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Ground State Solutions for A Quasilinear Elliptic Problem with a Convection Term

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Abstract

By a sub-supersolution argument and a perturbed argument, we improve the earlier results concerning the existence of ground state solutions to a quasilinear elliptic problem

$$-\text{div}(|\nabla u|^{p-2}\nabla u) + p(x)|\nabla u|^q = f(x,u), u > l \geq 0, x \in \mathbb{R}^N, \lim_{|x|\to\infty} u(x) = l,$$

where $p \geq 2, q > p-1, p(x) \in C^{\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$ is non-negative and $f : \mathbb{R}^N \times (0,\infty) \to [0,\infty)$ is a locally Hölder continuous function which may be singular at zero.

Keywords: existenceground solution, quasilinear elliptic, convection term.
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1 Introduction and The Main Results

In this paper,we are concerned with the existence of ground state solutions for the following problem

$$-\text{div}(|\nabla u|^{p-2}\nabla u) + p(x)|\nabla u|^q = f(x,u), u > l \geq 0, x \in \mathbb{R}^N, \lim_{|x|\to\infty} u(x) = l,$$

We first consider $l = 0$, then the problem becomes as follows

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) + p(x)|\nabla u|^q = f(x,u), \\ u > 0, x \in \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1}$$

where $p \geq 2, q > p-1, p(x) \in C^{\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$ is non-negative and $f : \mathbb{R}^N \times (0,\infty) \to [0,\infty)$ is a locally Hölder continuous function which may be singular at zero.

In recent years, the study of ground state solutions, that is, positive solutions defined in the whole space $\mathbb{R}^N$ and decaying to zero at infinity, has received a lot of interest and numerous existence

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results have been established. Concerning ground state solutions for elliptic problems with a convection term, we refer readers to (Xue and Shao, 2009; Xue, 2011; Dinu, 2003; Goncalves and Silva, 2010), and the reference therein. Meanwhile , we also see that most of these investigations focus on the following problem

$$\begin{cases} -\triangle u + p(x)|\nabla u|^q = b(x)g(u), \\ u > 0, x \in \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{2}$$

Throughout the papers, authors assume that $g \in C^1((0,\infty),(0,\infty))$, additionally, with regard to g, consider the hypothesis

(g1) g is increasing on $(0,\infty)$;

(g2) $\lim_{s\longrightarrow 0^+} g(s) = \infty$;

(g3) g is bounded in a neighborhood of ∞;

(g4) $\lim_{s\longrightarrow 0^+} \frac{g(s)}{s^{p-1}} = \infty$;

(g5) $\frac{g(s)}{(s+c_0)^{p-1}}$ is decreasing on $(0,\infty)$;

(g6) $\frac{g(s)}{s^{p-1}}$ is decreasing on $(0,\infty)$;

(g7)$\lim_{s\longrightarrow\infty} \frac{g(s)}{s^{p-1}} = 0$;

And $b(x)$ satisfies
(b1) $b : \mathbb{R}^N \to (0,\infty)$ is a locally Hölder continuous function,
(b2) the problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) = b(x), \\ u > 0, x \in \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{3}$$

has a solution $w \in C^{1+\alpha}_{loc}(\mathbb{R}^N)$.

(Dinu, 2003) showed that problem (2) has a unique solution in the case when $g(u) = u^{-\gamma}$ with $\gamma > 0$. Later the paper (Xue and Shao, 2009) has showed problem (2) has at least one solution if g satisfies (g4) and (g7) and b satisfies (b1) and (b2).

For corresponding quasilinear elliptic equations, more specifically, when $p(x) = 0$, in (Yang and Yu, 2010) studied the following model

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) = \rho(x)f(u), \\ u > l, x \in \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = l, \end{cases} \tag{4}$$

where N$\geq$ 3 and $l\geq$ 0 is a real number. (Yuan and Yang, 2010) has showed the existence and asymptotic behavior of radially symmetric ground states of (4). (Liu and Yang, 2010) studied

$$\begin{cases} -\text{div}(|\nabla u|^{m-2}\nabla u) - |\nabla u|^{q(m-1)} = b(x)g(u), \\ u > 0, x \in \Omega, \\ u|_{\partial\Omega} = +\infty, \end{cases} \tag{5}$$

where Ω is a C^2 bounded domain with smooth boundary, $m > 1$, $q \in (1, \frac{m}{m-1}]$. (Liu and Yang, 2010) has showed the existence of large solutions of (5).

But the results about the existence of ground state solutions for a quasilinear elliptic problem with a convection term are few. For the following problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) + h(x)|\nabla u|^q = b(x)g(u), \\ u > 0, x \in \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{6}$$

(Shen and Zhang, 2011) has showed that (6) has at least one solution if g satisfies $(g4)$ and $(g7)$, and b satisfies $(b1)$ and $(b2)$.

The purpose of this paper is to investigate the existence of ground state solutions for problem (1), which includes problem (6) as a particular case. And we modify the method developed in (Xue, 2011) and extends the results obtained in (Xue, 2011; Shen and Zhang, 2011).
In this paper, we suppose that

(f1) $f(x, s)$ is locally Hölder continuous on $\mathbb{R}^N \times (0, \infty)$ and continuously differentiable in the variable s;

(f2) $f(x, s) \le b(x)g(s)$ for all $(x, s) \in \mathbb{R}^N \times (0, \infty)$, where b satisfies $(b1)$,$(b2)$, g satisfies
$(g8)\ \lim_{s\longrightarrow\infty} \sup \frac{g(s)}{s^{p-1}} < ||w||_\infty^{1-p}$, where w is the solution of problem (1.3) and $||w||_\infty := \max_{x\in\mathbb{R}^N w(x)}$;

(f3) There exists $s_0 > 0$ such that $f(x, s) \ge a(x)n(s)$ for all $(x, s) \in \mathbb{R}^N \times (0, s_0)$, where $a : \mathbb{R}^N \to (0, \infty)$ is locally Hölder continuous and n satisfies
$(n1) n : (0, s_0) \to (0, \infty)$ is continuous.

Our main result is summarized in the following theorem.

Theorem 1. Let $q > p - 1$, $p(x) \in C^\alpha_{loc}(\mathbb{R}^N)$ be non-negative . If f satisfies $(f1) - (f3)$, g satisfies $(g8)$, n satisfies $(n1)$ and $(n2)\ \lim_{s\longrightarrow 0^+} \frac{n(s)}{s^{p-1}} = \infty$, then problem (1.1) has at least one solution $u \in C^{1+\alpha}(\mathbb{R}^N)$.

The paper is organized as follows. In Section 2, we provide a suitable supersolution for problem (1) and show the existence of positive solutions in bounded domain. In Section 3, we prove Theorem 1, moreover, we will study the case $l > 0$.

2 Preliminary

The result below will provide a suitable supersolution for problem (1).

Lemma 2.1. If b satisfy $(b1)$ and $(b2)$, and g satisfies $(g8)$, then there exists a function $v := \Psi(\gamma w(x)) \in C^1_{loc}(\mathbb{R}^N)$ such that

$$-\text{div}(|\nabla v|^{p-2}\nabla v) \ge b(x)g(v(x)), \quad v(x) > 0, x \in \mathbb{R}^N, \lim_{|x|\to\infty} v(x) = 0, \tag{7}$$

for large $\gamma \ge 1$, where w is the solution of problem (3).

Proof. Since g satisfies $(g8)$, we define

$$\widehat{g}(t) := \sup\{\frac{g(s)}{s^{p-1}} : s > t\}, \qquad t > 0. \tag{8}$$

we denote that $\widehat{g}$ is non-increasing, positive and $\widehat{g}(t) \ge \frac{g(t)}{t^{p-1}}$.
Furthermore, by $(g8)$ we have $\widehat{g}(t) < |w|_\infty^{1-p}$ for sufficiently large t.
Let

$$h(t) := \frac{2}{t}\int_{\frac{t}{2}}^{t} \widehat{g}(s)ds, \qquad t > 0, \tag{9}$$

It is shown in (Ladyzenskaja and Ural'tseva, 1968) that h is C^1, non-increasing and $\widehat{g}(t) \le h(t) \le \widehat{g}(\frac{t}{2})$ for all $t \in (0, \infty)$.

Since h is non-increasing, we note that $h(t) \to \alpha < |w|_\infty^{1-p}$ at $t \longrightarrow \infty$ for some $\alpha \in [0, \infty)$.
Now let us set

$$\eta(t) := \int_0^t \frac{1}{h^{\frac{1}{p-1}}(s)} ds, \qquad t > 0, \tag{10}$$

On using $\widehat{g}(t) < |w|_\infty^{1-p}$ in (9) for sufficiently large $t > 0$, we see from (10) that

$$\eta(\gamma) > \gamma |w|_\infty. \tag{11}$$

for a sufficiently large $\gamma \geq 1$.
Let $\Psi = \eta^{-1}$ be the inverse function of η , i.e, Ψ satisfies

$$\int_0^{\Psi(t)} \frac{1}{h^{\frac{1}{p-1}}(s)} ds = t, \qquad t \in [0, \infty), \tag{12}$$

By direct calculation ,we see that

$\Psi^{'}(t) = h^{\frac{1}{p-1}}(\Psi(t)) > 0$, $\Psi(t) > 0$, for $t > 0$ and $\Psi(0) = 0$.

By condition $(b2)$, we take a solution w of (3) with $\Omega = R^N$. Let us set $v(x) := \Psi(\gamma w(x))$ for all $x \in \Omega$, we note from (11) that

$$v(x) = \Psi(\gamma w(x)) \leq \Psi(\gamma |w|_\infty) < \gamma, \tag{13}$$

A simple computation shows that v has the desired properties.
Indeed , on recalling $-\text{div}(|\nabla w|^{p-2} \nabla w) = b(x)$, we see that

$$\begin{aligned}
-\text{div}(|\nabla v|^{p-2}\nabla v) &= -\text{div}(|h^{\frac{1}{p-1}}(v)\gamma \nabla w|^{p-2} \cdot h^{\frac{1}{p-1}}(v)\gamma \nabla w) \\
&= -\text{div}(h(v) \cdot \gamma^{p-1} \cdot |\nabla w|^{p-2} \nabla w) \\
&= -\gamma^{p-1} \cdot h(v)\text{div}(|\nabla w|^{p-2}\nabla w) - |\nabla w|^{p-2}\nabla w \cdot \gamma^{p-1} h^{'}(v)\Psi^{'}(\gamma w(x)) \cdot \gamma \nabla w \\
&= -\gamma^{p-1} \cdot h(v)\text{div}(|\nabla w|^{p-2}\nabla w) - \gamma^p |\nabla w|^p h^{'}(v) h^{\frac{1}{p-1}}(\Psi(\gamma w(x))) \\
&\geq -\gamma^{p-1} \cdot h(v)\text{div}(|\nabla w|^{p-2}\nabla w) \\
&= b(x)\gamma^{p-1} h(v) \\
&> v^{p-1} b(x) \frac{g(v)}{v^{p-1}} \\
&= b(x) g(v) \quad x \in \mathbb{R}^N,
\end{aligned}$$

We have used (13) in the last inequality. Since $\Psi(0) = 0$, it is clear that $v(x) \longrightarrow 0$, as $|x| \longrightarrow \infty$.

Remark 2.2. Since $\gamma \geq 1$, by Lemma 2.1, we have $-\text{div}(|\nabla v|^{p-2}\nabla v) \geq b(x)\widehat{g}(v)$,

Consider the following problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) + p(x)|\nabla u|^q = f(x, u), \\ u > 0, x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{14}$$

where Ω is a smooth bounded domain. Next, we show the existence of problem (14) by a sub-supersolution method.
For the convenience, we denote $|u|_\infty = \max_{x \in \Omega} |u(x)|$ whenever $u \in C(\overline{\Omega})$

Lemma 2.3. Let $p > 2, q > p-1,\ \ p(x) \in C^{\alpha}(\overline{\Omega}),\ \ p(x) \geq 0,$ If f satisfies $(f1)-(f3)$, g satisfies $(g8)$, n satisfies $(n1)$ and $(n2)$, then problem (14) has at least one solution $u \in C(\overline{\Omega}) \cap C^{1+\alpha}(\Omega)$.

Proof. Let $\phi_1 \in C(\overline{\Omega}) \cap C^{1+\alpha}(\Omega)$ be the first eigenfunction corresponding to the first eigenvalue λ_1 of

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u,\ \ u > 0,\ \ x \in \Omega;\ \ u = 0,\ \ x \in \partial\Omega, \tag{15}$$

Let $\beta = \frac{q}{q-p+1}$, It follows by $(n2)$ that there exists a positive constant $\delta_1 \in (0, \min\{1, s_0\})$ such that

$$\frac{n(s)}{s^{p-1}} \geq \frac{\lambda_1 \beta^{p-1} + |p(x)|_\infty \beta^q |\nabla\phi_1|_\infty^q}{\min_{x\in\overline{\Omega}} a(x)}, \quad \forall s \in (0, \delta_1),$$

Here $a(x)$ is the function in the condition $(f3)$. Let $\underline{u} = m\phi_1^\beta$ with $m \in (0, \min\{1, \frac{\delta_1}{|\phi_1|_\infty^\beta}\})$. Since $m^{q-(p-1)} < 1$, we see that

$$\begin{aligned}
-\text{div}(|\nabla \underline{u}|^{p-2}\nabla \underline{u}) + p(x)|\nabla \underline{u}|^q &= -\text{div}(\beta^{p-1}m^{p-1}|\phi_1^{\beta-1}\nabla\phi_1|^{p-2}\phi_1^{\beta-1}\nabla\phi_1) \\
&\quad + p(x)\beta^q m^q \phi_1^{(\beta-1)q}|\nabla\phi_1|^q \\
&= -\beta^{p-1}m^{p-1}\phi_1^{(\beta-1)(p-1)}\text{div}(|\nabla\phi_1|^{p-2}\nabla\phi_1) \\
&\quad - \beta^{p-1}m^{p-1}(\beta-1)(p-1)|\nabla\phi_1|^p\phi_1^{(\beta-1)(p-1)-1} \\
&\quad + p(x)\beta^q m^q \phi_1^{(\beta-1)q}|\nabla\phi_1|^q \\
&\leq \lambda_1\beta^{p-1}m^{p-1}\phi_1^{(p-1)\beta} + p(x)\beta^q m^q\phi_1^{(\beta-1)q}|\nabla\phi_1|^q \\
&\leq \min_{x\in\overline{\Omega}} a(x)n(m\phi_1^\beta) \\
&= a(x)n(\underline{u}) \\
&\leq f(x,\underline{u}), \quad x \in \Omega
\end{aligned}$$

i.e., $\underline{u} = m\phi_1^\beta$ is a subsolution to problem (14). Since $(m\phi_1^\beta)^{p-1} \leq 1$ and $\forall t > 0, \widehat{g}(t) \geq \frac{g(t)}{t^{p-1}}$, combining with $(f2)$, we get

$$-\text{div}(|\nabla \underline{u}|^{p-2}\nabla \underline{u}) \leq b(x)g(\underline{u}) \leq b(x)\underline{u}^{p-1}\widehat{g}(\underline{u}) \leq b(x)\widehat{g}(\underline{u}), \tag{16}$$

On the other hand, we construct a super-solution denoted by $\overline{u} := \Psi(\gamma w_\Omega)$, where γ and Ψ are defined as in Lemma 2.1, and w_Ω is the solution of the following problem

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = b(x),\ \ u > 0, x \in \Omega, u|_{\partial\Omega} = 0,$$

Therefore, proceed as in the proof of Lemma 2.1, we have

$$-\text{div}(|\nabla \overline{u}|^{p-2}\nabla \overline{u}) \geq b(x)g(\overline{u}),\ \ x \in \Omega,$$

By (f2),

$$-\text{div}(|\nabla \overline{u}|^{p-2}\nabla \overline{u}) \geq f(x,\overline{u}),\ \ x \in \Omega,$$

i.e, $\overline{u} = \Psi(\gamma w_\Omega)$ is a super-solution to problem (14). By Remark 2.2, we obtain that

$$-\text{div}(|\nabla \overline{u}|^{p-2}\nabla \overline{u}) \geq b(x)\widehat{g}(\overline{u}),\ \ x \in \Omega, \tag{17}$$

Since $\widehat{g}$ is non-increasing by the comparison principle argument, we can obtain (16) and (17) that $\underline{u}(x) \leq \overline{u}(x),\ \ x \in \Omega$. It follows by (Yang, 2006) that problem (14) has at least one solution $u \in C(\overline{\Omega}) \cap C^{1+\alpha}(\Omega)$ in the ordered interval $[\underline{u}, \overline{u}]$.

The proof of Lemma 2.3 is finished.

Remark 2.4. By a simple comparison argument, we have that $w_\Omega \le w$. Here, the function w is defined in condition (b2), and w_Ω is in Lemma 2.3. Therefore, $v_\Omega \le v$, where v is as Lemma 2.1.

3 Proof of Theorem 1.

Consider the perturbed problem

$$-\text{div}(|\nabla u_k|^{p-2}\nabla u_k) + p(x)|\nabla u_k|^q = f(x, u_k), u_k > 0, x \in B(0, k), u_k = 0, x \in \partial B(0, k), \quad (18)$$

where $B(0, k) = \{x \in \mathbb{R}^N : |x| < k\}, \quad k = 1, 2, 3, \cdots$. It follows by Lemma 2.3 that problem (18) has one solution $u_k \in C^{1+\alpha}(B(0, k)) \cap C(\overline{B}(0, k))$. Put

$$u_k(x) = 0, \quad \forall |x| > k.$$

Let v be as in Lemma 2.1, we assert that

$$u_k(x) \le v(x), \quad x \in \mathbb{R}^N, k = 1, 2, 3 \cdots. \quad (19)$$

Now, we need to estimate $\{u_k\}$. For any bonded $C^{2+\alpha}$-smooth domain$\Omega^{'} \subset \mathbb{R}^N$, take Ω_1 and Ω_2 with $C^{2+\alpha}$-smooth boundaries, and K_1 large enough, such that

$$\Omega^{'} \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset B_k, \quad k \ge K_1.$$

Note that

$$v(x) \ge u_k(x) \ge \underline{u}(x) > 0, \quad \forall \ \ x \in B(0, K_1), \quad (20)$$

when $B(0, K_1)$ is the substitution for Ω in the proof of Lemma 2.3. Let

$$\rho_k(x) = f(x, u_k(x)) - p(x)|\nabla u_k(x)|^q, x \in \overline{B}(0, K_1),$$

For $k \in N$ We consider the problem

$$-\text{div}(|\nabla u_k|^{p-2}\nabla u_k) = \rho_k(x), \ \ x \in \Omega^{'}, \ \ u_k = \underline{u}, \ \ x \in \partial\Omega^{'}, \quad (21)$$

Since $\underline{u}$ is a sub-solution and v is a super-solution, the above problem has at least a solution $\underline{u}(x) \le u_k(x) \le v(x)$. This in particular gives local bounds for the sequence $\{u_k\}$ which in turn leads to local bounds in $C^{1+\alpha}$. Thus for every $m \in N$, we can select a sequence $\{u_k^m\}$ which converges in $C^{1+\alpha}(\overline{\Omega'})$. A diagonal procedure gives a subsequence (denoted again by u_k) which converges to a function u in $C^1(\overline{\Omega'})$, and u satisfies

$$-\text{div}(|\nabla u|^{p-2}\nabla u) + p(x)|\nabla u|^q = f(x, u), \ \ x \in \overline{\Omega'},$$

By (20), we obtain that

$$u > 0, \ \ x \in \overline{\Omega'}$$

and we can obtain that $u \in C^{1+\alpha}(\overline{\Omega'})$. Since $\Omega^{'}$ is arbitrary, we also see that $u \in C^{1+\alpha}_{loc}(\mathbb{R}^N)$. It follows by (19) that

$$\lim_{|x| \longrightarrow \infty} u(x) = 0.$$

The proof is finished.

4 The case $l > 0$

Next,we will consider the following problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) + p(x)|\nabla u|^q = f(x,u), \\ u > l > 0, x \in \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = l, \end{cases} \tag{22}$$

Theorem 2. Let $q > p-1$, $p(x) \in C^{\alpha}_{loc}(\mathbb{R}^N)$ be non-negative . If f satisfies $(f1)-(f3)$, g satisfies $(g8)$, n satisfies $(n1)$, $(n2)$ $\lim_{s\longrightarrow 0^+} \frac{n(s)}{s^{p-1}} = \infty$, and (n3) $n(x)$ is increasing on (0,∞), then problem (22) has at least one solution $u \in C^{1+\alpha}(\mathbb{R}^N)$.

Lemma 2.5. If b satisfy $(b1)$ and $(b2)$, and g satisfies $(g8)$, then there exists a function $v := \Phi(\beta w(x)) + l \in C^1_{loc}(\mathbb{R}^N)$ such that

$$-\text{div}(|\nabla v|^{p-2}\nabla v) \geq b(x)g(v(x)), \quad v(x) > l, x \in \mathbb{R}^N, \lim_{|x|\to\infty} v(x) = l, \tag{23}$$

for large $\beta \geq 1$, where w is the solution of problem (3).

Proof. Since g satisfies $(g8)$, we define

$$\widehat{g}(t) := \sup\{\frac{g(s)}{s^{p-1}} : s > t\}, \qquad t > 0.$$

we denote that $\widehat{g}$ is non-increasing, positive and $\widehat{g}(t) \geq \frac{g(t)}{t^{p-1}}$.
Furthermore, by $(g8)$ we have $\widehat{g}(t) < |w|_{\infty}^{1-p}$ for sufficiently large t.

Let

$$h(t) := \frac{2}{t}\int_{\frac{t}{2}}^{t} \widehat{g}(s)ds, \qquad t > 0. \tag{24}$$

It is shown in (Ladyzenskaja and Ural'tseva, 1968) that h is C^1, non-increasing and $\widehat{g}(t) \leq h(t) \leq \widehat{g}(\frac{t}{2})$ for all $t \in (0,\infty)$.

Since h is non-increasing, we note that $h(t) \to \alpha < |w|_{\infty}^{1-p}$ at $t \longrightarrow \infty$ for some $\alpha \in [0,\infty)$.
Now, let we set $\Phi(t)$ satisfies

$$\int_0^{\Phi(t)} \frac{s}{h^{\frac{1}{p-1}}(s+l)(s+l)} ds = t.$$

By direct calculation, we see that

$$\Phi'(t) = \frac{h^{\frac{1}{p-1}}(\Phi(t)+l)(\Phi(t)+l)}{\Phi(t)},$$

and $\Phi(t) > 0$, for $t > 0$, $\Phi(0) = 0$. Let we set $v(x)$:=$\Phi(\beta w(x)) + l$, where β large enough and satisfies $\beta \geq v(x) - l$. A simple computation shows that v has the desired properties.

Indeed, on recalling $-\text{div}(|\nabla w|^{p-2}\nabla w) = b(x)$, we see that

$$\begin{aligned}
-\text{div}(|\nabla v|^{p-2}\nabla v) &= -\text{div}(|\frac{v}{v-l}h^{\frac{1}{p-1}}(v)\beta\nabla w|^{p-2}\frac{v}{v-l}h^{\frac{1}{p-1}}(v)\beta\nabla w) \\
&= -\text{div}((\frac{v}{v-l})^{p-1}h(v)\beta^{p-1}|\nabla w|^{p-2}\nabla w) \\
&= -\beta^{p-1}(\frac{v}{v-l})^{p-1}h(v)\text{div}(|\nabla w|^{p-2}\nabla w) \\
&- \beta^p|\nabla w|^p\frac{d[(\frac{v}{v-l})^{p-1}h(v)]}{dv}h^{\frac{1}{p-1}}(v)\frac{v}{v-l} \\
&= -\beta^{p-1}\frac{v^{p-1}}{(v-l)^{p-1}}h(v)\text{div}(|\nabla w|^{p-2}\nabla w) \\
&- (p-1)\beta^p|\nabla w|^p h^{\frac{1}{p-1}}(v)(1+\frac{l}{v-l})^{p-1}\frac{-l}{(v-l)^2}h(v) \\
&- \beta^p|\nabla w|^p h^{\frac{1}{p-1}}(v)(1+\frac{l}{v-l})^p h'(v) \\
&\geq -\beta^{p-1}\frac{v^{p-1}}{(v-l)^{p-1}}h(v)\text{div}(|\nabla w|^{p-2}\nabla w) \\
&= b(x)\beta^{p-1}\frac{v^{p-1}}{(v-l)^{p-1}}h(v) \\
&\geq b(x)(v-l)^{p-1}\frac{v^{p-1}}{(v-l)^{p-1}}\frac{g(v)}{v^{p-1}} \\
&= b(x)g(v) \quad x \in R^N,
\end{aligned}$$

Moreover, since $\Phi(0) = 0$, it is clear that $v(x) \to l$, as $|x| \to \infty$.

Proof of Theorem 2 Consider the perturbed problem

$$-\text{div}(|\nabla u_k|^{p-2}\nabla u_k) + p(x)|\nabla u_k|^q = f(x, u_k), u_k > l > 0, x \in B(0,k), u_k = l, x \in \partial B(0,k), \quad (25)$$

where $B(0,k) = \{x \in \mathbb{R}^N : |x| < k\}$, $k = 1, 2, 3, \cdots$. Let $U_k(x) = u_k(x) - l$, where $U_k(x)$ is the solution of (18). Then, it follows by Lemma 2.3 that problem (18) has one solution $U_k \in C^{1+\alpha}(B(0,k)) \cap C(\overline{B}(0,k))$, thus, problem (25) has one solution $u_k \in C^{1+\alpha}(B(0,k)) \cap C(\overline{B}(0,k))$. Put

$$u_k(x) = l, \quad \forall |x| > k.$$

Let v be as in Lemma 2.5, we assert that

$$u_k(x) \leq v(x), \quad x \in \mathbb{R}^N, k = 1, 2, 3 \cdots. \quad (26)$$

Indeed,

$$-\text{div}(|\nabla v|^{p-2}\nabla v) \geq b(x)g(v(x)) \geq f(x,v) - p(x)|\nabla v|^q,$$

By the comparison principle argument, we can obtain (26).

Now, we need to estimate $\{u_k\}$. For any bonded $C^{2+\alpha}$-smooth domain $\Omega' \subset \mathbb{R}^N$, take Ω_1 and Ω_2 with $C^{2+\alpha}$-smooth boundaries, and K_1 large enough, such that

$$\Omega' \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset B_k, \quad k \geq K_1,$$

Note that

$$v(x) \geq u_k(x) \geq \underline{U}(x) > l, \quad \forall \ x \in B(0, K_1), \quad (27)$$

where $\underline{U}(x) = \underline{u}(x) + l$, and $\underline{u}(x)$ is defined by Lemma 2.3. When $B(0, K_1)$ is the substitution for Ω in the proof of Lemma 2.3, by (n3) it is easy to see that $\underline{U}(x)$ is the sub-solution of (25). Let

$$\rho_k(x) = f(x, u_k(x)) - p(x)|\nabla u_k(x)|^q, x \in \overline{B}(0, K_1),$$

For $k \in N$ We consider the problem

$$-\text{div}(|\nabla u_k|^{p-2}\nabla u_k) = \rho_k(x), \quad x \in \Omega^{'}, \quad u_k = \underline{U}, \quad x \in \partial\Omega^{'}, \tag{28}$$

Since $\underline{U}$ is a sub-solution and v is a super-solution, the above problem has at least a solution $\underline{U}(x) \leq u_k(x) \leq v(x)$. This in particular gives local bounds for the sequence $\{u_k\}$ which in turn leads to local bounds in $C^{1+\alpha}$. Thus for every $m \in N$, we can select a sequence $\{u_k^m\}$ which converges in $C^{1+\alpha}(\overline{\Omega'})$. A diagonal procedure gives a subsequence (denoted again by u_k) which converges to a function u in $C^1(\overline{\Omega'})$, and u satisfies

$$-\text{div}(|\nabla u|^{p-2}\nabla u) + p(x)|\nabla u|^q = f(x, u), \quad x \in \overline{\Omega'},$$

By (27), we obtain that

$$u > l, \quad x \in \overline{\Omega'},$$

and we can obtain that $u \in C^{1+\alpha}(\overline{\Omega'})$. Since $\Omega^{'}$ is arbitrary, we also see that $u \in C^{1+\alpha}_{loc}(\mathbb{R}^N)$. It follows by (26) that

$$\lim_{|x|\longrightarrow\infty} u(x) = l,$$

The proof is finished.

5 Conclusions

The boundary value quasilinear differential equation systems (1) and (22) are mathematical models occurring in the studies of the p-Laplace equation, generalized reaction-diffusion theory, non-Newtonian fluid theory, and the turbulent flow of a gas in porous medium. In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If p = 2, they are Newtonain fluids. When p=/ 2, the problem becomes more complicated since certain nice properties in herent to the case m = 2 seem to be lost or at least difficult to verify. The main differences between m = 2 and m =/ 2 can be founded in (Guo, 1992; Guo and Webb, 1994). When p = 2, it is well known that all the positive solutions in C2(BR) of the problem
4u + f(u) = 0 in BR;
u(x) = 0 on @BR;
are radially symmetric solutions for very general f (see Gidas and Nirenberg, 1979). Unfortunately, this result does not apply to the case p =/ 2. Kichenassary and Smoller showed that there exist many positive nonradial solutions of the above problem for some f (see Kichenassamy and Smoller, 1990). The major stumbling block in the case of p = /2 is that certain nice features inherent to the case p = 2 seem to be lost or at least difficult to verify. In this paper, we first provide a suitable supersolution for problem (1) and show the existence of positive solutions in bounded domain. Then, we prove Theorem 1, moreover, we have studied the case $l > 0$.

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