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An Exact Solution of Telegraph Equations for Voltage Monitoring of Electrical Transmission Line

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Abstract

Telegraph equations are derived from the equations of transmission line theory. They describe the relationships between the currents and voltages on a portion of an electric line as a function of the linear constants of the conductor (resistance, conductance, inductance, capacitance). Their resolution makes it possible to determine the variation of the current and the voltage as a function of time at each point of the line. By adopting a general sinusoidal form, we propose a new exact solution to the telegraphers' partial differential equations. Different simulations have been carried out considering the parameter of the 12/20 (24) kV Medium Voltage Cable NF C 33,220. The curves of the obtained solution better fit the real voltage curves observed in the electrical networks in operation.

Keywords

Telegraph Equation, Electric Network, Voltage Variation, Electrical Transmission Lines, Analytical Solution

1. Introduction

Electrical transmission lines conduct single-phase, two-phase, or three-phase electrical voltage and current in opposite directions to each other [1] [2] [3]. They are generally modeled by a succession of identical quadrupoles or linear filters, each quadrupole comprising a linear resistance R, a linear inductance L, a linear capacitance C, and a linear conductance G. These lines are interconnected in the form of wire networks using distribution source stations. The wave transmission equations on a power line describe the evolution of the current and voltage as a function of time and space. They are also called telegrapher's equations [4]. Several solutions have been proposed to solve these equations among which we can note analytical solutions [5] [6] and numerical ones [7] [8] [9].

M. K. Smail [8] proposed a finite differences numerical method for solving these equations based on the time domain. M. Franchet [9] modeled and solved the problems of multiconductor lines by numerical matrix methods. A. Fall [10] used a frequency-domain method to study the propagation of the optical signal on a multimode coupler. In the same way, J. Biazar *et al.* [7] proposed an iterative method to solve the telegrapher's equations.

Other authors have proposed analytical solutions, notably C. R. Paul [5] and J. Ahmed *et al.* [6].

All of the above solutions have given interesting results in their respective fields of application. As for the studies presented in [8] [9] [10], the objective was to determine a solution to the telegrapher's equations to achieve fault detection and location in electrical networks.

Many of the analytical solutions are more mathematical than physical. Their representative curves do not always fit the actual voltage and current curves seen on power system control and supervision systems.

To this end, a new exact solution could improve electricity management by minimizing energy losses on the lines.

This paper proposes a new exact solution of telegraph equations for better electricity management. This solution describes, at a given location, the time-variation of the voltage amplitude in an electric cable. It is suitable for single-phase, two-phase, and three-phase voltage. The resulting solution allows the generation of the electrical wave and the monitoring of its propagation along the line. It allows following the variation of the wave as a function of time, space, and phase. Its curves better fit the shape of the instantaneous voltage in an electrical network in operation than those of the previous solutions. This new solution presents the variation of the signal shape as a function of time and the angle of the phase shift between the voltage and current signals in the network.

The remainder of the paper is structured in three parts: The first part presents the methodical approach adopted to describe the electrical model and the different equations. The second part deals with formulating the proposed exact solution of the telegraph equations and presents and discusses simulation results. The third part presents the conclusion and perspectives for further work.

2. Methodological Approach

Electrical transmission lines interconnect a source to an electrical charge by at least two conducting wires composed of a charge wire and a neutral one. The charge wire conducts the electrical signal while the neutral wire is used to form a circuit loop.

A portion of a two-wire line is usually modeled by a quadrupole as shown in Figure 1. At a given time, the voltage $v(x)$ and the current $i(x)$ vary with position *x*. These variations take into account the linear resistance *R*, the linear conductance *G*, the linear inductance *L*, and the linear capacitance *C*.

In the quasi-steady state, the equations relating voltage and current will be of the partial derivative type concerning time and space. Thus, by applying Kirchhoff's laws to Figure 1 we find the Relations (1.1) and (1.2):

$$v(x,t) - R \cdot dx \cdot i(x,t) - L \cdot dx \cdot \frac{\partial i(x,t)}{\partial t} - v(x+dx,t) = 0 \quad (1.1)$$

$$i(x,t) - G \cdot dx \cdot v(x,t) - C \cdot dx \cdot \frac{\partial v(x,t)}{\partial t} - i(x+dx,t) = 0 \quad (1.2)$$

By introducing the notion of finite differences, Equations (1.1) and (1.2) become (1.3) and (1.4):

$$-\frac{v(x+dx,t) - v(x,t)}{dx} = R \cdot i(x,t) + L \cdot \frac{\partial i(x,t)}{\partial t} \quad (1.3)$$

$$-\frac{i(x+dx,t) - i(x,t)}{dx} = G \cdot v(x,t) + C \cdot \frac{\partial v(x,t)}{\partial t} \quad (1.4)$$

Assuming that dx is infinitesimal, we obtain Equations (1.5) and (1.6):

$$-\frac{\partial v(x,t)}{\partial x} = R \cdot i(x,t) + L \cdot \frac{\partial i(x,t)}{\partial t} \quad (1.5)$$

$$-\frac{\partial i(x,t)}{\partial x} = G \cdot v(x,t) + C \cdot \frac{\partial v(x,t)}{\partial t} \quad (1.6)$$

The second derivative of Equations (1.5) and (1.6) concerning space gives Equations (1.7) and (1.8):

$$-\frac{\partial^2 v(x,t)}{\partial x^2} = R \cdot \frac{\partial i(x,t)}{\partial x} + L \cdot \frac{\partial^2 i(x,t)}{\partial x \partial t} \quad (1.7)$$

$$-\frac{\partial^2 i(x,t)}{\partial x^2} = G \cdot \frac{\partial v(x,t)}{\partial x} + C \cdot \frac{\partial^2 v(x,t)}{\partial x \partial t} \quad (1.8)$$

By combining Equations (1.1), (1.5), and (1.7), respectively (1.2), (1.6), and (1.8), we find Equations (1.9) and (1.10) below. These equations correspond to the one-dimensional wave equations called the telegraph equations of voltage (1.9) and current (1.10).

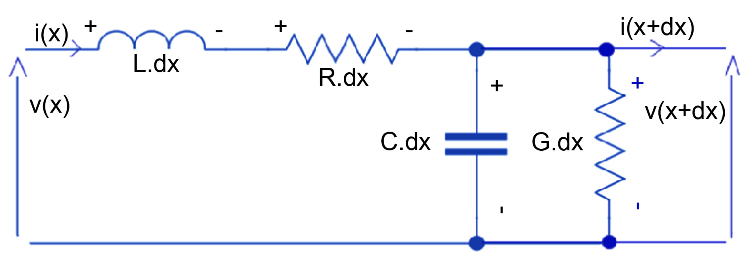

Figure 1. Model of a transmission line.

$$\frac{\partial^2 v(x,t)}{\partial x^2} = L\cdot C\cdot\frac{\partial^2 v(x,t)}{\partial t^2} + (R\cdot C + L\cdot G)\cdot\frac{\partial v(x,t)}{\partial t} + R\cdot G\cdot v(x,t) \tag{1.9}$$

$$\frac{\partial^2 i(x,t)}{\partial x^2} = L\cdot C\cdot\frac{\partial^2 i(x,t)}{\partial t^2} + (R\cdot C + L\cdot G)\cdot\frac{\partial i(x,t)}{\partial t} + R\cdot G\cdot i(x,t) \tag{1.10}$$

In this paper, we propose an exact solution to Equation (1.9). This latter solution depends on time, position, and phase. It can be used to simulate the variation of single-phase, two-phase, or three-phase voltage on electrical power lines.

3. Results and Discussions

3.1. Proposed Solution for Telegraph Equations

Equations (1.9) and (1.10) have similar forms. They well describe the propagation of voltage and current in electric cables.

For Equation (1.9), we propose a solution of the form:

$$v(x,\varphi,t) = u(x,\varphi)e^{(-j\omega t+\varphi)} \tag{1.11}$$

where t is the time variable, x is the position and φ is the phase. $u(x,\varphi)$ is the wave amplitude and ω is the wave pulsation.

The solution of Equation (1.10) can be deduced from the expression:

$$v(x,\varphi,t) = Z(x,\varphi)i(x,\varphi,t) \tag{1.12}$$

where $Z(x,\varphi)$ is the characteristic impedance of the transmission line. Therefore, finding the solution to one of the equations allows us to deduce the other solution.

Let's pose $a = LC$; $b = RC + LG$ and $c = RG$, then Equation (1.9) becomes:

$$\frac{\partial^2 v(x,t)}{\partial x^2} = a\frac{\partial^2 v(x,t)}{\partial t^2} + b\frac{\partial v(x,t)}{\partial t} + cv(x,t) \tag{1.13}$$

Let $v(x,\varphi,t)$ given by Equation (1.11) be the general form of the solution of Equation (1.9). By derivating two times $v(x,\varphi,t)$ according to space and time, we obtain:

$$\frac{\partial^2 u(x,\varphi)}{\partial x^2} + \left[\left(a\omega^2 - c\right) + b\,j\omega\right]u(x,\varphi) = 0 \tag{1.14}$$

Equation (1.14) is a second-order differential equation without a second member whose characteristic equation is given by (1.15)

$$r^2 = \left(c - a\omega^2\right) - jb\omega \tag{1.15}$$

Let $r = \alpha + j\beta$ such that $r^2 = \alpha^2 - \beta^2 + 2j\alpha\beta$, and $|r|^2 = \alpha^2 + \beta^2$.

We derive the system of equations given by:

$$\begin{cases} \alpha^2 - \beta^2 = c - a\omega^2 & \text{(a)} \\ \alpha^2 + \beta^2 = \sqrt{\left(c - a\omega^2\right)^2 + (b\omega)^2} & \text{(b)} \\ 2\alpha\beta = -b\omega & \text{(c)} \end{cases}$$

whose solution is given by:

$$\alpha=\pm\sqrt{\frac{c-a\omega^2+\sqrt{\left(c-a\omega^2\right)^2+\left(b\omega\right)^2}}{2}} \tag{d}$$

$$\beta=\pm\sqrt{\frac{a\omega^2-c+\sqrt{\left(c-a\omega^2\right)^2+\left(b\omega\right)^2}}{2}} \tag{e}$$

The product $\alpha\cdot\beta=-\frac{bw}{2}<0$. Since α denotes the attenuation coefficient we choose $\alpha<0$ and $\beta>0$. We derive a real solution of Equation (1.14) given by

$$u\left(x,\varphi\right)=\left(A\cdot\cos\left(\beta x+\varphi\right)+B\cdot\sin\left(\beta x+\varphi\right)\right)\mathrm{e}^{\left(\alpha\cdot x-\varphi\right)} \tag{1.16}$$

Then a solution to Equation (1.9) is:

$$v\left(x,\varphi,t\right)=\left(A\cos\left(\beta x+\varphi\right)+B\sin\left(\beta x+\varphi\right)\right)\mathrm{e}^{\left[\alpha\cdot x-j\omega t\right]} \tag{1.17}$$

where A and B are time-dependent parameters to be determined with boundary conditions on position, phase, and time.

3.1.1. Determination of Parameters *A* and *B*

The parameters A and B are time-dependent. To determine them let us apply the boundary conditions to x and φ. When $x=0$ and $\varphi=0$, the line is disconnected from the distribution source, therefore the voltage is equal to that of the production source which is $v_0\left(t\right)$.

Similarly, if $x=0$, and $t=0$, the voltage depends on the phase and is noted $v_0\left(\varphi\right)$.

Thus the expressions of the parameters A and B are determined as follows:

- If $x=0$ et $\varphi=0$ then $v\left(x,\varphi,t\right)=v\left(0,0,t\right)=v_0\left(t\right)=A\mathrm{e}^{-j\omega t}$ therefore $A=v_0\left(t\right)\mathrm{e}^{j\omega t}$
- If $x=0$ et $t=0$ then

$$v_0\left(\varphi\right)=A\cos\left(\varphi\right)+B\sin\left(\varphi\right)=\sqrt{A^2+B^2}\cos\left(\beta+\varphi\right)$$

with

$$\cos\left(\beta\right)=\frac{A}{\sqrt{A^2+B^2}} \quad \text{et} \quad \sin\left(\beta\right)=\frac{B}{\sqrt{A^2+B^2}}$$

Also $\frac{\mathrm{d}v_0\left(\varphi\right)}{\mathrm{d}\varphi}=-A\sin\left(\varphi\right)+B\cos\left(\varphi\right)=0$ therefore

$$v_0\left(\varphi\right)=\left(A+\frac{B}{A}\right)\cos\left(\varphi\right)$$

Assuming that β is very small compared to φ, we have:

$$A+\frac{B}{A}=\sqrt{A^2+B^2}\text{, hence}$$

$$B=\frac{2A^2}{A^2-1}=\frac{2v_0^2\left(t\right)\cdot\mathrm{e}^{j2\omega t}}{v_0^2\left(t\right)\cdot\mathrm{e}^{j2\omega t}-1}.$$

Given A and B the expression of the voltage is:

$$v(x,\varphi,t)=\left[v_0(t)\cos(\beta x+\varphi)+\frac{2v_0^2(t)\cdot \mathrm{e}^{j\omega t}}{v_0^2(t)\cdot \mathrm{e}^{j2\omega t}-1}\sin(\beta x+\varphi)\right]\mathrm{e}^{\alpha\cdot x} \tag{1.18}$$

3.1.2. Finding an Exact Real Solution

To find a real solution to the telegraph equations, we decompose Equation (1.18) into its real and imaginary parts. To do this, we write the expression of $B\mathrm{e}^{-j\omega t}$ in its real and imaginary forms.

$$B\mathrm{e}^{-j\omega t}=\frac{v_0^2(t)\cos\omega t-\cos\omega t-j\left[v_0^2(t)\sin\omega t+\sin\omega t\right]}{\frac{v_0^2(t)}{2}+\frac{1}{2\cdot v_0^2(t)}-\cos 2\omega t} \tag{1.19}$$

By replacing this expression of $B\mathrm{e}^{-jwt}$ in Equation (1.18) we deduce that:

$$v(x,\varphi,t)=\left[v_0(t)\cos(\beta x+\varphi)+\frac{v_0^2(t)\cos\omega t-\cos\omega t-j\left[v_0^2(t)\cdot\sin\omega t+\sin\omega t\right]}{\frac{v_0^2(t)}{2}+\frac{1}{2\cdot v_0^2(t)}-\cos 2\omega t}\sin(\beta x+\varphi)\right]\mathrm{e}^{\alpha\cdot x} \tag{1.20}$$

When we decompose the voltage expression into its real part ($Re[v]$) and imaginary one ($Im[v]$) where $v(x,\varphi,t)=Re[v]+jIm[v]$, we identify:

$$Re[v]=\left[v_0(t)\cos(\beta x+\varphi)+\frac{v_0^2(t)\cos\omega t-\cos\omega t}{\frac{v_0^2(t)}{2}+\frac{1}{2\cdot v_0^2(t)}-\cos 2\omega t}\right]\mathrm{e}^{\alpha x}$$

$$Im[v]=-\frac{v_0^2(t)\sin\omega t+\sin\omega t}{\frac{v_0^2(t)}{2}+\frac{1}{2v_0^2(t)}-\cos 2\omega t}\sin(\beta x+\varphi)\mathrm{e}^{\alpha\cdot x}$$

From this solution, we run simulations on $Re[v]$ and $Im[v]$ as a function of cable length, time, and the angle θ of the cosine of the electrical network.

3.2. Simulation Results

The solution of the telegraph equations depends on the linear constants *R*, *C*, *L*, and *G* of the power line. In this paper, we consider the Medium Voltage Cables NF C 33,220 standards of 12/20 (24) kV. It is an aluminum cable of a nominal cross-section of 150 mm^2 (square millimeter), whose linear constants in AC transmission current of frequency 50 Hz and temperature 90˚C are R = 0.265 Ω/km, C = 0.24 μF/km, L = 0.41 mH/km and G = 1/R Siemens/m. In the literature the phase shifts often considered are: φ = 60˚; 90˚; 120˚.

For simulations, the source voltage $v_0(t)$ considered is

$$v_0(t)=V_{\max}\sin(\omega t+\theta).$$

where:

- $V_{\max}$ is the maximum voltage on a medium voltage transmission power line. In Burkina Faso the effective voltages used for medium voltage transmission are 15,000 V; 20,000 V and 33,000 V with a tolerance of ±5%. For this study, we consider the RMS voltage of 20,000 V *i.e.* a maximum voltage $V_{\max} = 20000\sqrt{2}\ \text{V} \pm 1000\sqrt{2}\ \text{V}$.
- θ is the phase shift between voltage and current in the power system. It is determined by the cosine of the network which is $\cos\theta = 0.8$ or $\theta = 0.20\pi$ rad.
- The pulsation is $\omega = 100\pi$ rad/s with a transmission frequency $f = 50$ Hz.

Three situations are considered for the simulations:

Situation 1: Single-phase signal in three dimensions (3D) space.

We plot here the variation of the real and imaginary parts of the voltage $v(x,\varphi,t)$ as a function of time and length.

Situation 2: Single-phase, two-phase, and three-phase signals in the plane.

Firstly, we plot $v(x,\varphi,t)$ according to the position *x*, when the time is set to $t = 3$ s.

Secondly, we plot $v(x,\varphi,t)$ as a function of time. For this purpose, we set the wave speed to $v = 5$ m/s and fix the length *x* of the cable at 0.05 km.

Situation 3: Single-phase, two-phase, and three-phase signals in three dimensions projected on the plane.

In this third part, we vary $v(x,\varphi,t)$ as a function of time and length simultaneously and then plot a projection in the time plane. This corresponds to the curves visualized by electrical network supervision systems.

3.2.1. Simulation Results for the First Situation

Figure 2 shows the voltage curve in the first situation.

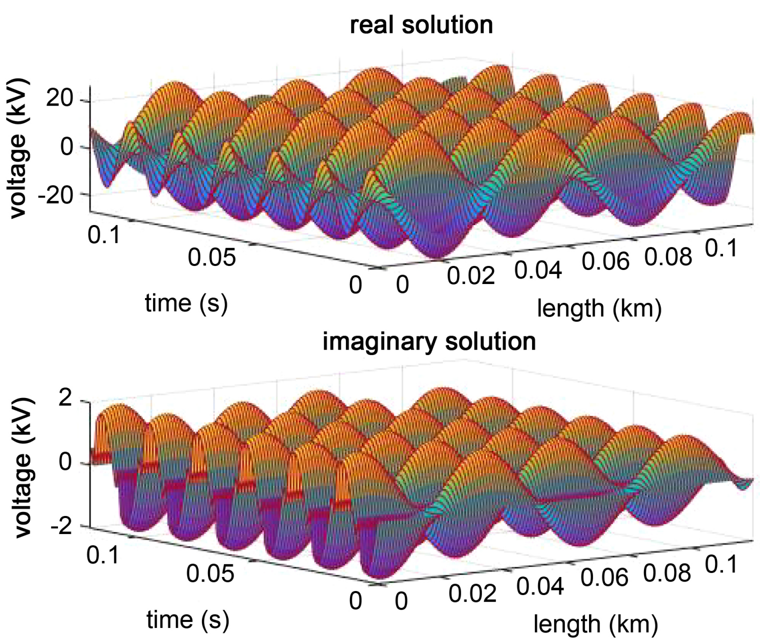

Figure 2. 3D representation of voltage in a single-phase case

Figure 2 shows a sinusoidal shape as a function of length and time. This dual sinusoidal behavior shows that the founded voltage solution $v(x,\varphi,t)$ is realistic. Indeed, in the literature, most of the proposed solutions give sinusoidal functions of time.

3.2.2. Simulation Results for the Second Situation

Figure 3 and **Figure 4** show respectively the variation of the electric voltage as a function of length and time, for single-phase, two-phase and three-phase signals.

These two figures show a sinusoidal character of the voltage according to the length and time.

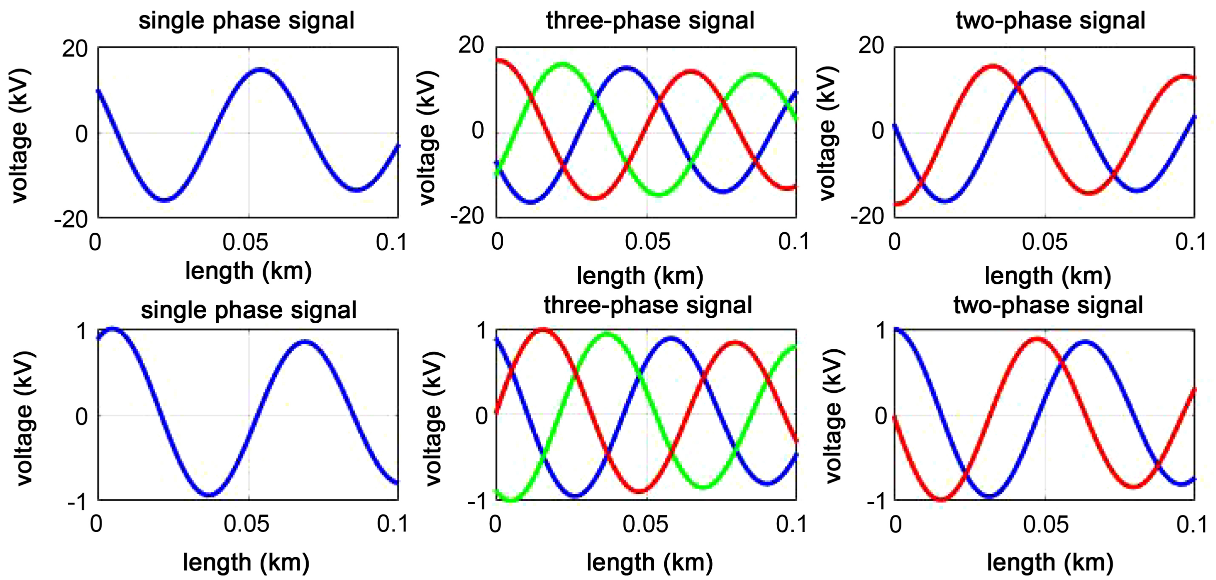

Figure 3. Voltage versus length for $t = 3\,\text{s}$.

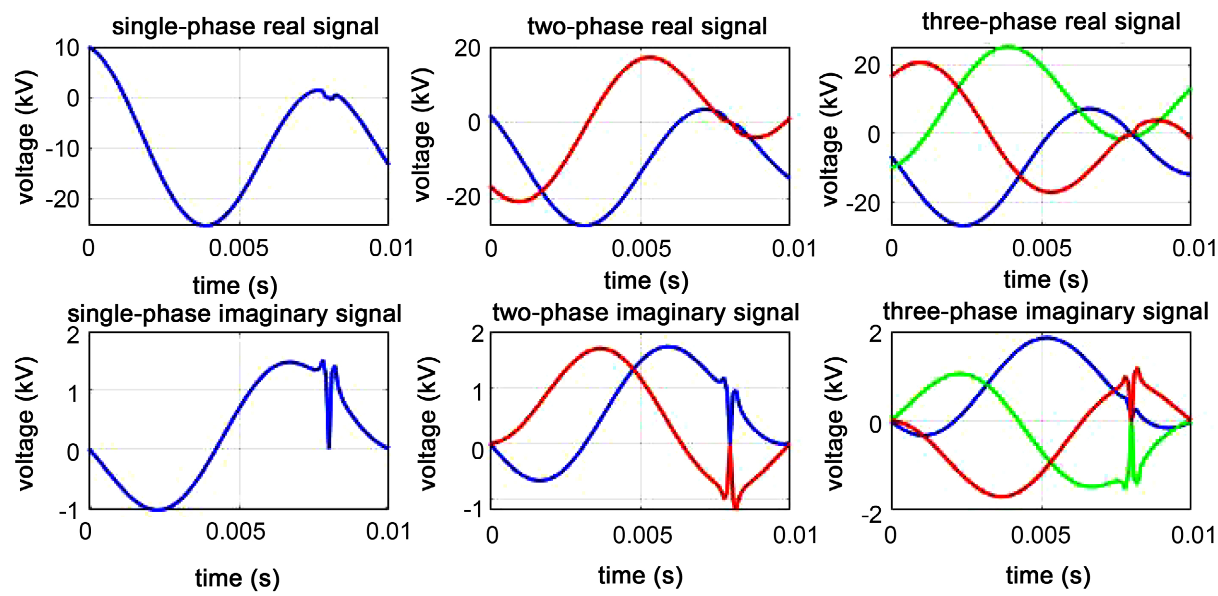

Figure 4. Voltage versus time for $x = 0.05\,\text{km}$ and $v = 5\,\text{m/s}$.

3.3. Simulation Results for the Third Situation

Figure 5 and Figure 6 show the voltages inside the cables in normal operation (Situation 3) for frequencies $f = 50\ \text{Hz}$ and $f = 25\ \text{Hz}$ respectively.

Figure 5 and Figure 6 show that, whatever the frequency, the amplitude of the voltage tends to decrease with time. This can be explained by the fact that the linear resistance of the cable is nonnull. This corresponds well to the real behavior of the voltage in electrical cables in operation.

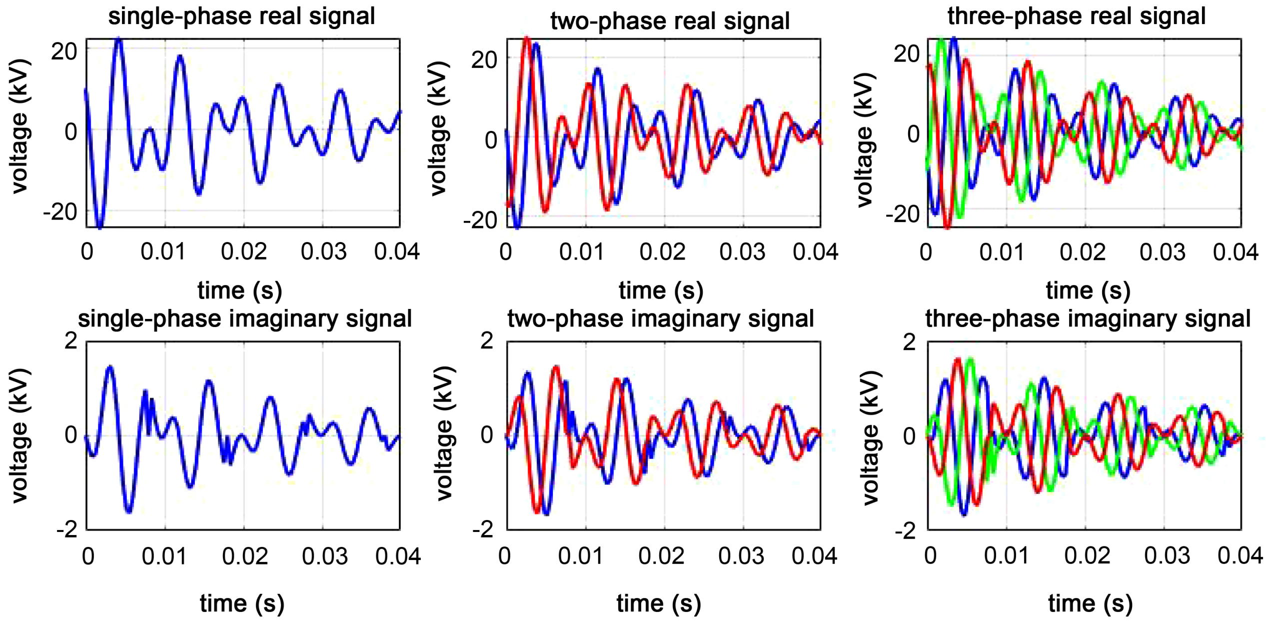

Figure 5. Projection in the plane of the voltage variation for the frequency $f = 50\ \text{Hz}$.

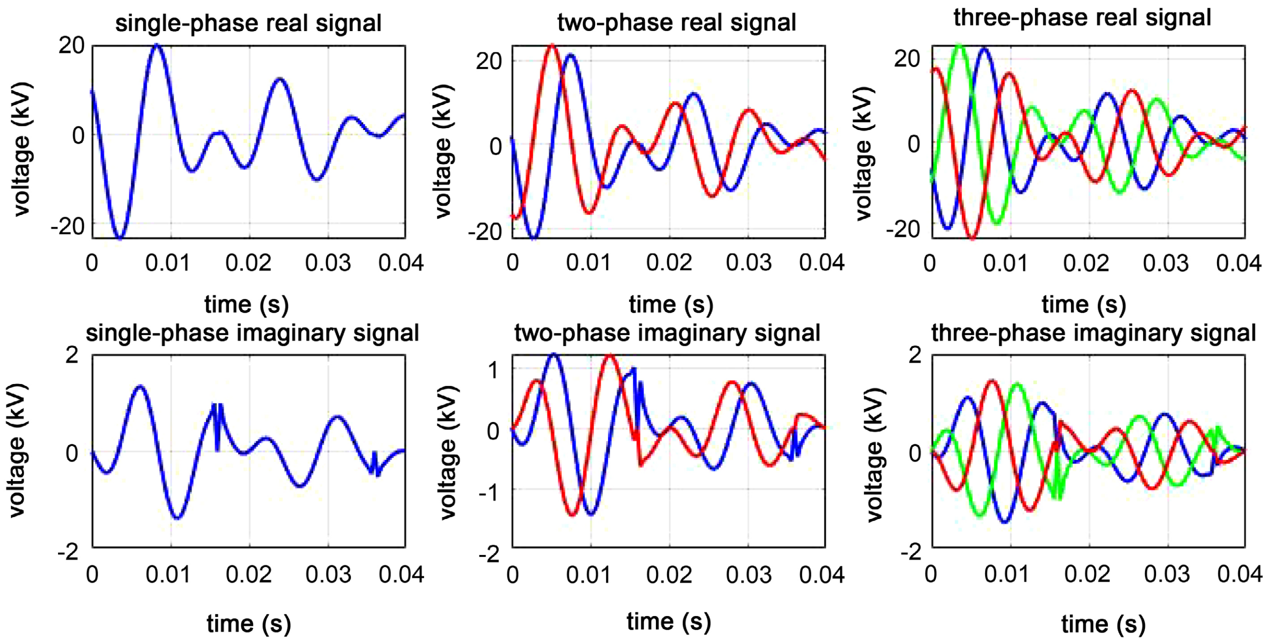

Figure 6. Projection in the plane of the voltage variation for the frequency $f = 25\ \text{Hz}$.

4. Conclusions

This paper proposes an exact solution to telegraph equations. For the founded solution, the amplitude of the voltage varies with position and time. By taking the phase into account, the proposed solution allows the simulation of single-phase, two-phase and three-phase voltages. The choice of the initial condition of the voltage is a factor that influences the shape of the solution. The simulations showed that the voltage variation corresponds well to the patterns observed by electrical network supervision systems.

This work could be extended to a multi-branch power line section to describe the voltage behavior in the medium voltage (MV) and low voltage (LV) networks. Also, this exact solution could be explored to study the stability of an electrical network in the case of an energy mix. The a priori knowledge of the voltage level as a function of position and time, provided by the proposed solution, could also be exploited to detect and locate faults on a power line.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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