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The $\mathcal{E}$-Hyperstability of an Euler-Lagrange Type Quadratic Functional Equation in Banach Spaces

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Abstract

The main goal of this paper is the investigation of the $\mathcal{E}$-hyperstability of an Euler-Lagrange type quadratic functional equation

$$f(x+y)+\frac{1}{2}\Big[f(x-y)+f(y-x)\Big]=2f(y)+2f(x)$$

in the class of functions from an abelian group into a Banach space.

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1 Introduction and Preliminaries

In 1940, S. M. Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which suggested the following stability problem, well-known as Ulam stability problem:
Let G_1 be a group and let G_2 be a metric group with the metric $d(.,.)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h\colon G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all

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$x, y \in G_1$, then there is a homomorphism $H: G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, D. H. Hyers provided in [2] a first partial answer to Ulam's problem for Banach spaces. Hyers' theorem was generalized by T. Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. P. Găvruta [5] provided a further generalization of the Rassias' theorem by using a general control function.

A functional equation of the form

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is called the quadratic functional equation. Every solution of the quadratic functional equation is said to be quadratic function. Quadratic functional equation was used to characterize inner product spaces [6, 7, 8]. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x) = B(x, x)$ for all x (see [6, 9]). The bi-additive mapping is given by

$$B(x, y) = \frac{1}{4}\left[f(x+y) - f(x-y)\right].$$

The Hyers-Ulam stability problem for the above quadratic functional equation was proved by F. Skof [10] for mapping $f: X \to Y$, where X is a normed space and Y is a Banach space. P. W. Cholewa [11] noticed that the theorem of F. Skof is still true if relevant domain X is replaced by an abelian group. In [12], S. Cherwik proved the generalized Hyers-Ulam stability of the quadratic functional equation as above. A. Grabiec [13] has generalized these results mentioned above. Several functional equations have been investigated in [14, 15, 16, 17, 18, 19].

M. J. Rassias [20] introduced the Euler-Lagrange type quadratic functional equation

$$f(x+y) + \frac{1}{2}\Big[f(x-y) + f(y-x)\Big] = 2f(y) + 2f(x) \tag{1.2}$$

and established the general solution and the "J. M. Rassias product-sum" stability for the functional equation (1.2).
The above equation and his stability results have many applications in Mathematical Statistics, Stochastic Analysis and Psychology. Every solution of (1.2) satisfies the quadratic functional equation (1.1).

In 2001, G. Maksa and Z. Páles [21] proved a new type of stability of a class of linear functional equation

$$f(x) + f(y) = \frac{1}{n}\sum_{i=1}^{n} f\big(x\varphi_i(y)\big), \tag{1.3}$$

where f is a real-valued mapping defined on a semigroup $S := (S, .)$ and where the maps $\varphi_1, \cdots, \varphi_n$: $S \to S$ are pairwise distinct automorphisms of S. More precisely, they proved that if the error bound for the difference of the two sides of (1.3) satisfies a certain asymptotic property, then in fact, the two sides have to be equal. Such a phenomenon is called the hyperstability of the functional equation on S. Further, J. Brzdęk and K. Ciepliński [22] introduced the following definition, which describes the main ideas of such a hyperstability notion for equations in several variables.

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, the set of integers by $\mathbb{Z}$ and the set of real numbers by $\mathbb{R}$. Let $\mathbb{N}_+$ be the set of positive integers. By $\mathbb{N}_m$, $m \in \mathbb{N}_+$, we will denote the set of all integers greater than or equal to m. Let $\mathbb{R}_0 := [0, \infty)$ be the set of nonnegative real numbers and $\mathbb{R}_+ := (0, \infty)$ the set of positive real numbers. We write B^A to mean " the family of all functions mapping from a nonempty set A into a nonempty set B ".

Definition 1.1. Let X be a nonempty set, (Y, d) be a metric space, $\varepsilon\colon X^n \to \mathbb{R}_0$ be an arbitrary function, and let $\mathcal{F}_1$, $\mathcal{F}_2$ be two operators mapping from a nonempty set $\mathcal{D} \subset Y^X$ into Y^{X^n}. We say that the operator equation

$$\mathcal{F}_1\varphi(x_1, \ldots, x_n) = \mathcal{F}_2\varphi(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n \in X) \tag{1.4}$$

is ε-hyperstable provided that every $\varphi_0 \in \mathcal{D}$ which satisfies

$$d\big(\mathcal{F}_1\varphi_0(x_1, \ldots, x_n), \mathcal{F}_2\varphi_0(x_1, \ldots, x_n)\big) \le \varepsilon(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n \in X)$$

fulfills equation (1.4) on X.

In this article, we introduce the following definition, which describes the main ideas of the concept of hyperstability for equations in several variables.

Definition 1.2. Let X be a nonempty set, (Y, d) be a metric space, $\mathcal{E} \subset \mathbb{R}_+^{X^n}$ be a nonempty subset, and $\mathcal{F}_1$, $\mathcal{F}_2$ be operators mapping from a nonempty set $\mathcal{D} \subset Y^X$ into Y^{X^n}. We say that the operator equation

$$\mathcal{F}_1\varphi(x_1, \ldots, x_n) = \mathcal{F}_2\varphi(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n \in X) \tag{1.5}$$

is $\mathcal{E}$-hyperstable for the pair (X, Y) provided for any $\varepsilon \in \mathcal{E}$ and $\varphi_0 \in \mathcal{D}$ satisfies the inequality

$$d\big(\mathcal{F}_1\varphi_0(x_1, \ldots, x_n), \mathcal{F}_2\varphi_0(x_1, \ldots, x_n\big) \le \varepsilon(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n \in X) \tag{1.6}$$

fulfills equation (1.5).

In [23], J. Brzdęk proved the hyperstability of the Cauchy's functional equation by an idea based on a fixed point result that can be derived from Theorem 1([24]). E. Gselmann [25] investigated the hyperstability of parametric fundamental equation of information. M. Piszczek in [26] proved the hyperstability of the general linear equation. In 2014, A. Bahyrycz and M. Piszczek in [27] studied the hyperstability of the Jensen's equation on a restricted domain. M. Piszczek and J. Szczawińska in [28] studied the hyperstability of the Drygas equation.

A function $H\colon \mathbb{R}_0^2 \to \mathbb{R}_0$ is called homogeneous of degree a real number p if it satisfies $H(tu, tv) = t^p H(u, v)$ for all $t \in \mathbb{R}_+$ and $u, v \in \mathbb{R}_0$. In the sequel, we assume that $G = (G, +)$ is an abelian group, E is an arbitrary real Banach space, $H\colon \mathbb{R}_0^2 \to \mathbb{R}_0$ is a symmetric homogeneous function of degree $p < 0$ for which there exists a positive integer n_0 such that

$$\inf\{\varepsilon((m+1)x, -mx) : m \in \mathbb{N}_{n_0}\} = 0 \tag{1.7}$$

for all $x \in G$, and $\gamma\colon G \to \mathbb{R}_+$ is a function satisfying:

(C1) $\gamma(kx) = |k|\,\gamma(x)$ for all $k \in \mathbb{Z}\backslash\{0\}$ and all $x \in G$,

(C2) $\gamma(x + y) \le \gamma(x) + \gamma(y)$ for all $x, y \in G$.

We will denote by $\mathcal{E}$ the set of all functions $\varepsilon\colon G^2 \to \mathbb{R}_0$ for which there exist a constant $c \in \mathbb{R}_0$ such that

$$\varepsilon(x, y) = cH(\gamma(x), \gamma(y)) \qquad x, y \in G. \tag{1.8}$$

Remark 1.1. Note that conditions (C1) and (C2) imply the following equality

$$\varepsilon(kx, ky) = |k|^p\,\varepsilon(x, y)$$

for all $k \in \mathbb{Z}\backslash\{0\}$ and all $x, y \in G$.

The aim goal of the paper is to establish the $\mathcal{E}$-hyperstability of (1.2) in the class of functions from a commutative group $(G, +)$ into a Banach space E by a fixed point method that can be derived from [22].

Before proceeding to the main results, we will prove the general solution of the functional equation (1.2) on an abelian group and state the fixed point theorem (Theorem 1.2) which is useful to our purpose.

We first obtain the general solution of the proposed functional equation (1.2).

Lemma 1.1. *Let $(G, +)$ be an abelian group and E be a real vector space. A function $f\colon G \to E$ satisfies the functional equation*

$$f(x+y) + \frac{1}{2}\Big[f(x-y) + f(y-x)\Big] = 2f(x) + 2f(y) \tag{1.9}$$

for all $x, y \in G$ if and only if it satisfies

$$f(x+y) + f(x-y) = 2f(x) + 2fy) \tag{1.10}$$

for all $x, y \in G$.

To present the fixed point theorem, we need the following three hypothesis [22]:

(H1) U is a nonempty set, E_2 is a Banach space, $f_1, \ldots, f_k\colon U \to U$ and $L_1, \ldots, L_k\colon U \to \mathbb{R}_+$ are given.

(H2) $\mathcal{T}\colon E_2^U \to E_2^U$ is an operator satisfying the inequality

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^{k} L_i(x)\, \|\xi(f_i(x)) - \mu(f_i(x))\| \qquad \xi, \mu \in E_2^U, x \in U.$$

(H3) $\Lambda\colon \mathbb{R}_+^U \to \mathbb{R}_+^U$ is defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)) \qquad \delta \in \mathbb{R}_+^U, x \in U.$$

Now we present the mentioned fixed point theorem.

Theorem 1.2. *[22]. Let hypotheses (H1)-(H3) be valid and functions $\varepsilon\colon U \to \mathbb{R}_+$ and $\varphi\colon U \to E_2$ satisfy the following two conditions*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \varepsilon(x) \qquad x \in U,$$
$$\varepsilon^* := \sum_{n=1}^{\infty} \Lambda^n \varepsilon(x) < \infty \qquad x \in U.$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x) \qquad x \in U.$$

Moreover,

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x) \qquad x \in U.$$

2 $\mathcal{E}$-Hyperstability of (1.2)

Using Theorem (1.2), we prove that the functional equation (1.2) is $\mathcal{E}$-hyperstable for the pair (G, E).

Theorem 2.1. *Let G be an abelian group, E be a Banach space. Let a function $f\colon G \to E$ satisfy*

$$\left\|f(x+y) + \frac{1}{2}\left[f(x-y) + f(y-x)\right] - 2f(y) - 2f(x)\right\| \leq \varepsilon(x,y) \tag{2.1}$$

for all $x, y \in G$ and for some $\varepsilon \in \mathcal{E}$. Then f satisfies the functional equation (1.2) *on G.*

Proof. Not that for some $p < 0$, we have

$$\lim_{m\to\infty} \left(2(m+1)^p + 2m^p + (2m+1)^p\right) = 0.$$

Then, there exists $n_0 \in \mathbb{N}$ such that

$$2(m+1)^p + 2(m)^p + (2m+1)^p < 1 \tag{2.2}$$

for all $m \geq n_0$.

Let $\varepsilon \in \mathcal{E}$, then there exists $c \in \mathbb{R}_0$ such that $\varepsilon(x,y) = cH(\gamma(x), \gamma(y))$. Let $f\colon G \to E$ be a function satisfy the inequality

$$\left\|f(x+y) + \frac{1}{2}\left[f(x-y) + f(y-x)\right] - 2f(y) - 2f(x)\right\| \leq \varepsilon(x,y)$$

for all $x, y \in G$. Let us fix $m \in \mathbb{N}_{n_0}$. Replacing x by $(m+1)x$ and y by $-mx$ in (2.1), we get

$$\begin{aligned} &\left\|f(x) - 2f((m+1)x) - 2f(-mx) + \frac{1}{2}f((2m+1)x) + \frac{1}{2}f((-2m-1)x)\right\| \\ &\leq \varepsilon\big((m+1)x, -mx\big) \\ &= H\big(\gamma((m+1)x), \gamma(-mx)\big) := \varepsilon_m(x) \end{aligned} \tag{2.3}$$

for all $x \in G$.
Putting

$$\mathcal{T}_m\xi(x) := 2\xi((m+1)x) + 2\xi(-mx) - \frac{1}{2}\xi((2m+1)x) - \frac{1}{2}\xi((-2m-1)x) \tag{2.4}$$

for all $x \in G$ and $\xi \in E^G$. Then the inequality (2.3) becomes

$$\|\mathcal{T}_m f(x) - f(x)\| \leq \varepsilon_m(x) \qquad x \in G. \tag{2.5}$$

Now, we define an operator $\Lambda_m\colon \mathbb{R}_+^G \to \mathbb{R}_+^G$ by

$$\Lambda_m\eta(x) := 2\eta((m+1)x) + 2\eta(-mx) + \frac{1}{2}\eta((2m+1)x) + \frac{1}{2}\eta((-2m-1)x) \tag{2.6}$$

for all $x \in G$ and $\eta \in \mathbb{R}_+^G$. This operator has the form described in $(H3)$ with $k = 4$ and $f_1(x) = (m+1)x$, $f_2(x) = -mx$, $f_3(x) = (2m+1)x$, $f_4(x) = (-2m-1)x$, $L_1(x) = L_2(x) = 2$ and $L_3(x) = L_4(x) = \frac{1}{2}$ for all $x \in G$.

Further ,

$$\begin{aligned}\Big\|\mathcal{T}_m\xi(x)-\mathcal{T}_m\mu(x)\Big\| &= \Big\|2\xi((m+1)x)+2\xi(-mx)\\ &\quad-\frac{1}{2}\xi((2m+1)x)-\frac{1}{2}\xi((-2m-1)x)\\ &\quad-2\mu((m+1)x)-2\mu(-mx)\\ &\quad+\frac{1}{2}\mu((2m+1)x)+\frac{1}{2}\mu((-2m-1)x)\Big\|\\ &\le 2\Big\|(\xi-\mu)((m+1)x)\Big\|+2\Big\|(\xi-\mu)(-mx)\Big\|\\ &\quad+\frac{1}{2}\Big\|(\xi-\mu)((2m+1)x)\Big\|\\ &\quad+\frac{1}{2}\Big\|(\xi-\mu)((-2m-1)x)\Big\|\\ &=\sum_{i=1}^{4}L_i(x)\Big\|(\xi-\mu)(f_i(x))\Big\|\end{aligned} \tag{2.7}$$

for all $x\in G$ and all $\xi,\mu\in E^G$. Therefore, In view of (2.6) and (1.8), it is easily to check that

$$\begin{aligned}\Lambda_m\varepsilon_m(x) &= 2\varepsilon_m((m+1)x)+2\varepsilon_m(-mx)+\frac{1}{2}\varepsilon_m((2m+1)x)\\ &\quad+\frac{1}{2}\varepsilon_m((-2m-1)x)\\ &=2H\Big(\gamma((m+1)(m+1)x),\gamma(-m(m+1)x)\Big)\\ &\quad+2H\Big(\gamma((m+1)(-mx)),\gamma(-m(-mx))\Big)\\ &\quad+\frac{1}{2}H\Big(\gamma((m+1)(2m+1)x),\gamma(-m(2m+1)x)\Big)\\ &\quad+\frac{1}{2}H\Big(\gamma((m+1)(-2m-1)x),\gamma(-m(-2m-1)x)\Big)\\ &=2H\Big((m+1)\gamma((m+1)x),(m+1)\gamma(-mx)\Big)+2H\Big(m\gamma((m+1)x),\\ &\quad m\gamma(-mx)\Big)+H\Big((2m+1)\gamma((m+1)x),(2m+1)\gamma(-mx)\Big)\\ &=2(m+1)^pH\Big(\gamma((m+1)x),\gamma(-mx)\Big)+2m^pH\Big(\gamma((m+1)x),\gamma(-mx)\Big)\\ &\quad+(2m+1)^pH\Big(\gamma((m+1)x),\gamma(-mx)\Big)\\ &=\Big(2(m+1)^p+2m^p+(2m+1)^p\Big)H\Big(\gamma((m+1)x),\gamma(-mx)\Big)\\ &=\Big(2(m+1)^p+2m^p+(2m+1)^p\Big)\varepsilon_m(x).\end{aligned} \tag{2.8}$$

Then,

$$\sum_{k=0}^{n}\Lambda_m^k\varepsilon_m(x)=\varepsilon_m(x)\sum_{k=0}^{n}\left(2(m+1)^p+2m^p+(2m+1)^p\right)^k \tag{2.9}$$

for all $x \in G$ and $n \in \mathbb{N}$. As $m \in \mathbb{N}_{n_0}$, we have

$$\begin{aligned} \varepsilon_m^*(x) &:= \sum_{k=0}^{\infty} \Lambda_m^k \varepsilon_m(x) \\ &= \varepsilon_m(x) \sum_{k=0}^{\infty} \Big(2(m+1)^p + 2m^p + (2m+1)^p\Big)^k \\ &= \frac{\varepsilon_m(x)}{1 - 2(m+1)^p - 2m^p - (2m+1)^p} \\ &< \infty \end{aligned}$$

for all $x \in G$. Now, it follows from Theorem (1.2) that there exists a unique solution $F_m \colon G \to E$ of the functional equation

$$\begin{aligned} F_m(x) &= 2F_m((m+1)x) + 2F_m(-mx) \\ &\quad - \frac{1}{2}F_m((2m+1)x) - \frac{1}{2}F_m((-2m-1)x) \quad x \in G, \end{aligned} \tag{2.10}$$

which is a fixed point of $\mathcal{T}_m$, such that

$$\|F_m(x) - f(x)\| \leq \frac{\varepsilon_m(x)}{1 - 2(m+1)^p - 2(m)^p - (2m+1)^p} \tag{2.11}$$

for all $x \in G$. Moreover,

$$F_m(x) = \lim_{n \to \infty} \mathcal{T}_m^n f(x) \qquad x \in G.$$

To prove that the function F_m satisfies the functional equation (1.2) on G, it suffices to prove the following inequality

$$\begin{aligned} &\Big\|\mathcal{T}_m^n f(x+y) + \frac{1}{2}\mathcal{T}_m^n f(x-y) + \frac{1}{2}\mathcal{T}_m^n f(y-x) - 2\mathcal{T}_m^n f(x) - 2\mathcal{T}_m^n f(y)\Big\| \\ &\leq \Big(2(m+1)^p + 2(m)^p + (2m+1)^p\Big)^n \varepsilon(x,y) \end{aligned} \tag{2.12}$$

for all $x, y \in G$, and $n \in \mathbb{N}$.
Indeed, if $n = 0$, then (2.12) is simply (2.1). So, take $n \in \mathbb{N}_+$ and suppose that (2.12) holds for $n \in \mathbb{N}_+$ and $x, y \in G$. Then, using (2.4) and (2.12), we have

$$\begin{aligned} &\Big\|\mathcal{T}_m^{n+1} f(x+y) + \tfrac{1}{2}\mathcal{T}_m^{n+1} f(x-y) + \tfrac{1}{2}\mathcal{T}_m^{n+1} f(y-x) - 2\mathcal{T}_m^{n+1} f(x) \\ &\quad -2\mathcal{T}_m^{n+1} f(y)\Big\| = \Big\|2\mathcal{T}_m^n f((m+1)(x+y)) + 2\mathcal{T}_m^n f(-m(x+y)) \\ &\quad -\tfrac{1}{2}\mathcal{T}_m^n f((2m+1)(x+y)) - \tfrac{1}{2}\mathcal{T}_m^n f((-2m-1)(x+y)) \\ &\quad +\mathcal{T}_m^n f((m+1)(x-y)) + \mathcal{T}_m^n f(-m(x-y)) \\ &\quad -\tfrac{1}{4}\mathcal{T}_m^n f((2m+1)(x-y)) - \tfrac{1}{4}\mathcal{T}_m^n f((-2m-1)(x-y)) \\ &\quad +\mathcal{T}_m^n f((m+1)(y-x)) + \mathcal{T}_m^n f(-m(y-x)) \\ &\quad -\tfrac{1}{4}\mathcal{T}_m^n f((2m+1)(y-x)) - \tfrac{1}{4}\mathcal{T}_m^n f((-2m-1)(y-x)) \end{aligned}$$

$$
\begin{aligned}
&-4\mathcal{T}_m^n f((m+1)x) - 4\mathcal{T}_m^n f(-mx)\\
&+\mathcal{T}_m^n f((2m+1)x) + \mathcal{T}_m^n f((-2m-1)x)\\
&-4\mathcal{T}_m^n f((m+1)y) - 4\mathcal{T}_m^n f(-my)\\
&+\mathcal{T}_m^n f((2m+1)y) + \mathcal{T}_m^n f((-2m-1)y)\Big\|\\
\leq\ & 2\Big\|\mathcal{T}^n f((m+1)(x+y)) + \tfrac{1}{2}\mathcal{T}^n f((m+1)(x-y))\\
&+\tfrac{1}{2}\mathcal{T}^n f((m+1)(y-x)) - 2\mathcal{T}^n f((m+1)x) - 2\mathcal{T}^n f((m+1)y)\Big\|\\
&+2\Big\|\mathcal{T}^n f(-m(x+y)) + \tfrac{1}{2}\mathcal{T}^n f(-m(x-y))\\
&+\tfrac{1}{2}\mathcal{T}^n f(-m(y-x)) - 2\mathcal{T}^n f(-mx) - 2\mathcal{T}^n f(-my)\Big\|\\
&+\tfrac{1}{2}\Big\|\mathcal{T}^n f((2m+1)(x+y)) + \tfrac{1}{2}\mathcal{T}^n f((2m+1)(x-y))\\
&+\tfrac{1}{2}\Big\|\mathcal{T}^n f((2m+1)(y-x)) - 2\mathcal{T}^n f((2m+1)x) - 2\mathcal{T}^n f((2m+1)y)\Big\|\\
&+\tfrac{1}{2}\Big\|\mathcal{T}^n f((-2m-1)(x+y)) + \tfrac{1}{2}\mathcal{T}^n f((-2m-1)(x-y))\\
&+\tfrac{1}{2}\mathcal{T}^n f((-2m-1)(y-x)) - 2\mathcal{T}^n f((-2m-1)x) - 2\mathcal{T}^n f((-2m-1)y)\Big\|\\
\leq\ & \Big(2(m+1)^p + 2m^p + (2m+1)^p\Big)^n \Big(2(m+1)^p + 2m^p + (2m+1)^p\Big)\varepsilon(x,y)\\
=\ & \Big(2(m+1)^p + 2(m)^p + (2m+1)^p\Big)^{n+1}\varepsilon(x,y).
\end{aligned}
$$

By induction, we have shown that (2.12) holds for all $x, y \in G$. Letting $n \to \infty$ in (2.12), we get

$$F_m(x+y) + \frac{1}{2}\left[F_m(x-y) + F_m(y-x)\right] = 2F_m(x) + 2F_m(y) \tag{2.13}$$

for all $x, y \in G$. Thus, we have proved that for every $m \in \mathbb{N}_{n_0}$ there exists a function $F_m : G \to E$ which is a solution of the functional equation (1.2) on G and satisfies

$$\|f(x) - F_m(x)\| \leq \frac{\varepsilon_m(x)}{1 - 2\lambda(m+1) - 2\lambda(m) - \lambda(2m+1)} \tag{2.14}$$

for all $x \in G$. Next, we prove that $F_m = F_k$ for all $m, k \in \mathbb{N}_{n_0}$.
Let us fix $m, k \in \mathbb{N}_{n_0}$. Note that F_m and F_k satisfy (2.13). Hence, by replacing x by $(m+1)x$ and y by $-mx$ in (2.13), we get,

$$
\begin{aligned}
F_m(x) &= 2F_m((m+1)x) + 2F_m(-mx) - \frac{1}{2}F_m((2m+1)x) - \frac{1}{2}F_m((-2m-1)x),\\
F_k(x) &= 2F_k((m+1)x) + 2F_k(-mx) - \frac{1}{2}F_k((2m+1)x) - \frac{1}{2}F_k((-2m-1)x)
\end{aligned}
$$

for all $x \in G$, that is $\mathcal{T}_m F_m = F_m$, $\mathcal{T}_m F_k = F_k$ and

$$
\begin{aligned}
\|F_m(x) - F_k(x)\| \quad &\leq \frac{\varepsilon_m(x)}{1 - 2\lambda(m+1) - 2\lambda(m) - \lambda(2m+1)}\\
&+\frac{\varepsilon_k(x)}{1 - 2\lambda(k+1) - 2\lambda(k) - \lambda(2k+1)}
\end{aligned}
$$

for all $x \in G$. Hence, by linearity of Λ_m and (2.8), we get

$$\begin{aligned} \|F_m(x) - F_k(x)\| &= \|\mathcal{T}_m^n F_m(x) - \mathcal{T}_m^n F_k(x)\| \\ &\leq \frac{\Lambda_m^n \varepsilon_m(x)}{1 - 2\lambda(m+1) - 2\lambda(m) - \lambda(2m+1)} \\ &\quad + \frac{\Lambda_m^n \varepsilon_k(x)}{1 - 2\lambda(k+1) - 2\lambda(k) - \lambda(2k+1)} \\ &\leq \frac{\Big(2(m+1)^p + 2(m)^p + (2m+1)^p\Big)^n \varepsilon_m(x)}{1 - 2\lambda(m+1) - 2\lambda(m) - \lambda(2m+1)} \\ &\quad + \frac{\Big(2(m+1)^p + 2(m)^p + (2m+1)^p\Big)^n \varepsilon_k(x)}{1 - 2\lambda(k+1) - 2\lambda(k) - \lambda(2k+1)} \end{aligned}$$

for all $x \in G$ and $n \in \mathbb{N}$. Now letting $n \to \infty$ we get $F_m = F_k =: F$. Thus, in view of (2.14), we have

$$\|f(x) - F(x)\| \leq \frac{\varepsilon_m(x)}{1 - 2\lambda(m+1) - 2\lambda(m) - \lambda(2m+1)}$$

for all $x \in G$ and all $m \in \mathbb{N}_{n_0}$.
Since (2.13), the function F is a solution of (1.2).

To prove the uniqueness of the function F, let us assume that there exists a function $F' : G \to E$ which satisfies (1.2) and the inequality

$$\left\|f(x) - F'(x)\right\| \leq \frac{\varepsilon_m(x)}{1 - 2\lambda(m+1) - 2\lambda(m) - \lambda(2m+1)}$$

for all $x \in G$ and all $m \in \mathbb{N}_{n_0}$. Then it follows easily that

$$\left\|F(x) - F'(x)\right\| \leq \frac{2\varepsilon_m(x)}{1 - 2\lambda(m+1) - 2\lambda(m) - \lambda(2m+1)}$$

for all $x \in G$ and all $m \in \mathbb{N}_{n_0}$. Further, $\mathcal{T}_m F' = F'$ for each $m \in \mathbb{N}_{n_0}$. Therefore, with a fixed $m \in \mathbb{N}_{n_0}$,

$$\begin{aligned} \left\|F(x) - F'(x)\right\| &= \left\|\mathcal{T}_m^n F(x) - \mathcal{T}_m^n F'(x)\right\| \\ &\leq \frac{2\Lambda_m^n \varepsilon_m(x)}{1 - 2\lambda(m+1) - 2\lambda(m) - \lambda(2m+1)} \\ &\leq \frac{\Big(2(m+1)^p + 2(m)^p + (2m+1)^p\Big)^n \varepsilon_m(x)}{1 - 2\lambda(m+1) - 2\lambda(m) - \lambda(2m+1)} \end{aligned}$$

for all $x \in G$ and $n \in \mathbb{N}$. By letting $n \to \infty$, we get $F' = F$, which yields

$$\|f(x) - F(x)\| \leq \frac{\varepsilon_m(x)}{1 - 2\lambda(m+1) - 2\lambda(m) - \lambda(2m+1)}$$

for all $x \in G$ and all $m \in \mathbb{N}_{n_0}$. Next, in view of (1.7), we have

$$\inf\left\{\frac{\varepsilon_m(x)}{1 - 2\lambda(m+1) - 2\lambda(m) - \lambda(2m+1)} : m \in \mathbb{N}_{n_0}\right\} = 0 \tag{2.15}$$

for all $x \in G$, this means that $f(x) = F(x)$ for $x \in G$, which implies that f satisfies the functional equation (1.2) on G and the proof of the theorem is complete.

□

In a similar way we can prove that Theorem (2.1) holds if the inequality (2.1) is defined on $G\backslash\{0\} := G_0$.

Theorem 2.2. *Let G be an abelian group, E be a Banach space. Let $\mathcal{E}$ be the set of all functions $\varepsilon\colon G_0^2 \to \mathbb{R}_0$ which satisfy the conditions as stated in the beginning of Section (2). Let a function $f\colon G \to E$ satisfy*

$$\left\| f(x+y) + \frac{1}{2}\left[f(x-y) + f(y-x)\right] - 2f(y) - 2f(x) \right\| \le \varepsilon(x,y)$$

for all $x, y \in G_0$ and for some $\varepsilon \in \mathcal{E}$. Then f satisfies the functional equation (1.2) *on G_0.*

Theorem 2.3. *Let G be an abelian group, E be a Banach space. Let $\mathcal{E}$ be the set of all functions $\varepsilon\colon G_0^2 \to \mathbb{R}_0$ which satisfy the conditions as stated in the beginning of Section (2). Let a function $f\colon G \to E$ satisfy $f(0) = 0$ and*

$$\left\| f(x+y) + \frac{1}{2}\left[f(x-y) + f(y-x)\right] - 2f(y) - 2f(x) \right\| \le \varepsilon(x,y)$$

for all $x, y \in G_0$ and for some $\varepsilon \in \mathcal{E}$. Then f satisfies the functional equation (1.2) *on G.*

Proof. It easy to see that if $f(0) = 0$, then f satisfies the functional equation (1.2) on the whole G. $\square$

From Theorem (2.2), we can obtain the following three corollaries with the cases $\varepsilon(x,y) = c\left(\|x\|^p + \|y\|^q\right)$, $\varepsilon(x,y) = c\,\|x\|^p\,\|y\|^q$ and $\varepsilon(x,y) = c\big(\|x\|^p + \|y\|^q + \|x\|^p\,\|y\|^q\big)$ as natural results.

Corollary 2.4. *Let E and F be a normed space and a Banach space, respectively. Assume that $X := (X,+)$ is a subgroup of the group $(E,+)$, $p < 0$, $q < 0$ and $c \ge 0$. If a function $f\colon X \to F$ satisfies the inequality*

$$\left\| f(x+y) + \frac{1}{2}\left[f(x-y) + f(y-x)\right] - 2f(x) - 2f(y) \right\| \le c\big(\|x\|^p + \|y\|^q\big) \tag{2.16}$$

for all $x, y \in X\backslash\{0\}$. Then the function f is a solution of the functional equation (1.2) *on $X\backslash\{0\}$.*

Proof. By taking $\mathcal{E}$ the set of all functions $\varepsilon\colon (X\backslash\{0\})^2 \to \mathbb{R}_0$ such that

$$\varepsilon(x,y) = c\big(\|x\|^p + \|y\|^q\big),$$

for some $c \in \mathbb{R}_0$ and for all $(x,y) \in (X\backslash\{0\})^2$. Define $H\colon \mathbb{R}_+^2 \to \mathbb{R}_0$ by $H(u,v) = c(u^p + v^p)$ for some $p < 0$ and for all $u, v \in \mathbb{R}_+$ and $\gamma\colon E \to \mathbb{R}_0$ by $\gamma(x) = \|x\|$ for all $x \in E$.
It is easily seen that H is monotonically symmetric homogeneous function of degree $p < 0$ and conditions indicated in the start of the second section are fulfilled. Therefore every function $f\colon E \to F$ satisfying (2.16) is a solution of the functional equation (1.2) on $X\backslash\{0\}$. $\square$

Note that if $f(0) = 0$ and f satisfies (2.16) on $X\backslash\{0\}$, then from Theorem (2.3) we obtain the following hyperstabilty result for (1.2) on X.

Corollary 2.5. *Let E and F be a normed space and a Banach space, respectively. Assume that X is a subgroup of the group $(E,+)$, and $p < 0$, $q < 0$ and $c \ge 0$. Let a function $f\colon X \to E$ satisfy $f(0) = 0$ and*

$$\left\| f(x+y) + \frac{1}{2}\left[f(x-y) + f(y-x)\right] - 2f(x) - 2f(y) \right\| \le c\big(\|x\|^p + \|y\|^q\big) \tag{2.17}$$

for all $x, y \in X\backslash\{0\}$. Then f satisfies the functional equation (1.2) *on X.*

In the case where functions $\varepsilon \in \mathcal{E}$ are given by

$$\varepsilon(x,y) = c\,\|x\|^p \,.\, \|y\|^q \qquad x,y \in X\backslash\{0\},$$

with some real $c \in \mathbb{R}_0$ and $p,q \in \mathbb{R}$ such that $p+q<0$, we also get an analogous conclusion.

Corollary 2.6. *Let E and F be a normed space and a Banach space, respectively. Assume that X is a subgroup of the group $(E,+)$, and $c \geq 0$, $p,q \in \mathbb{R}$, $p+q<0$ are given. If $f\colon X \to F$ satisfies*

$$\left\| f(x+y) + \frac{1}{2}\left[f(x-y)+f(y-x)\right] - 2f(y) - 2f(x) \right\| \leq c\,\|x\|^p \,.\, \|y\|^q \tag{2.18}$$

for all $x,y \in E\backslash\{0\}$, then f satisfies the functional equation (1.2) *on $X\backslash\{0\}$.*

Corollary 2.7. *Let E and F be a normed space and a Banach space, respectively. Assume that X is a subgroup of the group $(E,+)$, and $p<0$, $q<0$, $p+q<0$ and $c \geq 0$. If $f : X \to F$ satisfies*

$$\begin{aligned} &\left\| f(x+y) + \frac{1}{2}\left[f(x-y)+f(y-x)\right] - 2f(y) - 2f(x) \right\| \\ &\quad \leq c\Big(\|x\|^p + \|y\|^q + \|x\|^p \,.\, \|y\|^q \Big) \end{aligned} \tag{2.19}$$

for all $x,y \in X\backslash\{0\}$, then f satisfies the functional equation (1.2) *on $X\backslash\{0\}$.*

We know that any norm that satisfies the parallelogram law is bound to have been originated from a scalar product. The following corollary gives a characterization of the inner product space, which is one of the applications of Corollary (2.4).

Corollary 2.8. *Let E be a normed space and*

$$\sup_{x,y\in E\backslash\{0\}} \frac{\left| \|x+y\|^2 + \|x-y\|^2 - 2\,\|x\|^2 - 2\,\|y\|^2 \right|}{\|x\|^p + \|y\|^p} < \infty \tag{2.20}$$

for some $p<0$. Then E is an inner product space.

Proof. Write $f(x) = \|x\|^2$ for $x \in E$. Then from Corollary (2.5), we have

$$f(x+y) + \frac{1}{2}\left[f(x-y)+f(y-x)\right] = 2f(x) + 2f(y) \qquad x,y \in E.$$

That implies

$$\|x+y\|^2 + \frac{1}{2}\Big[\, \|x-y\|^2 + \|y-x\|^2 \Big] = 2\,\|x\|^2 + 2\,\|y\|^2 \qquad x,y \in E.$$

Thus, the norm $\|.\|$ on E obeys the parallelogram low:

$$\|x+y\|^2 + \|x-y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2 \qquad x,y \in E.$$

Therefore, E is an inner product space. □

Competing Interests

The authors declare that no competing interests exist.

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