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Simultaneous Remotality in Operator Spaces

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Abstract

The aim of this paper is to study simultaneous remotality in the Banach space of bounded linear operators on a Banach space H.

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1 Introduction

Let X be a Banach space, and $E \subset X$ be a closed bounded set of X. For $x \in X$ we let

$$D(x, E) = \sup\{\|x - e\| : e \in E\}$$

We call $D(x, E)$ attainable if there exists $e_1 \in E$ such that

$$D(x, E) = \|x - e_1\|$$

The point e_1 is called farthest point from x in E.

The set E is called **remotal** if $D(x, E)$ is attainable for all $x \in X$. If $D(x, E)$ is uniquely attainable for all $x \in X$, then E is called **uniquely remotal.**

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With each such set E, there is a set map associated with it, called the metric projection defined as follows

$$F(x,E) = \{e \in E : \|x-e\| = D(x,E)\}$$

If $F(x,E)$ is a singleton for all $x \in X$, then E is called uniquely remotal.

The problem of whether a set is remotal is an important problem in the theory of Banach spaces.

The study of remotal sets started in the sixties of the last century.

Edelstein and Lewis in [1], proved results on remotality of exposed points in Banach spaces.

Remotality has its application in the geometry of Banach spaces. Many recent papers have been published on the relation between the geometry of the Banach space and remotality of sets in that Banach space. We refer to [2], [3], [4], [5], [6] and [7].

Now, let X be a Banach space. On $X \times X$, one can define many norms. The most interesting ones are the p-norms for $1 \le p \le \infty$:

For $(x,y) \in X \times X$, we let $\|(x,y)\|_p = (\|x\|^p + \|y\|^p)^{\frac{1}{p}}$, $1 \le p < \infty$. For $p = \infty$, $\|(x,y)\|_\infty = \max\{\|x\|, \|y\|\}$. For $E \subset X$, let $D(E) = \{(e,e) : e \in E\}$, the diagonal of E.

The set E is called simultaneously remotal in the $p-$norm in X, if $D(E)$ is remotal in $(X \times X, \|.\|_p)$.

Few results are known on simultaneous remotality in Banach spaces. In [8], Abu Sirhan and Edely, studied simultaneous remotality in $L^1(\mu, X)$. Al-Sharif, and Rawashdeh, studied simultaneous remotality in $L^\infty(I,X)$ in [9].

In this paper we study simultaneous remotality in $L(H,H)$, the space of bounded linear operators on a Hilbert space H.

2 Simultaneous Remotality in *L(H,H)*

Let H be a separable Hilbert space and Y be a closed subspace of H. So $H = Y \oplus Y^\perp$, where $Y^\perp$ is the orthogonal complement of Y. We set $L(H,H)$ to denote the space of all bounded linear operators on H to H, and $E = L(B_1(Y), B_1(Y))$, the set of all bounded linear operators that takes the unit ball of Y into the unit ball of Y. Then E is a closed convex bounded subset of $L(H,H)$.

.

Now:

Any operator $T : H \to H$ has a matrix representation

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} : Y \oplus Y^\perp \to Y \oplus Y^\perp$$

where

$$T(y \oplus \hat{y}) = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} y \\ \hat{y} \end{bmatrix}$$

So

$$\begin{aligned} T_1 &: Y \to Y, \qquad T_2 : Y^\perp \to Y \\ T_3 &: Y \to Y^\perp, \qquad T_4 : Y^\perp \to Y^\perp \end{aligned}$$

Definition 2.1. An operator $T \in L(H,H)$ is called block diagonal if $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix}$.

A nice fact about the diagonal operator is the following:

Lemma 2.2
If $T \in L(H, H)$ and $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, then $\|T\| = \max\{\|A\|, \|B\|\}$.
Proof :
Let $T \in L(H, H)$ such that $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.
Then

$$\begin{aligned} \|T\|^2 &= \sup_{\|x\|=1} \|Tx\|^2 \\ &= \sup_{\|y\oplus\hat{y}\|=1} \|T(y \oplus \hat{y})\|^2 \\ &= \sup_{\|y\oplus\hat{y}\|=1} \left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} y \\ \hat{y} \end{bmatrix} \right\|^2 \\ &= \sup_{\|y\oplus\hat{y}\|=1} \left\| \begin{bmatrix} Ay \\ B\hat{y} \end{bmatrix} \right\|^2 \\ &= \sup_{\|y\|^2+\|\hat{y}\|^2=1} \left(\|Ay\|^2 + \|B\hat{y}\|^2\right) \\ &\leq \sup_{\|y\|+\|\hat{y}\|=1} \left(\|y\|^2 \left\| A\frac{y}{\|y\|} \right\|^2 + \|\hat{y}\|^2 \left\| B\frac{\hat{y}}{\|\hat{y}\|} \right\|^2 \right) \end{aligned}$$

But , $\|y\|^2 \left\| A\frac{y}{\|y\|} \right\|^2 + \|\hat{y}\|^2 \left\| B\frac{\hat{y}}{\|\hat{y}\|} \right\|^2$ is in the form $tf(t) + sg(s)$, with $s + t = 1$, where $t = \|y\|^2$ and $s = \|\hat{y}\|^2$. Consequently,

$$\begin{aligned} \|T\|^2 &\leq \max(\sup_{\|y\|=1} \left\| A\frac{y}{\|y\|} \right\|^2, \sup_{\|\hat{y}\|=1} \left\| B\frac{\hat{y}}{\|\hat{y}\|} \right\|^2) \\ &= \max(\|A\|^2, \|B\|^2 \end{aligned}$$

Now,

$$\begin{aligned} \|A\| &= \sup_{\|y\|=1} \|Ay\| \\ &= \sup_{\|y\|=1} \|Ay + B0\| \\ &= \sup_{\|y\|=1} \|T(y \oplus 0)\| \\ &\leq \|T\| \end{aligned}$$

Similarly

$$\|B\| \leq \|T\|$$

Thus,

$$\max\{\|A\|, \|B\|\} \leq \|T\|$$

This ends the proof. □

Now, for $S, T \in L(H, H)$, let $\|(S, T)\|_\infty = \max\{\|S\|, \|T\|\}$. Our goal now is to prove that E is simultaneously remotal in a certain class of operators in $L(H, H)$, in the $\|.\|_\infty$ norm. The remotality of
E was proved in [10].

Theorem 2.3

The set E is simultaneously remotal with respect to diagonal operators in $L(H, H)$.

Proof:

Let T be a diagonal matrix in $L(H, H)$.Then $T = T_1 + T_2$, where $T_1 : Y \to Y$ and $T_2 : Y^{\perp} \to Y^{\perp}$. Now, T_1 has a farthest point in $L(B_1(Y), B_1(Y))$, say $A = -\frac{T_1}{\|T_1\|}$.

Then $T - A = T_1 - A + T_2$. And

$$\begin{aligned} \|T - A\| &= \max\{\|T_1 - A\|, \|T_2\|\} \\ &\geq \max\{\|T_1 - B\|, \|T_2\|\} \\ &= \|T - B\| \text{ for any } B \in E. \end{aligned}$$

Thus, E is remotal in the diagonal operators in $L(H, H)$.

Now let $S, T \in L(H, H)$, such that both are diagonal. Hence $S = S_1 + S_2$, and $T = T_1 + T_2$, with $S_1, T_1 \in L(Y, Y)$. Let B_1 be the farthest point of S_1 in E, and B_2 be the farthest point of T_1 in E.

Assume without loss of generality that

$$\max\{\|S_1 - B_1\|, \|S_2\|, \|T_1 - B_2\|, \|T_2\|\} = \|S_1 - B_1\| \text{...........}(*)$$

Let $B = B_1 + 0$. Then $B \in E$. Further

$$\begin{aligned} \|(S - B, T - B)\|_{\infty} &= \max\{\|S - B\|, \|T - B\|\} \\ &= \max\{\|S_1 - B_1\|, \|S_2\|, \|T_1 - B_1\|, \|T_2\|\} \\ &= \|S_1 - B_1\| \end{aligned}$$

Since from $(*)$, we have $\|S_1 - B_1\| \geq \|T_1 - B_2\| \geq \|T_1 - C\|$ for any $C \in E$, we get

$$\begin{aligned} \|(S - B, T - B)\|_{\infty} &\geq \max\{\|S_1 - C\|, \|S_2\|, \|T_1 - C\|, \|T_2\|\}, \forall C \in E \\ &= \|(S - C, T - C)\|_{\infty}. \end{aligned}$$

Now assume that

$$\max\{\|S_1 - B_1\|, \|S_2\|, \|T_1 - B_2\|, \|T_2\|\} = \|S_2\| \text{......}(**)$$

Then

$$\begin{aligned} \|(S - B, T - B)\|_{\infty} &= \max\{\|S - B\|, \|T - B\|\} \\ &= \max\{\|S_1 - B_1\|, \|S_2\|, \|T_1 - B_1\|, \|T_2\|\} \\ &= \|S_2\| \qquad \text{(by } (**) \text{ and } \|T_1 - B_1\| \leq \|T_1 - B_2\| \leq \|S_2\|\text{)} \end{aligned}$$

For all $C \in E$

$\|S_1 - C\| \leq \|S_1 - B_1\| \leq \|S_2\| = \|(S - B, T - B)\|_{\infty}$

$\|T_1 - C\| \leq \|T_1 - B_1\| \leq \|T_1 - B_2\| \leq \|S_2\| = \|(S - B, T - B)\|_{\infty}$

$\|T_2\| \leq \|S_2\| = \|(S - B, T - B)\|_{\infty}$

So

$$\begin{aligned} \|(S - C, T - C)\|_{\infty} &= \max\{\|S_1 - C\|, \|S_2\|, \|T_1 - C\|, \|T_2\|\} \\ &\leq \|S_2\| \\ &= \|(S - B, T - B)\|_{\infty} \end{aligned}$$

Similarly, if $\max\{\|S_1 - B_1\|, \|S_2\|, \|T_1 - B_2\|, \|T_2\|\} = \|T_2\|$. This completes the proof.

Theorem 2.4 :
$B_1\left(L(H,Y)\right)$ is simultaneously remotal in $L(H,Y)$.

Proof :
Let $A_1, A_2 \in L(H,Y)$. Then there exist $J_1, J_2 \in B_1\left(L(H,Y)\right)$ such that J_1 is a farthest point of A_1, and J_2 is a farthest point of A_2.
We have two cases :
If

$$\max\{\|A_1 - J_1\|, \|A_2 - J_2\|\} = \|A_1 - J_1\|,$$

then

$$\|(A_1 - J, A_2 - J)\|_\infty = \max\{\|A_1 - J\|, \|A_2 - J\|\} = \|A_1 - J_1\|$$

For any $C \in B_1\left(L(H,Y)\right)$

$$\begin{aligned}
\|(A_1 - C, A_2 - C)\|_\infty &= \max\{\|A_1 - C\|, \|A_2 - C\|\} \\
&\leq \max\{\|A_1 - J_1\|, \|A_2 - J_2\|\} \\
&= \|A_1 - J_1\| \\
&= \|(A_1 - J_1, A_2 - J_1)\|_\infty .
\end{aligned}$$

.If

$$\max\{\|A_1 - J_1\|, \|A_2 - J_2\|\} = \|A_2 - J_2\|,$$

then

$$\|(A_1 - J_2, A_2 - J_2)\|_\infty = \max\{\|A_1 - J_2\|, \|A_2 - J_2\|\} = \|A_2 - J_2\| .$$

For all $C \in B_1\left(L(H,Y)\right)$

$$\begin{aligned}
\|(A_1 - C, A_2 - C)\|_\infty &= \max\{\|A_1 - C\|, \|A_2 - C\|\} \\
&\leq \max\{\|A_1 - J_1\|, \|A_2 - J_2\|\} \\
&= \|A_2 - J_2\| \\
&= \|(A_1 - J_2, A_2 - J_2)\|_\infty .
\end{aligned}$$

Thus (A_1, A_2) has a simultaneous farthest point in $B_1\left(L(H,Y)\right)$. This ends the proof.

3 Conclusions

For any subspace Y of a Hilbert space H, we prove that $L(B_1(Y), B_1(Y))$ is simultaneously remotal in $L(H,H)$ when the $\|\|_\infty$ is considered for the pair of operators. Here $L(E,F)$ stands for the set of bounded linear operators in $L(H,H)$ that takes the subset $E \sqsubseteq H$, into the subset F.

Competing Interests

The authors declare that no competing interests exist.

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