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On the Number of Primes in the Interval (*x*, 2*x*) by an Elementary Method

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Abstract

An elementary formula to know the number of primes in the interval $(x,2x)$ close to the exact figure for a fixed x is given here. A new elementary **equation** is derived (a relation between prime numbers and composite numbers distributed in the interval $[1,2x]$). An elementary method to know the number of primes in a given magnitude is suitably placed in the form of a general formula, and we have proved it. The general formula is applied to the terms of the **equation**, and a tactical simplification of the terms gives rise to an **expression** whose verification envisages scope for its further studies.

Keywords

Prime Numbers, Composite Numbers, Expression, Integers

1. Introduction

There are many questions concerning primes which are sophisticated. One of them is knowing how many primes are there in the interval $(x,2x)$ for a fixed x. Joseph Bertrand, in 1845, conjectured that there is at least one prime p in the interval $(x,2x)$ after checking this numerically up to $x=300000$ in ref. ([1], p: 498). In 1850, P. Tschebyschew proved it, and simplified later by P. Erdös in 1932 as in ref. [2]. Ramanujan proved it using gamma function in 1919 as in ref. [3]. Since then, many efforts have been made to show that there is a prime in a fixed interval.

Generally, to know $\pi(2x)-\pi(x)$, it is done by computing each term by means of a sieve method which involves more labor. The other option is to have a general formula for evaluation & simplify it to get an **expression** to provide an estimated value.

Earnest Meissel, in 1870-85, published a series of articles where he was able to count exactly the number of primes up to a given magnitude x in ref. [4] [5] [6] [7]. Derrick Henry Lehmer, in 1959, simplified it and introduced a modified counting method of composite numbers as in ref. [8].

The ultimate goal of this research is to show that there exists an **expression** that even shows that there are more than 60,000 primes in the interval $\left[10^6, 2\times 10^6\right]$. We can also call it an elementary formula of primes in the interval $(x, 2x)$ for a fixed x. The idea behind this investigation is the identified relationship between composite numbers and prime numbers distributed in the interval $[1, 2x]$.

On the basis of the computational technique of Legendre, E. Meissel, and D H Lehmer for composite numbers in ref. [4]-[9] respectively, a general formula for $\pi(x)$ is written in a modified pattern along with its proof using an appropriate lemma. An illustrated example for an "example" number 100 is provided for reference. This general formula and the identified relationship were utilized, and a tactical simplification of terms enabled us to obtain the **expression** for $\pi(2x)-\pi(x)$.

2. Statement of the Result

The number of primes in the interval $(x, 2x)$ for a fixed x is

$$\pi(x,2x) > \pi(x) - \frac{\mu(\mu-1)}{2}, \quad \text{where } \mu = \pi\left(\sqrt{x}\right),$$

a still better approximation,

$$\pi(x,2x) > \pi(x) - \left(\frac{\mu'(\mu'-1)}{2} - \frac{\mu(\mu-1)}{2}\right) \quad \text{where } \mu' = \pi\left[\sqrt{2x}\right].$$

We call these results as an **expression** for $\pi(2x)-\pi(x)$.

3. Results

We make readers familiarize themselves with the notations and symbols used here.

- We define a new symbol "*C*" where $\mathbb{C}(x)$ stands for the number of composite numbers not exceeding a given positive integer x, e.g. $\mathbb{C}(10)=5$, they are 4, 6, 8, 9, 10.
- $\pi(x)$ denote the number of primes not exceeding a fixed integer x, e.g. $\pi(10)=4$, they are 2, 3, 5, 7.
- $\pi(x,2x)$ denote the number of primes in the interval x to $2x$, *i.e.* $\pi(10,20)=4$, they are 11, 13, 17, 19; $\pi(11,22)=3$, they are 13, 17, 19, here "11" is not included.
- Integer means positive integer, c runs for composite number, and p runs for prime number throughout.

3.1. Idea

We obtain a relation between prime numbers and composite numbers distributed in the interval $[1,2x]$. This identified relation was the basis for obtaining an expression for $\pi(2x)-\pi(x)$. We have by computational technique of Legendre in ref. [9]

$$\pi(x)=(x-1)-\mathbb{C}(x) \tag{1}$$

Substitute $x=2x$, we get

$$\pi(2x)=(2x-1)-\mathbb{C}(2x) \tag{2}$$

Multiply Equation (1) by 2, we get

$$2\pi(x)=2x-2-2\mathbb{C}(x) \tag{3}$$

(2)-(3) gives

$$\begin{aligned}\pi(2x)-2\pi(x)&=(2x-1)-\mathbb{C}(2x)-2x+2+2\mathbb{C}(x)\\&=-\mathbb{C}(2x)+2\mathbb{C}(x)+1\end{aligned} \tag{4}$$

On rearraging the terms of (4), we get

$$\pi(2x)-\pi(x)=\pi(x)-(\mathbb{C}(2x)-2\mathbb{C}(x))+1 \tag{5}$$

$$\pi(2x)-\pi(x)=\pi(x)-a(x)+1$$

$$\pi(x,2x)=\pi(x)-a(x)+1 \tag{6}$$

where $a(x)=\mathbb{C}(2x)-2\mathbb{C}(x)$, we call Equation (5) as our basic tool.

3.2. Preliminaries

We revert back to the notations, symbols, and definitions used here.

- We define a new symbol π' where $\pi'(x)$ denote number of primes strictly less than *x*, e.g. $\pi'(5)=2$, they are 2,3, here "5" is not included.
- $p_1,p_2,p_3,p_4,\cdots$ denotes primes $2,3,5,7,\cdots$, *x* denote a fixed integer.
- *m* denote a positive integer < *x*.
- $\left[\frac{x}{p}\right]$ stands for an integral part of *x* when *x* is divided by a "*p*", e.g. $\frac{10}{3}=3.33$, here $\left[\frac{10}{3}\right]=3$, similarly for $\left[\frac{x}{m}\right]$.
- p_{r_1} is the largest prime in $\left[\frac{x}{p_1}\right]$ means when *x* is divided by p_1, p_{r_1} is the largest prime less than $\left[\frac{x}{p_1}\right]$.

Definition 1. *We define an integer* $m=p_1^a\cdot p_2^b\cdot p_3^c\cdots p_r^l$, *where* $a,b,c,\cdots,l\geq 0$, *and* p_r *is the largest prime* $|m$.

Keeping in mind the computational techniques of ref. [4] [5] [6] [7] [8] for computation of $\mathbb{C}(x)$, we place it in the form of a general formula. This computational technique is the modified pattern of ref. [4] [5] [6] [7] [8], where Legendre sum in ref. [9] was utilized for a certain range, and for the rest, the technique. We place it here in the following lemma where the technique is extended for the entire range of $m<x$, conditionally and give its proof.

Lemma 1. *The number of composite numbers not exceeding a given integer x is computed as given below.*

$$\mathbb{C}(x)=\sum_{2\le m<x}^{\pi\left[\frac{x}{m}\right]-\pi'[p_r]>0}\left(\pi\left[\frac{x}{m}\right]-\pi'[p_r]\right) \tag{7}$$

an elaborated form of Lemma (1), *expressed as in* (7), *is given below.*

$$\mathbb{C}(x)=\sum_{a\ge1}^{\left(\pi\left[\frac{x}{p^a}\right]-\pi'[p]\right)>0}\ \sum_{2\le p\le\sqrt{x}}^{\left(\pi\left[\frac{x}{p}\right]-\pi'[p]\right)>0}\left(\pi\left[\frac{x}{p^a}\right]-\pi'[p]\right) \quad \text{step I}$$

$$+\sum_{a,b\ge1}^{\left(\pi\left[\frac{x}{p^a\cdot q^b}\right]-\pi'[q]\right)>0}\ \sum_{\substack{6\le p\cdot q<x\\ q>p}}^{\left(\pi\left[\frac{x}{p\cdot q}\right]-\pi'[q]\right)>0}\left(\pi\left[\frac{x}{p^a\cdot q^b}\right]-\pi'[q]\right) \quad \text{step II}$$

Similarly for $m=p^a\cdot q^b\cdot r^c$ *and so on.*

We write a lemma necessary in proving the lemma 1. This lemma is based on the standard pattern of the fundamental theorem of arithmetic as in ref. ([1], p. 3). The proof of this lemma proves lemma 1 in the elaborated form.

Lemma 2. *Every* $c\le x$ *can be expressed as*

$$\begin{aligned}&\textstyle\sum_{1<c\le x}1=\mathbb{C}(x)\\ &=\Big\{\{p_1\cdot p_1,p_1\cdot p_2,p_1\cdot p_3,p_1\cdot p_4,\cdots,p_1\cdot p_{r_1}\},\{p_2\cdot p_2,p_2\cdot p_3,p_2\cdot p_4,p_2\cdot p_5,\cdots,p_2\cdot p_{r_2}\},\\ &\{p_3\cdot p_3,p_3\cdot p_4,p_3\cdot p_5,p_3\cdot p_6,\cdots,p_3\cdot p_{r_3}\},\cdots,\{p_1^2\cdot p_1,p_1^2\cdot p_2,p_1^2\cdot p_3,\cdots\},\\ &\{p_2^2\cdot p_2,p_2^2\cdot p_3,p_2^2\cdot p_4,\cdots\},\{p_3^2\cdot p_3,p_3^2\cdot p_4,p_3^2\cdot p_5,\cdots\},\\ &\{p_1\cdot p_2^2,p_1\cdot p_2\cdot p_3,p_1\cdot p_2\cdot p_4,\cdots\},\{p_2\cdot p_3^2,p_2\cdot p_3\cdot p_4,p_2\cdot p_3\cdot p_5,\cdots\},\cdots,\\ &\{p_1\cdot p_2\cdot p_3^2,p_1\cdot p_2\cdot p_3\cdot p_4,p_1\cdot p_2\cdot p_3\cdot p_5,\cdots\},\cdots,\{\forall\, p_1^a\cdot p_2^b\cdot p_3^c\cdots\le x;a,b,c,\cdots\ge0\}\Big\}\end{aligned}$$

Proof. Let $m<x$ as per definition in (1). Let p_{r_1} is the largest prime in $\left[\frac{x}{p_1}\right]$, p_{r_2} is the largest prime in $\left[\frac{x}{p_2}\right]$, $\cdots$. Let us obtain the pattern as in the lemma (2) $\forall\, c\le x$. We start with $m=p_1$, $\cdots$, and continue for $m=p_1^2$, $\cdots$, and so on $\forall\, m\le x$, conditionally. We write

$$\pi\left[\frac{x}{p_1}\right]=\{p_1\cdot p_1,p_1\cdot p_2,p_1\cdot p_3,p_1\cdot p_4,\cdots,p_1\cdot p_{r_1}\} \quad \text{(as required in the lemma 2)}$$

$$\pi\left[\frac{x}{p_2}\right]=\{p_2\cdot p_1,p_2\cdot p_2,p_2\cdot p_3,p_2\cdot p_4,p_2\cdot p_5,\cdots,p_2\cdot p_{r_2}\} \quad \text{(not as required in lemma 2)}$$

$$\begin{aligned}&\therefore\{p_2\cdot p_2,p_2\cdot p_3,p_2\cdot p_4,p_2\cdot p_5,\cdots,p_2.p_{r_2}\}\\ &=\{p_2\cdot p_1,p_2\cdot p_2,p_2\cdot p_3,p_2\cdot p_4,p_2\cdot p_5,\cdots,p_2.p_{r_2}\}\\ &-\{p_2\cdot p_1\} \qquad \text{(to make it as required in lemma 2)}\end{aligned}$$

$$=\pi\left[\frac{x}{p_2}\right]-\sum_{p_2\cdot p_1<x}1$$

here in the second term, instead of counting $p_2 \cdot p_1 < x$, only p_1, strictly less than p_2 is counted to reflect the counting of $p_2 \cdot p_1$, similarly in further steps, we write

$$= \pi\left[\frac{x}{p_2}\right] - \sum_{p_1 < p_2} 1$$

$$= \pi\left[\frac{x}{p_2}\right] - \pi'\left[p_2\right]$$

$\ldots m = p_j \le \sqrt{x}$; such that $\left(\pi\left[\frac{x}{p_j}\right] - \pi'\left[p_j\right]\right) > 0$ and $\left(\pi\left[\frac{x}{p_j}\right] - \pi'\left[p_{j+1}\right]\right) \le 0$ because, here $p_j \cdot p_j \le x$ but $p_{j+1} \cdot p_j > x$; similarly in further steps.

Next $\pi\left[\frac{x}{p_1^2}\right] = \left\{p_1^2 \cdot p_1, p_1^2 \cdot p_2, p_1^2 \cdot p_3, p_1^2 \cdot p_4, \cdots\right\}$ (as required in lemma 2).

Similarly

$$\left\{p_2^2 \cdot p_2, p_2^2 \cdot p_3, p_2^2 \cdot p_4, \cdots\right\} = \left\{p_2^2 \cdot p_1, p_2^2 \cdot p_2, p_2^2 \cdot p_4, \cdots\right\} - \left\{p_2^2 \cdot p_1\right\}$$

$$= \pi\left[\frac{x}{p_2^2}\right] - \sum_{p_1 \cdot p_2^2 < x} 1$$

$$= \pi\left[\frac{x}{p_2^2}\right] - \sum_{p_1 < p_2} 1$$

$$= \pi\left[\frac{x}{p_2^2}\right] - \pi'\left[p_2\right]$$

$\ldots \forall\, m = p^a, a > 1$, such that $\left(\pi\left[\frac{x}{m}\right] - \pi'\left[p\right]\right) > 0$ where p is the largest prime $|m$.

Similarly $\pi\left[\frac{x}{p_1 \cdot p_2}\right] = \left\{p_1^2 \cdot p_2, p_1 \cdot p_2^2, p_1 \cdot p_2 \cdot p_3, p_1 \cdot p_2 \cdot p_4, p_1 \cdot p_2 \cdot p_5, \cdots\right\}$

$$\left\{p_1 \cdot p_2^2, p_1 \cdot p_2 \cdot p_3, p_1 \cdot p_2 \cdot p_4, \cdots\right\}$$

$$= \left\{p_1^2 \cdot p_2, p_1 \cdot p_2^2, p_1 \cdot p_2 \cdot p_3, p_1 \cdot p_2 \cdot p_4, p_1 \cdot p_2 \cdot p_5, \cdots\right\} - \left\{p_1^2 \cdot p_2\right\}$$

& $$= \pi\left[\frac{x}{p_1 \cdot p_2}\right] - \sum_{p_1^2 \cdot p_2 < x} 1$$

$$= \pi\left[\frac{x}{p_1 \cdot p_2}\right] - \sum_{p_1 < p_2} 1$$

$$= \pi\left[\frac{x}{p_1 \cdot p_2}\right] - \pi'\left[p_2\right];\ p_2,\ \text{largest } p \mid p_1 \cdot p_2$$

$\ldots \forall\, m = p^a \cdot q^b$; such that $\left(\pi\left[\frac{x}{m}\right] - \pi'\left[q\right]\right) > 0$ where q is the largest prime $|m$.

$$\left\{p_1\cdot p_2\cdot p_3^2, p_1\cdot p_2\cdot p_3\cdot p_4, p_1\cdot p_2\cdot p_3\cdot p_5,\cdots\right\}$$
$$=\left\{p_1^2\cdot p_2\cdot p_3, p_1\cdot p_2^2\cdot p_3, p_1\cdot p_2\cdot p_3^2, p_1\cdot p_2\cdot p_3\cdot p_4,\cdots\right\}-\left\{p_1^2\cdot p_2\cdot p_3, p_1\cdot p_2^2\cdot p_3\right\}$$
$$\& \quad =\pi\left[\frac{x}{p_1\cdot p_2\cdot p_3}\right]-\sum_{p_1^2\cdot p_2\cdot p_3,\ p_1\cdot p_2^2\cdot p_3<x}1$$
$$=\pi\left[\frac{x}{p_1\cdot p_2\cdot p_3}\right]-\sum_{p_1,p_2<p_3}1$$
$$=\pi\left[\frac{x}{p_1\cdot p_2\cdot p_3}\right]-\pi'\left[p_3\right];$$

where p_3 is the largest $p\,|\,p_1\cdot p_2\cdot p_3$.

$\ldots\ \forall\ m=p^a\cdot q^b\cdot r^c\cdots$; such that $\left(\pi\left[\frac{x}{m}\right]-\pi'\left[p_r\right]\right)>0$, where p_r is the largest prime $|m$. Now, we ascertain that the entire pattern of lemma 2 is obtained. The condition $\left(\pi\left[\frac{x}{m}\right]-\pi'\left[p_r\right]\right)>0$, is by determination of the computational technique. This completes the proof of lemma 1. □

We illustrate the lemma 1 by taking a trivial example for example No "100", in the way it is written in the elaborated form. Following are the steps how to proceed.

- The fixed number (say 100 here) is to be divided by $m<x$ and find out the integral part.
- Choose all $m=p\le\sqrt{100}$ for which the condition $\left(\pi\left[\frac{x}{p}\right]-\pi'\left[p\right]\right)>0$, satisfies.
- Choose all $m=p\cdot q$, product of two primes, say $2\cdot 2, 2\cdot 3, 2\cdot 5,\cdots$ for which the condition $\left(\pi\left[\frac{x}{p\cdot q}\right]-\pi'\left[q\right]\right)>0$, $q>p$, satisfies.
- Choose all $m=p\cdot q\cdot r$, product of three primes, say $2\cdot 2\cdot 2, 2^2\cdot 3, 2\cdot 3^2,\cdots$ for which the condition $\left(\pi\left[\frac{x}{p\cdot q\cdot r}\right]-\pi'\left[r\right]\right)>0$, $r>q>p$, satisfies.

All possibilities of $m=p^a\cdot q^b\cdot r^c\cdots$, for all combinations of $p,q,r,\cdots,a,b,c,\cdots$ are to be covered and care is taken that no non-trivial m is left out, ultimately, these are $m<x$ for which the condition $\left(\pi\left[\frac{x}{m}\right]-\pi'\left[p_r\right]\right)>0$, satisfies where p_r is the largest prime $|m$.

Example. *For* $x=100$, *to work out for* $\pi(100)$, *from* (1), *we have*

$$\pi(100)=100-1-\mathbb{C}(100) \tag{8}$$

Computation of $C(100)$: *For Step I,* $m=p^a$: $a=1$, $m=p\le\sqrt{100}=\{2,3,5,7\}$; *not for* $m=11$ *because* $\pi\left[\frac{100}{11}\right]-\pi'[11]\le 0$; *for* $a=2,3,\cdots$, $m=\left\{2^2,3^2,2^3,3^3,2^4,2^5\right\}$; *not for* $m=2^6$ *because* $\left(\pi\left[\frac{100}{2^6}\right]-\pi'[2]\right)\le 0$, *simi-*

larly for $m=3^4$ *and so on; we proceed for doing calculations, we have*

$$\begin{aligned}&+\pi\left[\frac{100}{2}\right]-\pi'[2]+\pi\left[\frac{100}{3}\right]-\pi'[3]+\pi\left[\frac{100}{5}\right]-\pi'[5]+\pi\left[\frac{100}{7}\right]-\pi'[7]\\&=\pi[50]-0+\pi[33]-1+\pi[20]-2+\pi[14]-3\\&=15-0+11-1+8-2+6-3=15+10+6+3=34\end{aligned}\tag{9}$$

$$\begin{aligned}&+\pi\left[\frac{100}{2^2}\right]-\pi'[2]+\pi\left[\frac{100}{3^2}\right]-\pi'[3]\\&=\pi[25]-0+\pi[11]-1=9-0+4-1=9+4=13\end{aligned}\tag{10}$$

$$\begin{aligned}&+\pi\left[\frac{100}{2^3}\right]-\pi'[2]+\pi\left[\frac{100}{3^3}\right]-\pi'[3]\\&=\pi[12]-0+\pi[3]-1=5-0+2-1=5+1=6\end{aligned}\tag{11}$$

$$\begin{aligned}&+\pi\left[\frac{100}{2^4}\right]-\pi'[2]+\pi\left[\frac{100}{2^5}\right]-\pi'[2]\\&=\pi[6]-0+\pi[3]-1=3-0+2-0=5\end{aligned}\tag{12}$$

For Step II: $m=p^a\cdot q^b=\{2\cdot3,2\cdot5,2\cdot7,2^2\cdot3,2^2\cdot5,2^3\cdot3,2\cdot3^2,3\cdot5\}$; *not for* $m=2\cdot11$ *because* $\left(\pi\left[\frac{100}{2\cdot11}\right]-\pi'[11]\right)\le0$; *similarly for* $m=2^2\cdot7,2^3\cdot5,3\cdot7,2\cdot5^2$ *and so on; we proceed for doing calculations, we have*

$$\begin{aligned}&+\pi\left[\frac{100}{2\cdot3}\right]-\pi'[3]+\pi\left[\frac{100}{2\cdot5}\right]-\pi'[5]+\pi\left[\frac{100}{2\cdot7}\right]-\pi'[7]\\&=\pi[16]-1+\pi[10]-2+\pi[7]-3\\&=6-1+4-2+4-3=5+2+1=8\end{aligned}\tag{13}$$

$$\begin{aligned}&+\pi\left[\frac{100}{2^2\cdot3}\right]-\pi'[3]+\pi\left[\frac{100}{2^2\cdot5}\right]-\pi'[5]+\pi\left[\frac{100}{2^3\cdot3}\right]-\pi'[3]\\&=\pi[8]-1+\pi[5]-2+\pi[4]-1\\&=4-1+3-2+2-1=3+1+1=5\end{aligned}\tag{14}$$

$$\begin{aligned}&+\pi\left[\frac{100}{2\cdot3^2}\right]-\pi'[3]+\pi\left[\frac{100}{3\cdot5}\right]-\pi'[5]\\&=\pi[5]-1+\pi[6]-2=3-1+3-2=2+1=3.\end{aligned}\tag{15}$$

No Step III: for $m=2\cdot3\cdot5$, *because* $\left(\pi\left[\frac{100}{2\cdot3\cdot5}\right]-\pi'[5]\right)\le0$ and so on; the *sum of all* (9) *to* (15) $=34+13+6+5+8+5+3=74$ *substituting in* (8), *we get* $\pi(100)=100-1-74=25$.

We close this section with the following assertion.

Assertion 3. *Primes near* $2x$ *"thin out" compared to near x. We write* $\pi(x,2x)<\pi(x)$. *There are a few rare intervals like* $(2,4)$, $(4,8)$, $(10,20)$ *where this assertion is not true is without loss of generality. Further, adding* $\pi(x)$ *to both sides of this, we have* $\pi(x,2x)+\pi(x)<\pi(x)+\pi(x)$ *which* $\Rightarrow$ $\pi(2x)<2\pi(x)$.

3.3. Expression

Now, we show, how we arrive to an expression for $\pi(x,2x)$ using our basic tool at (5), and the general formula for $\mathbb{C}(x)$ in the lemma 1 in its elaborated form, on evaluating all the terms of $\mathbb{C}(2x)-2\mathbb{C}(x)$ & substituting in (5), the exact value of $\pi(x,2x)$ can be obtained, but due to huge labor, it is kept aside at this stage. On observation, evaluation gives three types of terms:

- Negative values & zero, say $\mathbb{B}_j(x)$;
- Redundant;
- Positive values, say $\mathbb{A}_k(x)$.

We proceed as per methodology of lemma 1 in an elaborated form, considering Step I terms for $a=1$, for $p\le\sqrt{2x}$ and $a=1$, for $p\le\sqrt{x}$ for each term of $\mathbb{C}(2x)-2\mathbb{C}(x)$ respectively, we have a term for a "p" as

$$\left(\pi\left[\frac{2x}{p}\right]-\pi'[p]\right)-2\left(\pi\left[\frac{x}{p}\right]-\pi'[p]\right) \tag{16}$$

$$=\pi\left[\frac{2x}{p}\right]-\pi'[p]-2\pi\left[\frac{x}{p}\right]+2\pi'[p]$$

$=\pi\left[\frac{2x}{p}\right]-2\pi\left[\frac{x}{p}\right]$ is a term of negative value, by assertion 3.

$-\pi'[p]+\pi'[p]$ is redundant.

$+\pi'[p]$ is a term of positive value.

Taking the sum of all positive values $\forall\, p\le\sqrt{x}$ and balancing the term, $-\sum_{\sqrt{x}<p\le\sqrt{2x}}\pi'[p]$ with other terms, we get $\pi'[p_1]+\pi'[p_2]+\pi'[p_3]+\pi'[p_4]+\cdots+\pi'[p_r]$, where $p_r\le\sqrt{x}$ and $p_{r+1}>\sqrt{x}$ $=0+1+2+3+\cdots+r-1$ where $\pi'[p_r]=r-1$ and $\pi[p_r]=r$, $=\frac{(r-1)r}{2}$ on substituting $r=\pi[p_r]=\pi\left[\sqrt{x}\right]$, we get

$$\sum_{2\le p\le\sqrt{x}}\pi'[p]=\frac{\mu(\mu-1)}{2} \quad \text{where} \quad \mu=\pi\left[\sqrt{x}\right] \quad \text{say} \quad \mathbb{A}_1(x),$$

and is a term of positive value. that can be calculated by knowing the value of μ from the table of primes. Similarly considering all positive values and negative values of all the steps I, II, III,..., and their sum total, we have

$$\mathbb{C}(2x)-2\mathbb{C}(x)=\left(\sum\mathbb{A}_k(x)-\sum\mathbb{B}_j(x)\right)=a(x) \tag{17}$$

Substituting $a(x)$ in (6), we get

$$\pi(x,2x)$$
$$=\pi(x)-\left(\mathbb{A}_1(x)+\mathbb{A}_2(x)+\cdots+\mathbb{A}_k(x)-\mathbb{B}_1(x)-\mathbb{B}_2(x)-\cdots-\mathbb{B}_j(x)\right)+1$$
$$=\pi(x)-\mathbb{A}_1(x)-\left(\mathbb{A}_2(x)+\cdots+\mathbb{A}_k(x)-\mathbb{B}_1(x)-\mathbb{B}_2(x)-\cdots-\mathbb{B}_j(x)\right)+1$$

Knowing the values of all the terms of $a(x)$ in (17) is of huge labor. Hence, considering only $A_1(x)$, the highest positive value, and neglecting all other values, we get the expression

$$\pi(x,2x) > \pi(x) - \mathbb{A}_1(x)$$

$$\pi(x,2x) > \pi(x) - \frac{\mu(\mu-1)}{2} \quad \text{where } \mu = \pi\left[\sqrt{x}\right] \tag{18}$$

Value of the term μ can be known from the table of primes. On careful observation of terms, the expression (18) can further be improved as given below to get a still better approximation. On reconsidering a term for a "*p*" at (16) as in the proof of (18) and further simplify. Taking a term for a "*p*", for $\mathbb{C}(2x)-2\mathbb{C}(x)$, we write

$$\left(\pi\left[\frac{2x}{p}\right]-\pi'[p]\right)-2\left(\pi\left[\frac{x}{p}\right]-\pi'[p]\right)$$
$$=\pi\left[\frac{2x}{p}\right]-\pi'[p]-2\pi\left[\frac{x}{p}\right]+2\pi'[p]$$
$$=\pi\left[\frac{2x}{p}\right]-\pi'[p]-2\pi\left[\frac{x}{p}\right]+\pi'[p]+\pi'[p]$$
$$=\left(\pi\left[\frac{2x}{p}\right]-\pi'[p]\right)-\left\{\left(2\pi\left[\frac{x}{p}\right]-\pi'[p]\right)-\pi'[p]\right\}$$

a term for a "*p*" (16) is >0 by the computational technique of lemma 1. Hence, $\pi\left[\frac{2x}{p}\right] > \pi'(p)$ and $\left(2\pi\left[\frac{x}{p}\right]-\pi'[p]\right) > \pi'[p]$, taking the sum of all positive values $\forall\ p \le \sqrt{2x}$ for the first term, $\forall\ p \le \sqrt{x}$ for the second term, and proceed as in the proof of (18), we write

$$\sum_{2\le p\le\sqrt{2x}} \pi\left[\frac{2x}{p}\right] > \sum_{2\le p\le\sqrt{2x}} \pi'[p] = \frac{\mu'(\mu'-1)}{2} \quad \text{where } \mu' = \pi\left[\sqrt{2x}\right]$$

denote this value by $\mathbb{A}_1''(x)$, and

$$\sum_{2\le p\le\sqrt{x}} \left(2\pi\left[\frac{x}{p}\right]-\pi'[p]\right) > \sum_{2\le p\le\sqrt{x}} \pi'[p] = \frac{\mu(\mu-1)}{2} \quad \text{where } \mu = \pi\left[\sqrt{x}\right]$$

replacing the larger value with smaller one, and keeping the middle minus sign, the resulting term $\left(\frac{\mu'(\mu'-1)}{2}-\frac{\mu(\mu-1)}{2}\right)$ may be taken as a better positive value, denote it by $\mathbb{A}_1'(x)$, then we write (17) as

$$\mathbb{C}(2x)-2\mathbb{C}(x) = \left(\sum \mathbb{A}_k'(x) - \sum \mathbb{B}_j'(x)\right) = a(x)$$

where $\mathbb{A}_2'(x),\cdots$ and $\mathbb{B}_1'(x),\cdots$ are resulting positive & negative values here, and we write (6) as

$$\pi(x,2x)$$
$$=\pi(x)-\left(\mathbb{A}_1'(x)+\mathbb{A}_2'(x)+\cdots+\mathbb{A}_k'(x)-\mathbb{B}_1'(x)-\mathbb{B}_2'(x)-\cdots-\mathbb{B}_j'(x)\right)+1$$
$$=\pi(x)-\mathbb{A}_1'(x)-\left(\mathbb{A}_2'(x)+\cdots+\mathbb{A}_k'(x)-\mathbb{B}_1'(x)-\mathbb{B}_2'(x)-\cdots-\mathbb{B}_j'(x)\right)+1$$

proceeding as before at (18), we get a better approximation

$$\pi(x,2x) > \pi(x) - \left(\frac{\mu'(\mu'-1)}{2} - \frac{\mu(\mu-1)}{2}\right) \quad (19)$$

Denote the value of expression (18) as $\mathbb{K}_1(x)$ and its error with actual figure of $\pi(x,2x)$ as $\mathbb{K}_1'(x)$, of (19) as $\mathbb{K}_2(x)$ and its error as $\mathbb{K}_2'(x)$. The value of $\pi(x,2x)$ calculated from (18) & (19) and their error obtained for $x = 10^n$ up to a trivial $n = 6$, verified from the website in [10], is shown in the **Table 1**. (18) & (19) are obtained results, there is no need to verify their truth but the data obtained from them helps in their further studies.

Remark. *The following are calculations for other large non-trivial intervals from* (18).

$$\pi\left(10^{12}, 2\cdot 10^{12}\right) > 34526983265$$

$$\pi\left(10^{24}, 2\cdot 10^{24}\right) > 17728422244191190407713$$

here the value of the terms required in R.H.S of (18) *for calculation are from ref.* [10].

4. Further Studies

On observing the data of results obtained from the expression at (18) & (19) in **Table 1**, the % error for knowing the exact value of $\pi(x,2x)$ gradually decreases when x increases and sometimes it varies. There is an improvement in % error of (19) compared to that of (18). Observation shows that smaller the value of $a(x)$, the value of $\pi(x,2x)$ obtained is closer to its exact value. The plotted graph in **Figure 1** showing the graph of data (from **Table 1**) for the value of $K_1(x)$, $K_2(x)$, and the actual figure of $\pi(x,2x)$, shows that the results of the

Table 1. Table showing the results of $\pi(x,2x)$ from (18) and (19), see in ref. [10] for the value of $\pi(x)$.

x	$\pi(\sqrt{x})$ μ	$\pi(\sqrt{2x})$ μ'	$\mathbb{A}_1(x)$ $\frac{\mu(\mu-1)}{2}$	$\mathbb{A}_1''(x)$ $\frac{\mu'(\mu'-1)}{2}$	$\mathbb{A}_1'(x)$ $\mathbb{A}_1''(x)-\mathbb{A}_1(x)$	$\mathbb{K}_1(x)$ $\pi(x)-\mathbb{A}_1(x)$	$\mathbb{K}_2(x)$ $\pi(x)-\mathbb{A}_1'(x)$	$\pi[x,2x]$	$\mathbb{K}_1'(x)$ (%error)	$\mathbb{K}_2'(x)$ (%error)
10	2	2	1	1	0	3	4	4	1 (25)	0 (0)
10^2	4	6	6	15	9	19	16	21	2 (9.5)	5 (23.8)
10^3	11	14	55	91	36	113	132	135	22 (16.3)	3 (2.2)
10^4	25	34	300	561	261	929	968	1033	104 (10.1)	65 (6.3)
10^5	65	86	2080	3655	1575	7512	8017	8392	880 (10.4)	375 (4.6)
10^6	168	223	14,028	24,753	10,725	64,470	67,773	70,435	5965 (8.4)	2662 (3.8)

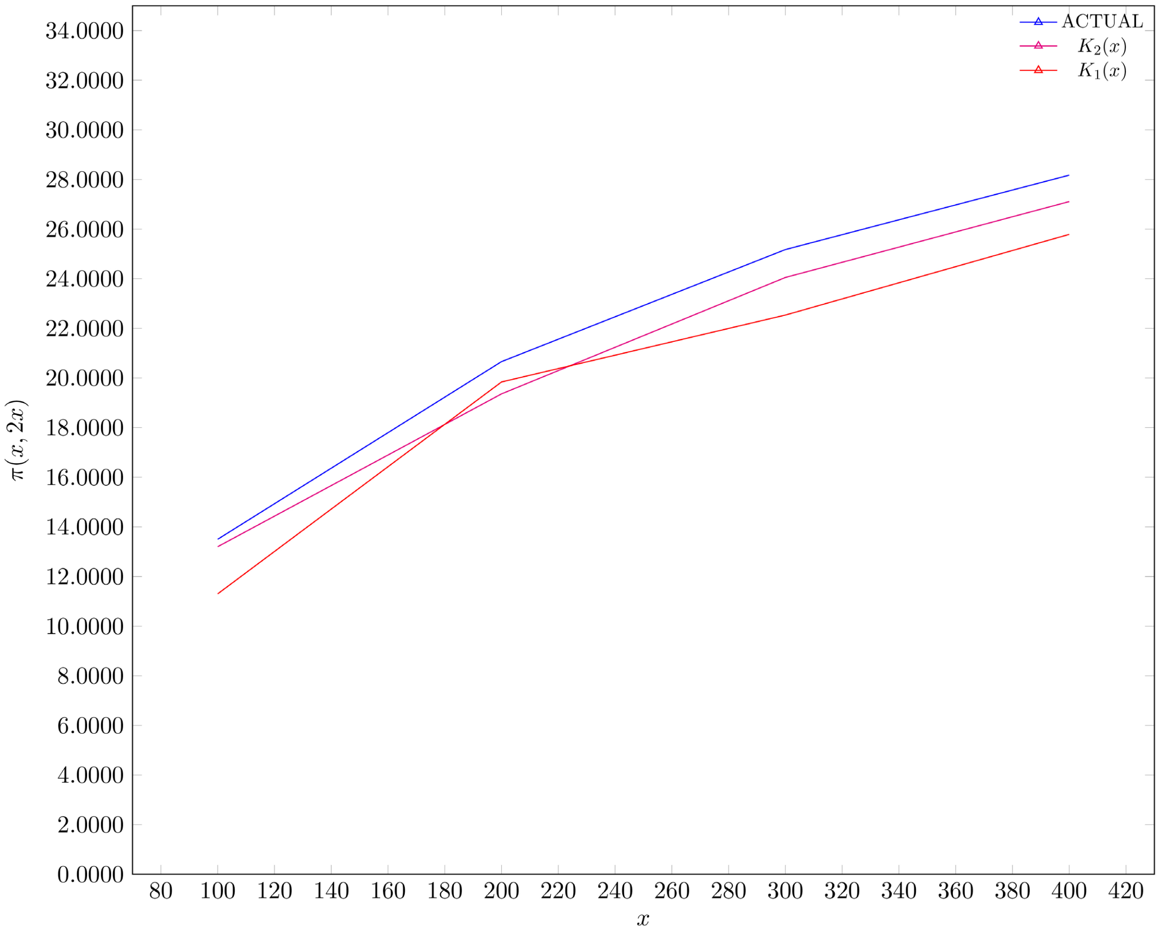

Figure 1. Figure showing the graph of $K_1(x)$, $K_2(x)$, and actual figure of $\pi(x,2x)$.

expression (18), (19) are inconsistent with $\pi(x,2x)$ (though not showing in the initial stage), and $K_2(x)$ runs close to $\pi(x,2x)$ than $K_1(x)$ is running. Knowing $\pi(x)$, $\pi(\sqrt{x})$, and $\pi(\sqrt{2x})$ from the table of primes can be avoided by applying the prime number theorem as in ref. ([1], p: 10) to the terms in R.H.S of (18), (19) for knowing their estimated values, again for (18), we write

$$\begin{aligned}\pi(x,2x) &> \frac{x}{\log x} - \frac{1}{2}\left(\frac{\sqrt{x}}{\log\sqrt{x}}\left(\frac{\sqrt{x}}{\log\sqrt{x}} - 1\right)\right)\\ &= \frac{x}{\log x} - \frac{1}{2}\left(\frac{\sqrt{x}}{\frac{1}{2}\log x}\left(\frac{\sqrt{x}}{\frac{1}{2}\log x} - 1\right)\right)\\ &= \frac{x}{\log x} - \frac{1}{2}\left(\frac{2\sqrt{x}}{\log x}\left(\frac{2\sqrt{x}}{\log x} - 1\right)\right)\\ &= \frac{x}{\log x} - \frac{\sqrt{x}}{\log x}\left(\frac{2\sqrt{x}}{\log x} - 1\right)\end{aligned} \tag{20}$$

Similarly, for (19) we write

$$\pi(x,2x) > \frac{x}{\log x} - \left\{ \frac{\sqrt{2x}}{\log 2x}\left(\frac{2\sqrt{2x}}{\log 2x} - 1\right) - \frac{\sqrt{x}}{\log x}\left(\frac{2\sqrt{x}}{\log x} - 1\right) \right\} \tag{21}$$

The workouts for an estimated value of $\pi(x,2x)$ from (20), (21) are shown in the Table 2 and Remark below.

Remark. *Calculation for other large intervals from* (20), *we have*

$$\pi\left(10^{12}, 2\cdot 10^{12}\right) > 33571660417$$

$$\pi\left(10^{24}, 2\cdot 10^{24}\right) > 17440714514802363566469$$

$$\pi\left(10^{25}, 2\cdot 10^{25}\right) > 167695213406436062322306$$

from (21), we have

Table 2. Table showing the results of $\pi(x,2x)$ from (20) and (21).

x	$\frac{x}{\log x}$	Calculations from(20)	% Error	Calculations from (21)	% Error
10^3	144	107	20	119	11.85
10^4	1085	850	17.71	927	10.26
10^5	8685	7207	14.12	7521	10.37
10^6	72,382	61,997	11.97	63,899	9.27

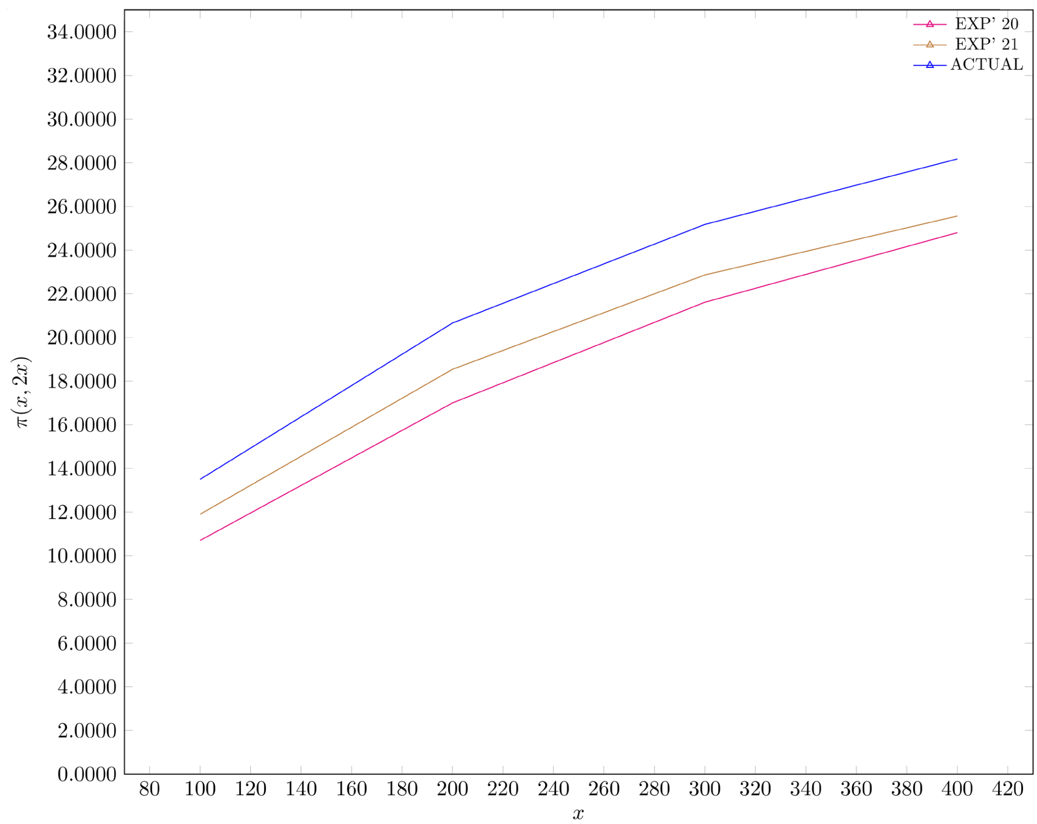

Figure 2. Figure showing the graph of EXP' (20), EXP' (21), and actual value of $\pi(x,2x)$.

$$\pi\left(10^{12}, 2\cdot 10^{12}\right) > 33823600357$$

$$\pi\left(10^{24}, 2\cdot 10^{24}\right) > 17472729013238329469058$$

$$\pi\left(10^{25}, 2\cdot 10^{25}\right) > 167979542770512164505980$$

On observing the data of results obtained from the expressions at (20), (21) in **Table 2**, the % error are quite similar to that of EXP' (18), (19), and the graph plotted in this case in **Figure 2** shows that they are inconsistent at regular intervals but not form a part of an inconsistent system, may be they take a curved path when $x \to \infty$. However, it shows that the path of EXP' (20), (21) are with that of path of $\pi(x, 2x)$.

5. Conclusions and Suggestions

However, the estimated value of $\pi(x, 2x)$ where actual values are not yet known for a large fixed x can be known from (20), (21) to the extent machine can be used to calculate the value of the terms. Expressions (20), (21) are a suitable answer to the query raised in Section 22.9, on "Primes in an interval", in ref. ([1], p: 494). We suggest the following future work.

- The expressions (18), (19) are subject to some mathematical operations in x, the results of which may help in answering questions concerning primes.
- To prove the statement that the number of primes in the interval $(x, 2x)$ is asymptotic to $\frac{x}{\log x} - \frac{\sqrt{x}}{\log x}\left(\frac{2\sqrt{x}}{\log x} - 1\right)$, and to $\frac{x}{\log x} - \left\{\frac{\sqrt{2x}}{\log 2x}\left(\frac{2\sqrt{2x}}{\log 2x} - 1\right) - \frac{\sqrt{x}}{\log x}\left(\frac{2\sqrt{x}}{\log x} - 1\right)\right\}$.

The discoveries and study conducted in this article shows that the "Assertion 3", is a truth.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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