^{-\alpha\theta y_{(s)}}\left(1+\frac{\theta\left(y_{(s)}\right)^2}{2}\right)}{\varphi^*(1+\theta)^\alpha\left[d_2-(m+1)\sum_{i=1}^{n-1}ln\left(u_i^*\right)-kln\left(u_n^*\right)+\sum_{i_1=0}^{s-1}\varpi_{i_1}\ln u^*_{y_{(s)}}\eta_{i_1}\right]^{c_2+n+1}}$$
$$\times\left(\delta_{y_{(s)}}\right)^{\alpha-1}\left[u^*_{y_{(s)}}\right]^{-1}e^{-\alpha\left(d_1+\theta\sum_{i=1}^n x_i+nln(1+\theta)\right)}e^{-\theta d_3}\prod_{i=1}^n\rho_i\left(u_i^*\right)^{-1}\left(\alpha^{c_1+n-1}\theta^{c_3+2n-1}\right)d\alpha d\theta\,. \qquad (46)$$

Case two: for $m_y = -1$

Substituting (39) and (44) in (45), the BPD of $y_{(s)}$ given $\underline{x}$ is

$$f_{2^*}\left(y_{(s)}|\underline{x}\right) = \int_0^\infty\int_0^\infty\int_0^\infty\frac{k_y^s\alpha\beta^s\theta^2 e^{-\alpha\theta y_{(s)}}}{(s-1)!\,(1+\theta)^\alpha}\left(1+\frac{\theta\left(y_{(s)}\right)^2}{2}\right)\left(\delta_{y_{(s)}}\right)^{\alpha-1}\left[-\ln\left(u^*_{y_{(s)}}\right)\right]^{s-1}$$
$$\times\frac{\left(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1}\right)}{\varphi^*\Gamma(c_2+n)}e^{-\alpha\left(d_1+\theta\sum_{i=1}^n x_i+nln(1+\theta)\right)}e^{-\theta d_3}$$
$$\times\prod_{i=1}^n\rho_i\left(u_i^*\right)^{-1}e^{-\beta\left[d_2-(m+1)\sum_{i=1}^{n-1}ln\left(u_i^*\right)-kln\left(u_n^*\right)+k_y lnu^*_{y_{(s)}}\right]}d\beta d\alpha d\theta$$

$$= \int_0^\infty\int_0^\infty\frac{k_y^s\alpha\beta^s\theta^2 e^{-\alpha\theta y_{(s)}}}{(s-1)!\,(1+\theta)^\alpha}\left(1+\frac{\theta\left(y_{(s)}\right)^2}{2}\right)\left(\delta_{y_{(s)}}\right)^{\alpha-1}\left[-\ln\left(u^*_{y_{(s)}}\right)\right]^{s-1}$$
$$\times\frac{\left(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1}\right)e^{-\alpha\left(d_1+\theta\sum_{i=1}^n x_i+nln(1+\theta)\right)}e^{-\theta d_3}\prod_{i=1}^n\rho_i\left(u_i^*\right)^{-1}\Gamma(c_2+n+s)}{\varphi^*\left(d_2-(m+1)\sum_{i=1}^{n-1}ln\left(u_i^*\right)-kln\left(u_n^*\right)+k_y lnu^*_{y_{(s)}}\right)^{c_2+n+s}\Gamma(c_2+n)}d\alpha d\theta. \qquad (47)$$

The BP of the future dgos $Y_{(s)}$ can be derived under SE and LINEX loss functions as given below

Case one: for $m_y \neq -1$

The BP of the future dgos $Y_{(s)}$ under SE loss function using (46) is

$$\tilde{y}_{(s)2SE} = E\left(y_{(s)}|\underline{x}\right) = \int_0^\infty y_{(s)}\, f_2\left(y_{(s)}|\underline{x}\right)\, dy_{(s)}$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty \frac{\xi\alpha y_{(s)}\theta^{c_3+2n+1}(c_2+n)e^{-\alpha\theta y_{(s)}}\left(1+\frac{\theta\left(y_{(s)}\right)^2}{2}\right)\left(\delta_{y_{(s)}}\right)^{\alpha-1}\left[u^*_{y_{(s)}}\right]^{-1}}{\varphi^*(1+\theta)^\alpha\left[d_2-(m+1)\sum_{i=1}^{n-1} ln(u_i^*)-kln(u_n^*)+\sum_{i_1=0}^{s-1}\varpi_{i_1}\ln u^*_{y_{(s)}}\eta_{i_1}\right]^{c_2+n+1}}$$
$$\times\, e^{-\alpha\left(d_1+\theta\sum_{i=1}^n x_i+nln(1+\theta)\right)}e^{-\theta d_3}\prod_{i=1}^n \rho_i\,(u_i^*)^{-1}\,(\alpha^{c_1+n-1}\theta^{c_3+2n-1})\;d\alpha d\theta dy_{(s)}. \qquad (48)$$

The BP of the future dgos $Y_{(s)}$ can be obtained under LINEX loss function using (46) as follows:

$$\tilde{y}_{(s)2LNX} = \frac{-1}{\vartheta}\ln E\left(e^{-\vartheta y_{(s)}}|\underline{x}\right)$$
$$= \frac{-1}{\vartheta} ln\left[\int_0^\infty\int_0^\infty\int_0^\infty e^{-\vartheta y_{(s)}}\frac{\xi\alpha\theta^{c_3+2n+1}(c_2+n)e^{-\alpha\theta y_{(s)}}\left(1+\frac{\theta\left(y_{(s)}\right)^2}{2}\right)\left(\delta_{y_{(s)}}\right)^{\alpha-1}\left[u^*_{y_{(s)}}\right]^{-1}}{\varphi^*(1+\theta)^\alpha\left[d_2-(m+1)\sum_{i=1}^{n-1} ln(u_i^*)-kln(u_n^*)+\sum_{i_1=0}^{s-1}\varpi_{i_1}\ln u^*_{y_{(s)}}\eta_{i_1}\right]^{c_2+n+1}}\right.$$
$$\left.\times\, e^{-\alpha\left(d_1+\theta\sum_{i=1}^n x_i+nln(1+\theta)\right)}e^{-\theta d_3}\prod_{i=1}^n \rho_i\,(u_i^*)^{-1}\,(\alpha^{c_1+n-1}\theta^{c_3+2n-1})d\alpha d\theta dy_{(s)}\right]. \qquad (49)$$

Case two: for $m_y = -1$

The BP of dgos $Y_{(s)}$ under SE loss function using (47) is

$$\tilde{y}_{(s)2^*SE} = E\left(y_{(s)}|\underline{x}\right) = \int_0^\infty y_{(s)}\, f_{2^*}\left(y_{(s)}|\underline{x}\right)\, dy_{(s)}$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty \frac{y_{(s)}k_y^s\alpha\beta^s\theta^2 e^{-\alpha\theta y_{(s)}}}{(s-1)!\,(1+\theta)^\alpha}\left(1+\frac{\theta\left(y_{(s)}\right)^2}{2}\right)\left(\delta_{y_{(s)}}\right)^{\alpha-1}\left[-\ln\left(u^*_{y_{(s)}}\right)\right]^{s-1}$$
$$\times\frac{(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1})\; e^{-\alpha\left(d_1+\theta\sum_{i=1}^n x_i+nln(1+\theta)\right)}e^{-\theta d_3}\prod_{i=1}^n\rho_i\,(u_i^*)^{-1}\Gamma(c_2+n+s)}{\varphi^*\left(d_2-(m+1)\sum_{i=1}^{n-1} ln(u_i^*)-kln(u_n^*)+k_y lnu^*_{y_{(s)}}\right)^{c_2+n+s}\Gamma(c_2+n)}d\alpha d\theta dy_{(s)}. \quad (50)$$

The BP of the future dgos Y_s can be obtained under LINEX loss function using (47),

$$\tilde{y}_{(s)2^*LNX} = \frac{-1}{\vartheta}\ln E\left(e^{-\vartheta y_{(s)}}|\underline{x}\right)$$
$$= \frac{-1}{\vartheta} ln\left[\int_0^\infty\int_0^\infty\int_0^\infty \frac{e^{-\vartheta y_{(s)}}k_y^s\alpha\beta^s\theta^2 e^{-\alpha\theta y_{(s)}}}{(s-1)!\,(1+\theta)^\alpha}\left(1+\frac{\theta\left(y_{(s)}\right)^2}{2}\right)\left(\delta_{y_{(s)}}\right)^{\alpha-1}\left[-\ln\left(u^*_{y_{(s)}}\right)\right]^{s-1}\right.$$
$$\left.\times\frac{(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1})\; e^{-\alpha\left(d_1+\theta\sum_{i=1}^n x_i+nln(1+\theta)\right)}e^{-\theta d_3}\prod_{i=1}^n\rho_i\,(u_i^*)^{-1}\Gamma(c_2+n+s)}{\varphi^*\left(d_2-(m+1)\sum_{i=1}^{n-1} ln(u_i^*)-kln(u_n^*)+k_y lnu^*_{y_{(s)}}\right)^{c_2+n+s}\Gamma(c_2+n)}\right]d\alpha d\theta dy_{(s)}. \quad (51)$$

3.2 Credible interval prediction

The BPB of $Y_{(s)}$ can be obtained using (46) and (47) as given below

Case one: for $m_y \neq -1$

$$P\left[Y_{(s)} > L_{(s)2}(\underline{x}) \middle| \underline{x}\right]$$
$$= \int_{L_{(s)2}(\underline{x})}^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{\xi \alpha \theta^{c_3+2n+1}(c_2+n) e^{-\alpha\theta y_{(s)}} \left(1 + \frac{\theta\left(y_{(s)}\right)^2}{2}\right) \left(\delta_{y_{(s)}}\right)^{\alpha-1} \left[u^*_{y_{(s)}}\right]^{-1}}{\varphi^*(1+\theta)^{\alpha} \left[d_2 - (m+1)\sum_{i=1}^{n-1} ln(u_i^*) - kln(u_n^*) + \sum_{i_1=0}^{s-1} \varpi_{i_1} \ln u^*_{y_{(s)}} \eta_{i_1}\right]^{c_2+n+1}}$$
$$\times e^{-\alpha\left(d_1 + \theta\sum_{i=1}^{n} x_i + nln(1+\theta)\right)} e^{-\theta d_3} \prod_{i=1}^{n} \rho_i \, (u_i^*)^{-1} \left(\alpha^{c_1+n-1}\theta^{c_3+2n-1}\right) d\alpha d\theta \, dy_{(s)}$$
$$= 1 - \frac{\omega}{2}, \tag{52}$$

and

$$P\left[Y_{(s)} > U_{(s)2}(\underline{x}) \middle| \underline{x}\right]$$
$$= \int_{U_{(s)2}(\underline{x})}^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{\xi \alpha \theta^{c_3+2n+1}(c_2+n) e^{-\alpha\theta y_{(s)}} \left(1 + \frac{\theta\left(y_{(s)}\right)^2}{2}\right) \left(\delta_{y_{(s)}}\right)^{\alpha-1} \left[u^*_{y_{(s)}}\right]^{-1}}{\varphi^*(1+\theta)^{\alpha} \left[d_2 - (m+1)\sum_{i=1}^{n-1} ln(u_i^*) - kln(u_n^*) + \sum_{i_1=0}^{s-1} \varpi_{i_1} \ln u^*_{y_{(s)}} \eta_{i_1}\right]^{c_2+n+1}}$$
$$\times e^{-\alpha\left(d_1 + \theta\sum_{i=1}^{n} x_i + nln(1+\theta)\right)} e^{-\theta d_3} \prod_{i=1}^{n} \rho_i \, (u_i^*)^{-1} \left(\alpha^{c_1+n-1}\theta^{c_3+2n-1}\right) d\alpha d\theta \, dy_{(s)} = \frac{\omega}{2}. \tag{53}$$

Case two: for $m_y = -1$

$$P\left[Y_{(s)} > L_{(s)2^*}(\underline{x}) \middle| \underline{x}\right]$$
$$= \int_{L_{(s)2^*}(\underline{x})}^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{k_y^s \alpha \beta^s \theta^2 e^{-\alpha\theta y_{(s)}}}{(s-1)!\,(1+\theta)^{\alpha}} \left(1 + \frac{\theta\left(y_{(s)}\right)^2}{2}\right) \left(\delta_{y_{(s)}}\right)^{\alpha-1} \left[-\ln\left(u^*_{y_{(s)}}\right)\right]^{s-1}$$
$$\times \frac{\left(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1}\right) e^{-\alpha\left(d_1+\theta\sum_{i=1}^{n} x_i + nln(1+\theta)\right)} e^{-\theta d_3} \prod_{i=1}^{n} \rho_i \, (u_i^*)^{-1} \Gamma(c_2+n+s)}{\varphi^* \left(d_2 - (m+1)\sum_{i=1}^{n-1} ln(u_i^*) - kln(u_n^*) + k_y lnu^*_{y_{(s)}}\right)^{c_2+n+s} \Gamma(c_2+n)} d\alpha d\theta dy_{(s)}$$
$$= 1 - \frac{\omega}{2}, \tag{54}$$

and

$$P\left[Y_{(s)} > U_{(s)2^*}(\underline{x}) \middle| \underline{x}\right]$$
$$= \int_{U_{(s)2^*}(\underline{x})}^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{k_y^s \alpha \beta^s \theta^2 e^{-\alpha\theta y_{(s)}}}{(s-1)!\,(1+\theta)^{\alpha}} \left(1 + \frac{\theta\left(y_{(s)}\right)^2}{2}\right) \left(\delta_{y_{(s)}}\right)^{\alpha-1} \left[-\ln\left(u^*_{y_{(s)}}\right)\right]^{s-1}$$
$$\times \frac{\left(\alpha^{c_1+n-1}\beta^{c_2+n-1}\theta^{c_3+2n-1}\right) e^{-\alpha\left(d_1+\theta\sum_{i=1}^{n} x_i + nln(1+\theta)\right)} e^{-\theta d_3} \prod_{i=1}^{n} \rho_i \, (u_i^*)^{-1} \Gamma(c_2+n+s)}{\varphi^* \left(d_2 - (m+1)\sum_{i=1}^{n-1} ln(u_i^*) - kln(u_n^*) + k_y lnu^*_{y_{(s)}}\right)^{c_2+n+s} \Gamma(c_2+n)} d\alpha d\theta dy_{(s)}$$
$$= \frac{\omega}{2}. \tag{55}$$

4 Numerical Results

This section aims to illustrate the theoretical results of Bayesian prediction (point and interval) for a future observation of EG-Xg distribution based on lower record values through a simulation study and real data sets.

4.1 Simulation study

The lower record values can be obtained as special case from dgos by setting $m = -1,\ k = 1,\ m_y = -1$ and $k_y = 1$. The predictors for the future of the lower record values $Y_{(s)}$ are computed through Monte Carlo simulation study according to the following steps:

a. To generate random number from EG-Xg with shape parameters α, β and scale parameter θ, the following steps may be used

- Specify the values of α, β, θ and n.
- Generate U_i from uniform (0, 1) distribution $(i = 1,2,3, \dots, n)$.
- Generate V_i from gamma (1,θ) distribution$(i = 1,2,3, \dots, n)$.
- Generate W_i from gamma (3,θ) distribution $(i = 1,2,3, \dots, n)$.
- If $U_i \leq \frac{\theta}{1+\theta}$, set $X_i = V_i$,otherwise set $X_i = W_i$.

b. For each sample size n, consider the first observation is the first lower record value x_1 denote it by R_1 and the second observation x_2 denote it by R_2 ; which is smaller than the maximum $(x_1 > x_2)$ record and if $x_1 \leq x_2$ ignore it and repeat until you get sample of *record values* (Rv) records.
c. Determine the value of $s,\ 1 \leq s \leq n_y$, which is the index of the future unobserved lower record value from the second sample.
d. Using (50), (51), (54) and (55), the BP for the future lower records is calculated under SE and LINEX loss functions, respectively.
e. Table 1 displays the point and 95% interval two sample predictors for the future lower record values $Y_{(s)}$ from EG-Xg distribution, where $\mathrm{Rv} = 7,\ \alpha = 0.2,\ \beta = 0.9$ and $\theta = 0.3$.

4.2 Applications

In this subsection, the application to real data set is provided to illustrate the importance of the EG-Xg distribution based on lower records. The point and 95% interval two sample predictors for the future lower record values $Y_{(s)}$ from the poverty headcount ratio data are given in Table 2. The point and 95% interval predictors for the future lower record values $Y_{(s)}$ from the COVID-19 data are given in Table 3. To check the validity of the fitted model, Kolmogorov-Smirnov goodness of fit test is performed for each data set and the p values in each case indicates that the model fits the data very well. Fig. 2 presents the fitted pdf, PP-Plot and Q-Q plot of the EG-Xg distribution for the poverty headcount ratio data. Fig. 3: displays the fitted pdf, PP-Plot and Q-Q plot of the EG-Xg distribution for the first real data of COVID-19. Fig. 4 shows the fitted pdf, PP-Plot and Q-Q plot of the EG-Xg distribution for the second real data of COVID-19. The plots indicate that the EG-Xg distribution provides better fits to these data.

4.2.1 Application of Economic data

The application is the poverty headcount ratio at $1.90 a day (2011 purchasing power parity) (% of population) from https://data.worldbank.org/topic/poverty.

a) The world data (2000-2017) is: 27.7, 26.9, 25.7, 24.7, 22.9, 20.9, 20.3, 19.1, 18.4, 17.6, 16, 13.8, 12.9, 11.3, 10.7, 9.7, and 9.2.

From the original data, one can observe that the lower record values are: 27.7, 26.9, 25.7, 24.7, 22.9, 20.9, 20.3, 19.1, 18.4, 17.6, 16, 13.8, 12.9, 11.3, 10.7, 9.7, and 9.2.

b) The lower middle income data (2000-2017) is: 37.6, 36.6, 35, 33, 31.8, 30.1, 29, 28, 25.1, 21.4, 19.9, 18.5, and 16.9.

From the original data, one can notice that the lower record values are: 37.6, 36.6, 35, 33, 31.8, 30.1, 29, 28, 25.1, 21.4, 19.9, 18.5, and 16.9.

c) The Middle East and North Africa data is: 3.7, 3.4, 3.4, 3.4, 3.4, 3.2, 3, 2.9, 2.8, 2.5, 2.1, 2.3, 2.3, 2.3, 2.7, 3.8, 5.1, and 6.3.

From the original data, one can observe that the lower record values are: 3.7, 3.4, 3.2, 3, 2.9, 2.8, 2.5, and 2.1.

d) The Egypt data (2000-2017) is: 5.2, 4.7, 2.2, 1.5, 1.6, and 3.8.

From the original data, one can observe that the lower record values are: 5.2, 4.7, 2.2, and 1.5.

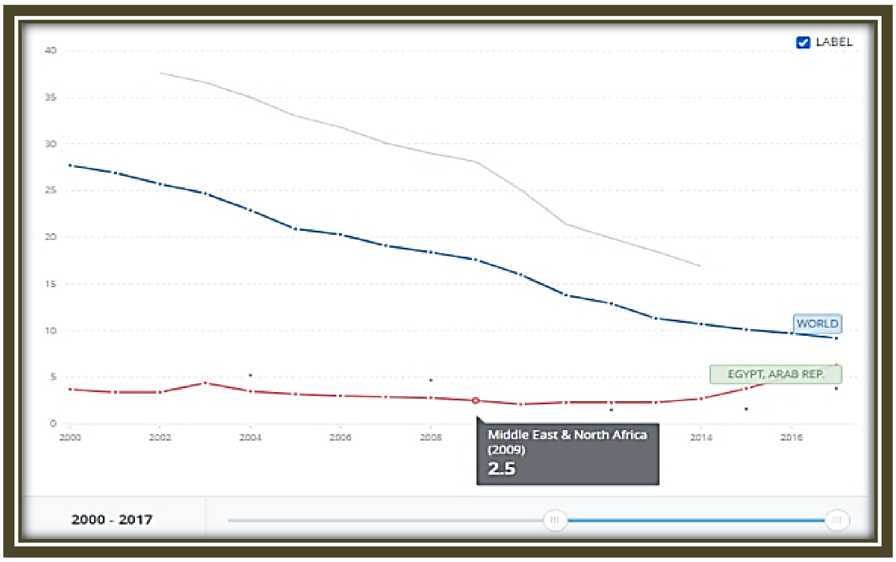
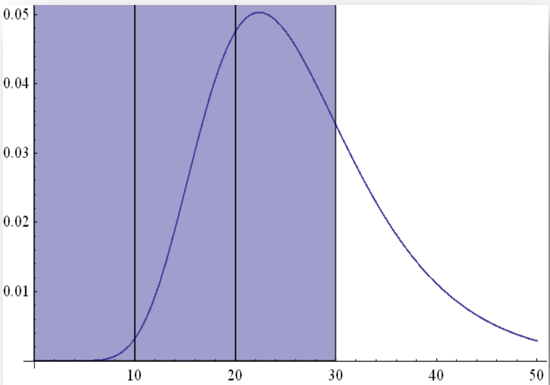
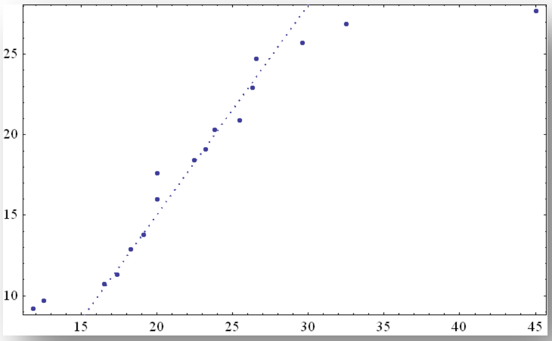


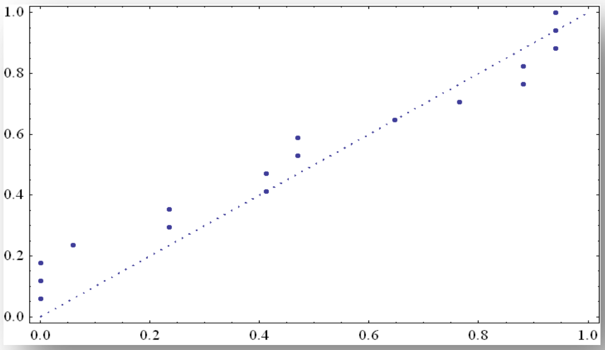
Fig. 1. Poverty headcount ratio at $1.90 a day (2011 purchasing power parity) (% of population)



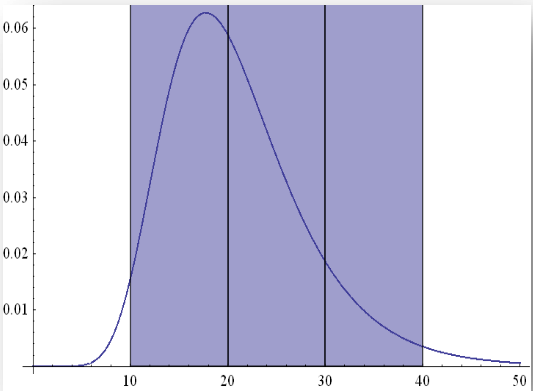
Histogram of world data



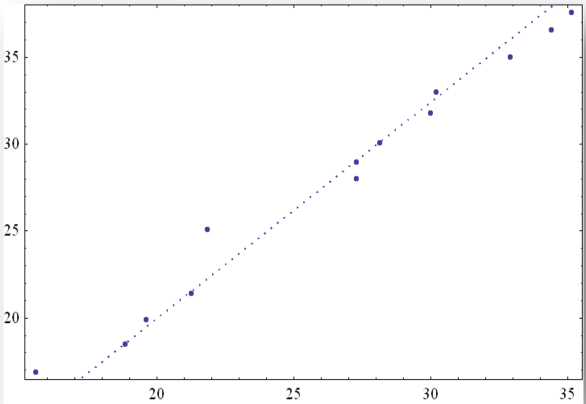
Q-Q Plot of world data



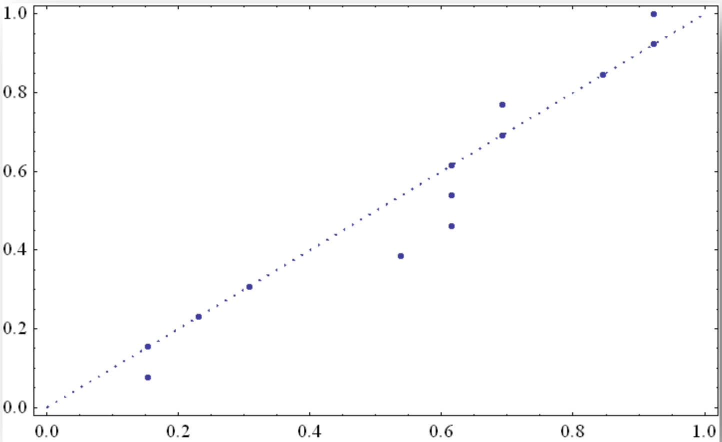
PP- Plot of world data



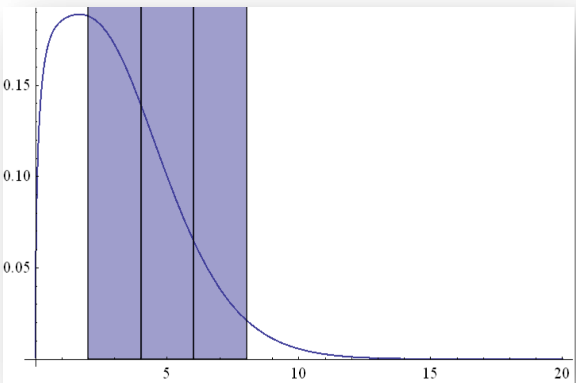
Histogram of lower middle income data



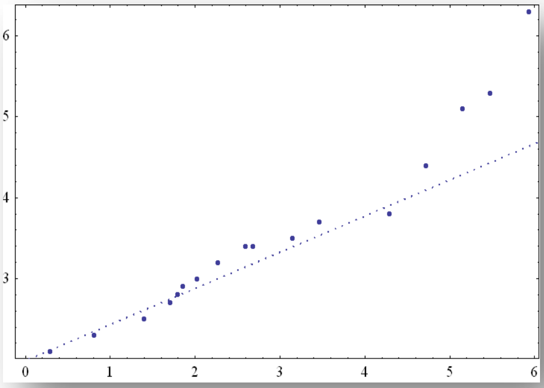
Q-Q Plot of lower middle income data



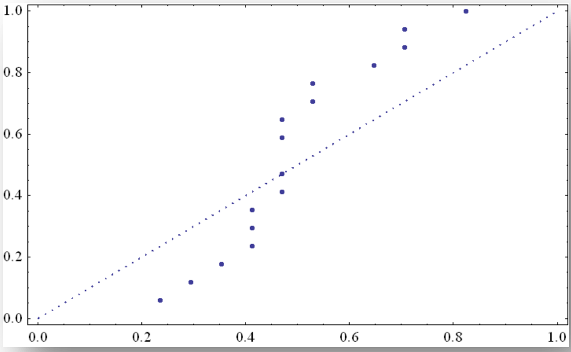
PP- Plot of lower middle income data



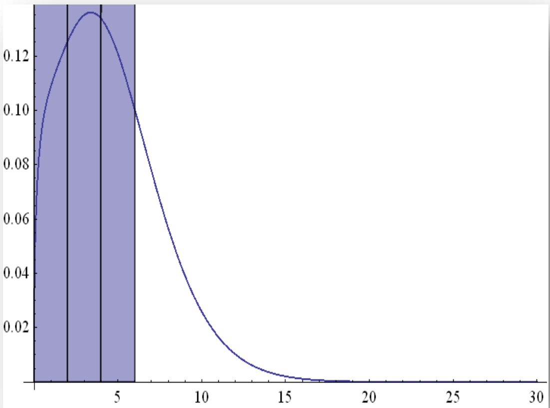
Histogram of North Africa data



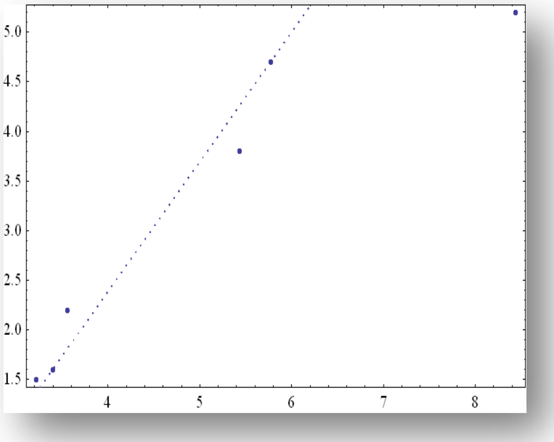
Q-Q Plot of North Africa data



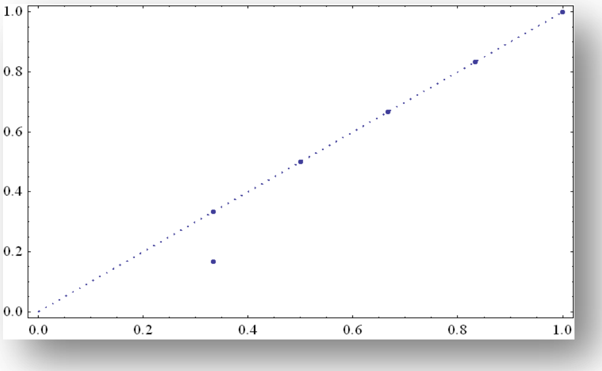
PP- Plot of North Africa data



Histogram of Egypt data



Q-Q Plot of Egypt data



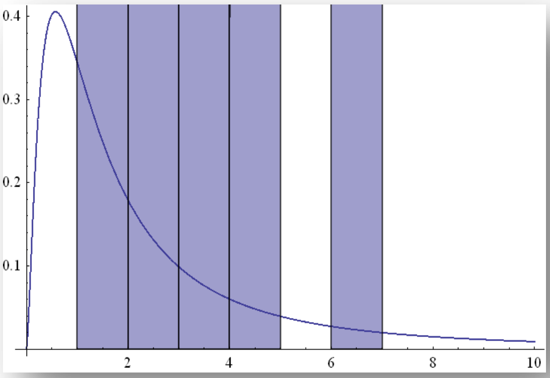
PP- Plot of Egypt data

Fig. 2. The fitted pdf, PP-Plot and Q-Q plot of the EG-Xg distribution for the poverty headcount ratio data

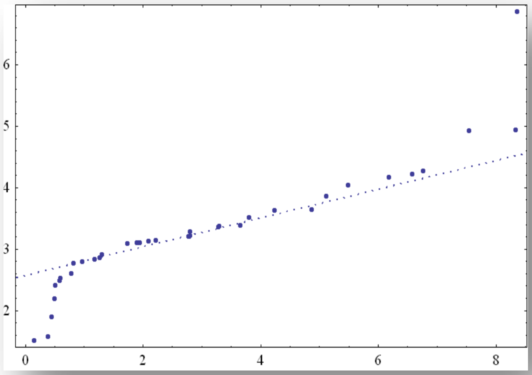
4.2.2 Application of COVID-19 data

The first data of this application represents a COVID-19 data belong to Canada of 36 days, from 10 April to 15 May 2020 see the link [https://covid19.who.int/]. These data formed of drought mortality rate, used by Almetwally et al. [22]. The data are: 3.1091, 3.3825, 3.1444, 3.2135, 2.4946, 3.5146, 4.9274, 3.3769, 6.8686, 3.0914, 4.9378, 3.1091, 3.2823, 3.8594, 4.0480, 4.1685, 3.6426, 3.2110, 2.8636, 3.2218, 2.9078, 3.6346, 2.7957, 4.2781, 4.2202, 1.5157, 2.6029, 3.3592, 2.8349, 3.1348, 2.5261, 1.5806, 2.7704, 2.1901, 2.4141 and 1.9048.

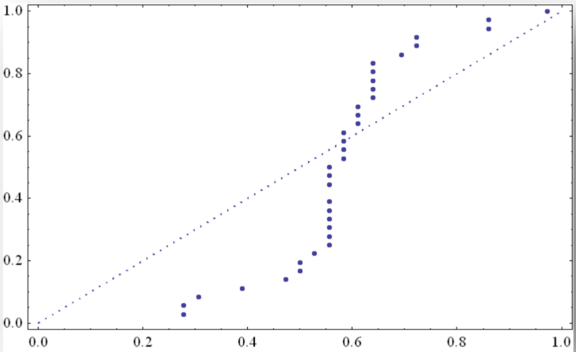
From the original data, one can observe that the lower record values are: 3.1091, 2.4946, 1.5157, p-value = 0.1246



Histogram of Canada data



Q-Q Plot of Canada data

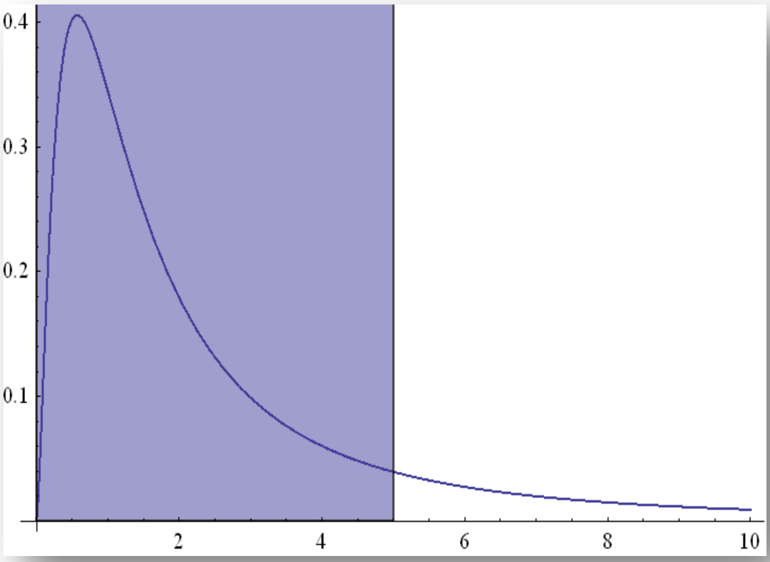


PP- Plot of Canada data

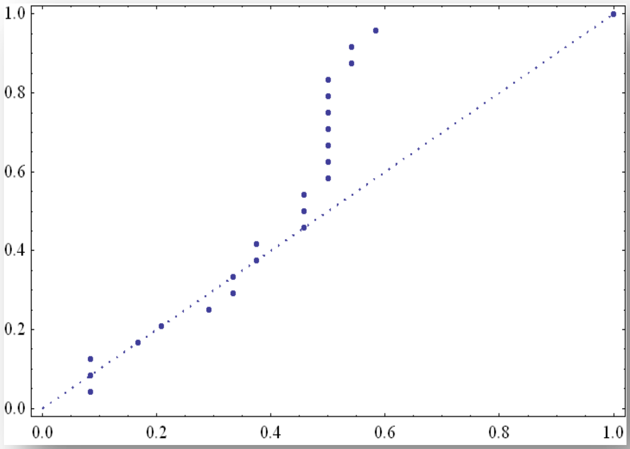
Fig. 3. The fitted pdf, PP-Plot and Q-Q plot of the EG-Xg distribution for the first real data of COVID-19

The second data represents a COVID-19 data which belong to the United Kingdom of 24 days, from 15 October to 7 November 2020 [https://covid19.who.int/]. These data formed of drought mortality rate, used by Almetwally et al. [22]. The data are: 0.2240, 0.2189, 0.2105, 0.2266, 0.0987, 0.1147, 0.3353, 0.2563, 0.2466, 0.2847, 0.2150, 0.1821, 0.1200, 0.4206, 0.3456, 0.3045, 0.2903, 0.3377, 0.1639, 0.1350, 0.3866, 0.4678, 0.3515, and 0.3232.

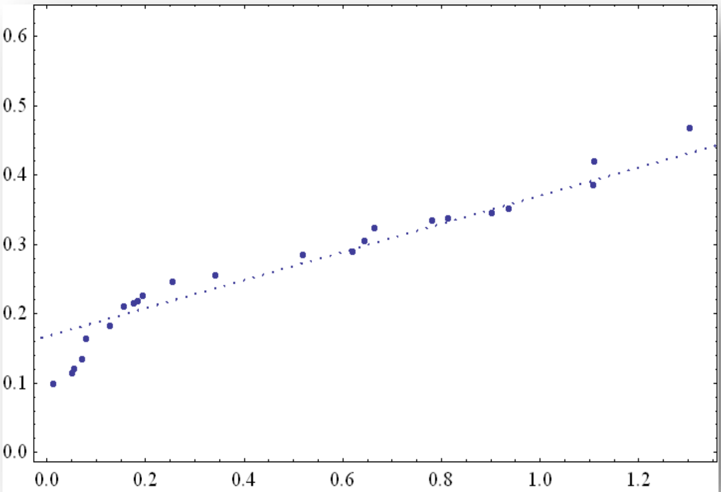
From the original data, one can notice that the lower record values are: 0.2240, 0.2189, 0.2105, 0.0987, p-value = 0.1398



Histogram of United Kingdom data



Q-Q Plot of United Kingdom data



PP- Plot of United Kingdom data

Fig. 4. The fitted pdf, PP-Plot and Q-Q plot of the EG-Xg distribution for the second real data of COVID-19

Table 1. Point and 95% interval two-sample predictors for the future lower record values $Y_{(s)}$ from EG-Xg distribution, Rv = 7, $\alpha = 0.2$, $\beta = 0.9, \theta = 0.3$

s	Loss function	$\tilde{y}_{(s)}$	Predictive intervals		
			LL	UL	Length
1	**SE**	1.9003	1.8990	1.9023	0.0033
	LINEX	1.9010	1.8990	1.9019	0.0028
3	**SE**	1.9995	1.9972	2.0008	0.0036
	LINEX	1.9017	1.8999	1.9030	0.0031
6	**SE**	2.5022	2.5000	2.5046	0.0046
	LINEX	2.4976	2.4961	2.4993	0.0032

Table 2. Point and 95% interval two-sample predictors for the future lower record values $Y_{(s)}$ for the poverty headcount ratio data

Application I	s	SE				LINEX			
		$\tilde{y}_{(s)(SE)}$	Predictive intervals			$\tilde{y}_{(s)(LNX)}$	Predictive intervals		
			LL	UL	Length		LL	UL	Length
World	1	0.9001	0.8985	0.9015	0.0029	0.9010	0.8996	0.9019	0.0022
	4	0.8986	0.8970	0.9000	0.0031	0.8992	0.8977	0.9005	0.0027
The lower middle income	1	0.9009	0.8995	0.9020	0.0024	0.8999	0.8990	0.9003	0.0012
	4	0.9004	0.8984	0.9019	0.0035	0.8985	0.8969	0.8999	0.0030
The middle east &North Africa	1	0.9008	0.8992	0.9022	0.0029	0.9000	0.8985	0.9009	0.0024
	4	0.9020	0.8997	0.9044	0.0047	0.9009	0.8992	0.9024	0.0031
Egypt	1	0.9005	0.8989	0.9015	0.0026	0.9004	0.8992	0.9011	0.0018
	4	0.8981	0.8964	0.8999	0.0035	0.8982	0.8970	0.8999	0.0028

Table 3. Point and 95% interval two-sample predictors for the future lower record values $Y_{(s)}$ for COVID-19 data

Application II	s	SE				LINEX			
		$\tilde{y}_{(s)(SE)}$	Predictive intervals			$\tilde{y}_{(s)(LNX)}$	Predictive intervals		
			LL	UL	Length		LL	UL	Length
Canada	1	1.4972	1.4947	1.4988	0.0040	1.4996	1.4985	1.5005	0.0020
	3	1.5987	1.5948	1.6001	0.0053	1.5992	1.5978	1.6002	0.0024
United	1	1.5026	1.4997	1.5036	0.0038	1.4992	1.4975	1.5001	0.0026
Kingdom	3	1.6018	1.5996	1.6036	0.0040	1.5992	1.5975	1.6006	0.0031

4.3 Concluding remarks

- It is clear that the BPs and the lengths of the BPB increase when s increases.
- One can notice that the lengths of the BPB under LINEX loss function have values less than the corresponding lengths under SE loss function.
- The results obtained in this chapter can be modified to obtain special results for sub-models of EG-Xg distribution as follows:
  - i. The exponentiated xgamma distribution, if $\alpha = 1$.
  - ii. The xgamma distribution, if $\alpha = 1$ and $\beta = 1$.

5 Conclusions

In this paper, the EG-Xg distribution is introduced as an application to the EGGC of distributions. The Bayes point and interval prediction of EG-Xg distribution based on dgos are considered. The results are verified using simulation study to assess the performance of predictors. The EGGC of distributions and the EG-Xg distribution as special case can be widely applied in various areas of biology, engineering and economics. Applications to poverty and COVID-19 mortality rates are provided to illustrate the importance of the EG-Xg distribution based on lower records. Also, the Histogram, PP-Plot and Q-Q plot of the EG-Xg distribution provide better fits for these real data.

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Competing Interests

Authors have declared that no competing interests exist.

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