mathematics

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Dynamic Analysis of the M/G/1 Stochastic Clearing Queueing Model in a Three-Phase Environment

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Abstract: In this paper, we consider the M/G/1 stochastic clearing queueing model in a three-phase environment, which is described by integro-partial differential equations (IPDEs). Our first result is semigroup well-posedness for the dynamic system. Utilizing a C_0-semigroup theory, we prove that the system has a unique positive time-dependent solution (TDS) that satisfies the probability condition. As our second result, we prove that the TDS of the system strongly converges to its steady-state solution (SSS) if the service rates of the servers are constants. For this asymptotic behavior, we analyze the spectrum of the system operator associated with the system. Additionally, the stability of the semigroup generated by the system operator is also discussed.

Keywords: stochastic clearing queueing model; integro-differential equations; C_0-semigroup; spectrum; stability

MSC: 35B40; 45K05; 47D06; 47A10; 60K25

Citation: Yiming, N. Dynamic Analysis of the M/G/1 Stochastic Clearing Queueing Model in a Three-Phase Environment. *Mathematics* **2024**, *12*, 805. https://doi.org/10.3390/math12060805

Academic Editor: János Sztrik

Received: 12 February 2024
Revised: 4 March 2024
Accepted: 7 March 2024
Published: 8 March 2024

1. Introduction

A stochastic clearing system receives and accumulates random inputs at random time intervals until certain predetermined criteria are met; then, some or all of these inputs are immediately cleared. This type of system has many practical applications in manufacturing, healthcare industries, tourism, transportation, etc. The stochastic clearing systems have a wide range of applications in queuing systems, see, e.g., [1–5].

In this paper, we investigate the M/G/1 stochastic clearing queueing model in a three-phase environment. The system is described as follows:

- There are three phases in this system, phase 1 and phase 3 are working phases, and phase 2 is a deterministic time phase without service.
- The Poisson arrival rate is λ_j $(j = 1, 2, 3)$ in phase j, the service times in phase i $(i = 1, 3)$ follow a general distribution and the system residing in phase i follows an exponential distribution with parameter θ_i.
- After phase 1, the system will enter phase 2, and if a new customer arrives during phase 2, the customer enters the system with a probability p or leaves the system with probability $1 - p$, the customers in phase 2 do not receive service, and they experience a fixed duration d.
- After phase 2, the system will enter phase 3. Once phase 3 is complete, the current customer is forced to leave the system without receiving service, and then the system enters phase 1 and restarts a new service cycle.
- $\mu_j(\cdot)$ $(j = 1, 3)$ is the service rate in phase j and satisfies $\mu_j(\cdot) \geq 0$, $\int_0^\infty \mu_j(x)dx = \infty$.
- $N_r := e^{-\lambda_2 pd}(\lambda_2 pd)^r (r!)^{-1}$ $(r \geq 0)$ denotes the probability of r customers arriving in phase 2.

Additionally, we assume that

$$Q_{j,0}(t) = Prob\left\{ \begin{array}{l} \text{the server being idle at time } t \text{ and no} \\ \text{customer being present in phase } j \text{ , } j = 1, 3 \end{array} \right\},$$

$$V_n(t) = Prob\left\{ \begin{array}{l} \text{the system in phase 2 and there are } n\ (n \geq 0) \\ \text{customers in the system at time } t \end{array} \right\},$$

$$p_{j,n}(x,t)dx = Prob\left\{ \begin{array}{l} \text{there are } n\ (n \geq 1) \text{ customers in the system at time } t \\ \text{in phase } j\ (j = 1,3) \text{ and the service time} \\ \text{used for providing services to customers is } x \end{array} \right\}.$$

Then, the system can be described by the following IPDEs:

$$\begin{cases} \dfrac{dQ_{1,0}(t)}{dt} = -(\lambda_1 + \theta_1)Q_{1,0}(t) + \displaystyle\int_0^\infty p_{1,1}(x,t)\mu_1(x)dx + \theta_3 \sum_{n=1}^\infty \int_0^\infty p_{3,n}(x,t)dx \\ \qquad +\theta_3 Q_{3,0}(t) \\ \dfrac{dQ_{3,0}(t)}{dt} = -(\lambda_3 + \theta_3)Q_{3,0}(t) + N_0 V_0(t) + \displaystyle\int_0^\infty p_{3,1}(x,t)\mu_3(x)dx, \\ \dfrac{dV_0(t)}{dt} = -V_0(t) + \theta_1 Q_{1,0}(t), \\ \dfrac{dV_n(t)}{dt} = -V_n(t) + \theta_1 \displaystyle\int_0^\infty p_{1,n}(x,t)dx,\ n \geq 1, \\ \partial_t p_{j,1}(x,t) + \partial_x p_{j,1}(x,t) = -[\lambda_j + \theta_j + \mu_j(x)]p_{j,1}(x,t),\ j = 1,3, \\ \partial_t p_{j,n}(x,t) + \partial_x p_{j,n}(x,t) = -[\lambda_j + \theta_j + \mu_j(x)]p_{j,n}(x,t) + \lambda_j p_{j,n-1}(x,t),\ n \geq 2, \end{cases} \tag{1}$$

with the following boundary and initial conditions:

$$\begin{cases} p_{1,1}(0,t) = \lambda_1 Q_{1,0}(t) + \displaystyle\int_0^\infty p_{1,2}(x,t)\mu_1(x)dx, \\ p_{1,n}(0,t) = \displaystyle\int_0^\infty p_{1,n+1}(x,t)\mu_1(x)dx,\ n \geq 2, \\ p_{3,1}(0,t) = \lambda_3 Q_{3,0}(t) + N_1 V_0(t) + N_0 V_1(t) + \displaystyle\int_0^\infty p_{3,2}(x,t)\mu_3(x)dx, \\ p_{3,n}(0,t) = \displaystyle\sum_{k=0}^n N_k V_{n-k}(t) + \int_0^\infty p_{3,n+1}(x,t)\mu_3(x)dx,\ n \geq 2, \\ V_n(0) = u_{2,n} \geq 0,\ n \geq 0; Q_{j,1}(0) = \overline{u}_{2,j} \geq 0, p_{j,n}(x,0) = u_{j,n}(x) \geq 0,\ n \geq 1. \end{cases} \tag{2}$$

Here, $(x,t) \in [0,\infty) \times [0,\infty)$, $\overline{u}_{2,1} + \overline{u}_{2,3} + \sum_{n=0}^\infty u_{2,n} + \sum_{j=1,3}\sum_{n=1}^\infty \int_0^\infty u_{j,n}(x)dx = 1$.

In [6], some steady-state indices for system (1) and (2) such as the steady-state queue length and the steady-state sojourn time distribution of customers were developed under the following hypothesis:

- $\lim_{t\to\infty} Q_{j,0}(t) = Q_{j,0},\ j = 1,3,$
- $\lim_{t\to\infty} V_n(t) = V_n,\ n \geq 0,$
- $\lim_{t\to\infty} p_{j,n}(\cdot,t) = p_{j,n}(\cdot),\ j = 1,3,\ n \geq 1,$

which means the following hypotheses from the perspective of PDEs, see, e.g., [7,8]:

Hypothesis 1 (H1). *System (1) and (2) admit a nonnegative TDS.*

Hypothesis 2 (H2). *The TDS converges to its SSS.*

In this paper, we show that the above hypotheses (**H1**) and (**H2**) are satisfied, and we discuss the stability of the corresponding semigroup of system (1) and (2). The difficulty in the well-posedness and asymptotic behavior is that the system is composed of an infinite number of IPDEs, with integral boundary conditions. However, the system (1) and (2) we consider in our work are problems in an L^1-based nonreflexive Banach space, and proving semigroups in this space is not easy. Hence, we provide the asymptotic behavior of system (1) and (2) when the service rates $\mu_1(\cdot)$ and $\mu_3(\cdot)$ are constants, which simplifies many calculation processes. If we consider system (1) and (2) only in phase 2 and phase 3, then system (1) and (2) become the M/G/1 queueing model with optional deterministic server vacations, see, e.g., [9,10]. In [9], they proved that hypotheses (**H1**) and (**H2**) hold true for the optional deterministic server vacations queueing model. According to [11], it can be seen that if the reliable retrial queueing model (see, e.g., [12], p. 9), is extended to the case of the retrial queueing model with server breakdowns (see, e.g., [13]), the steady-state assumption of this server breakdowns queueing model does not hold true. Therefore, it is necessary to investigate whether the hypotheses (**H1**) and (**H2**) of system (1) and (2) are valid.

In this paper, first of all, we transform system (1) and (2) into a Cauchy problem in a Banach space. Second, by using the semigroup theory, we verify that the system operator of system (1) and (2) generates a positive C_0-semigroup of contractions in the space, which is isometric under certain conditions. Thus, we show that the system has a unique positive TDS that satisfies the probability condition, which means (**H1**) holds under certain conditions.

To answer (**H2**), that is, to obtain the asymptotic properties of the TDS of system (1) and (2), we need to calculate the spectrum of the system operator, see, e.g., [7,11,14–17]. When the service rates $\mu_1(\cdot)$ and $\mu_3(\cdot)$ are constants, the M/G/1 stochastic clearing queueing model in a three-phase environment is called the M/M/1 stochastic clearing queueing model in a three-phase environment. In this case, by studying the spectral distribution of a system operator on the imaginary axis, we show that zero is a point spectrum of the operator and its adjoint operator with geometric multiplicity one, and other points on the imaginary axis are not spectrums of the operator. Hence, we obtain that the TDS of system (1) and (2) strongly converges to its non-zero SSS. This implies (**H2**) holds under strong convergence.

Furthermore, we consider the spectrum-determined growth condition (SDGC), as well as the stability of the semigroup corresponding to system (1) and (2). We obtain that the SDGC is equal to zero, and the semigroup is not asymptotically stable.

The organization of this paper is as follows. In Section 2, we write system (1) and (2) as a Cauchy problem in a Banach space and present the well-posedness. Based on the spectrum analysis, Section 3 is devoted to the asymptotic behavior of the TDS of Equations (1) and (2). The stability of the corresponding semigroup of system (1) and (2) is further investigated in Section 4, which leads to the SDGC. Section 5 concludes the paper.

2. Well-Posedness Of The System

We consider system (1) and (2) in the following state Banach space

$$X = \left\{ (V, p_1, p_3) \left| \begin{array}{l} V = (Q_{1,0}, Q_{3,0}, V_0, V_1, V_2, \cdots) \in l^1, p_j = (p_{j,1}, p_{j,2}, \cdots), \\ p_{j,n} \in L^1[0,\infty), \|(V, p_1, p_3)\| = \sum_{j=1,3} |Q_{j,0}| \\ + \sum_{n=0}^{\infty} |V_n| + \sum_{j=1,3} \sum_{n=1}^{\infty} \|p_{j,n}\|_{L^1[0,\infty)} < \infty \end{array} \right. \right\}.$$

In the following, we define the system operator and its domain. We define the operator A and its domain by

$$A(V, p_1, p_3) = \left(\begin{pmatrix} -(\lambda_1+\theta_1)Q_{1,0} + \theta_3 Q_{3,0} \\ -(\lambda_3+\theta_3)Q_{3,0} + N_0 V_0 \\ -V_0 \\ -V_1 \\ \vdots \end{pmatrix}, \begin{pmatrix} -\frac{dp_{1,1}}{dx} \\ -\frac{dp_{1,2}}{dx} \\ -\frac{dp_{1,3}}{dx} \\ \vdots \end{pmatrix}, \begin{pmatrix} -\frac{dp_{3,1}}{dx} \\ -\frac{dp_{3,2}}{dx} \\ -\frac{dp_{3,3}}{dx} \\ \vdots \end{pmatrix} \right),$$

$$D(A) = \left\{ (V, p_1, p_3) \left| \begin{array}{l} \frac{dp_{j,n}}{dx} \in L^1[0,\infty),\ p_{j,n} \text{ are absolutely continuous} \\ \text{functions and } \ p_1(0) = \int_0^\infty \Gamma_{1,1} V dx + \int_0^\infty \Gamma_{1,2} p_1(x) dx, \\ p_3(0) = \int_0^\infty \Gamma_{3,1} V dx + \int_0^\infty \Gamma_{3,2} p_3(x) dx + \Gamma V \end{array} \right. \right\},$$

where $j = 1,3;\ n \geq 1$, and

$$\Gamma_{1,1} = \begin{pmatrix} \lambda_1 e^{-x} & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad \Gamma_{3,1} = \begin{pmatrix} 0 & \lambda_3 e^{-x} & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\Gamma_{j,2} = \begin{pmatrix} 0 & \mu_j(x) & 0 & 0 & \cdots \\ 0 & 0 & \mu_j(x) & 0 & \cdots \\ 0 & 0 & 0 & \mu_j(x) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 & 0 & N_1 & N_0 & 0 & 0 & \cdots \\ 0 & 0 & N_2 & N_1 & N_0 & 0 & \cdots \\ 0 & 0 & N_3 & N_2 & N_1 & N_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \end{pmatrix}.$$

Define the operators U and E and their domains by

$$U(V, p_1, p_3) = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} -\eta_1(x)p_{1,1}(x) \\ \lambda_1 p_{1,1}(x) - \eta_1(x)p_{1,2}(x) \\ \lambda_1 p_{1,2}(x) - \eta_1(x)p_{1,3}(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} -\eta_3(x)p_{3,1}(x) \\ \lambda_3 p_{3,1}(x) - \eta_3(x)p_{3,2}(x) \\ \lambda_3 p_{3,2}(x) - \eta_3(x)p_{3,3}(x) \\ \vdots \end{pmatrix} \right),$$

$$E(V, p_1, p_3) = \left(\begin{pmatrix} \int_0^\infty p_{1,1}(x)\mu_1(x)dx + \theta_3 \sum_{n=1}^\infty \int_0^\infty p_{3,n}(x)dx \\ \int_0^\infty p_{3,1}(x)\mu_3(x)dx \\ 0 \\ \theta_1 \int_0^\infty p_{1,1}(x)dx \\ \theta_1 \int_0^\infty p_{1,2}(x)dx \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix} \right),$$

$$D(U) = X, \quad D(E) = X,$$

where $\eta_j(\cdot) = \lambda_j + \theta_j + \mu_j(\cdot),\ j = 1,3$. Then, system (1) and (2) can be written as a Cauchy problem in X:

$$\begin{cases} \dfrac{d(V(t), p_1(\cdot,t), p_3(\cdot,t))}{dt} = (A + U + E)(V(t), p_1(\cdot,t), p_3(\cdot,t)), \quad t \in (0,\infty), \\ (V(0), p_1(\cdot,0), p_3(\cdot,0)) = (u_2, u_1(\cdot), u_3(\cdot)), \\ u_2 = (\overline{u}_{2,1}, \overline{u}_{2,3}, u_{2,0}, u_{2,1}, \cdots), u_j(\cdot) = (u_{j,1}(\cdot), u_{j,2}(\cdot), \cdots),\ j = 1,3. \end{cases} \tag{3}$$

Now, we study the well-posedness of system (3). Firstly, we prove that the operator $A + U + E$ generates a positive C_0-semigroup $e^{(A+U+E)t}$ of contractions on X. Secondly, we show that $A + U + E$ is a conservative operator, from which we obtain that $e^{(A+U+E)t}$ is an isometric semigroup on X. Hence, we present the well-posedness of system (3).

Theorem 1. *If $\lambda_j, \theta_j > 0$ and $0 < \overline{\mu}_j = \sup_{x\in[0,\infty)} \mu_j(x) < \infty,\ j = 1,3,$ then $A + U + E$ generates a positive C_0-semigroup $e^{(A+U+E)t}$ of contractions on X.*

Proof. The proof will be finished by four steps.

Step 1. If $\gamma > \mathbb{M} = \max\{\overline{\mu}_1, \overline{\mu}_3\}$ and $\overline{\mu}_3 \geq N_0$, then $(\gamma I - A)^{-1}$ exists and is bounded.

For any $(w, y_1, y_3) \in X$, we consider $(\gamma I - A)(V, p_1, p_3) = (w, y_1, y_3)$ for unknown (V, p_1, p_3), where $w = (z_{1,0}, z_{3,0}, w_0, w_1, \cdots)$, $y_j = (y_{j,1}, y_{j,2}, y_{j,3}, \cdots), j = 1,3$. This is equivalent to

$$(\gamma + \lambda_1 + \theta_1)Q_{1,0} = z_{1,0} + \theta_3 Q_{3,0}, \tag{4a}$$

$$(\gamma + \lambda_3 + \theta_3)Q_{3,0} = z_{3,0} + N_0 V_0, \tag{4b}$$

$$(\gamma + 1)V_0 = w_0 + \theta_1 Q_{1,0}, \tag{4c}$$

$$(\gamma + 1)V_n = w_n, \quad n \geq 1, \tag{4d}$$

$$\frac{dp_{j,n}(x)}{dx} = -\gamma p_{j,n}(x) + y_{j,n}(x), \quad j = 1,3;\ n \geq 1, \tag{4e}$$

$$p_{1,1}(0) = \lambda_1 Q_{1,0} + \int_0^\infty p_{1,2}(x)\mu_1(x)dx \tag{4f}$$

$$p_{1,n}(0) = \int_0^\infty p_{1,n+1}(x)\mu_1(x)dx, \quad n \geq 2, \tag{4g}$$

$$p_{3,1}(0) = \lambda_3 Q_{3,0} + N_1 V_0 + N_0 V_1 + \int_0^\infty p_{3,2}(x)\mu_3(x)dx, \tag{4h}$$

$$p_{3,n}(0) = \sum_{k=0}^{n} N_k V_{n-k} + \int_0^\infty p_{3,n+1}(x)\mu_3(x)dx, \quad n \geq 2. \tag{4i}$$

Solve system (4e) to obtain

$$p_{1,n}(x) = a_n e^{-\gamma x} + e^{-\gamma x}\int_0^x y_{1,n}(\tau)e^{\gamma\tau}d\tau, \quad n \geq 1, \tag{5a}$$

$$p_{3,n}(x) = b_n e^{-\gamma x} + e^{-\gamma x}\int_0^x y_{3,n}(\tau)e^{\gamma\tau}d\tau, \quad n \geq 1, \tag{5b}$$

where $a_n = p_{1,n}(0), b_n = p_{3,n}(0)$. Combining the boundary conditions system (4f)–(4i) with system (5a) and (5b), we obtain that a_n and b_n satisfy

$$a_1 - a_2\int_0^\infty \mu_1(x)e^{-\gamma x}dx = \lambda_1 Q_{1,0} + \int_0^\infty \mu_1(x)e^{-\gamma x}\int_0^x y_{1,2}(\tau)e^{\gamma\tau}d\tau dx, \tag{6a}$$

$$a_n - a_{n+1}\int_0^\infty \mu_1(x)e^{-\gamma x}dx = \int_0^\infty \mu_1(x)e^{-\gamma x}\int_0^x y_{1,n+1}(\tau)e^{\gamma\tau}d\tau dx, \quad n \geq 2, \tag{6b}$$

$$b_1 - b_2\int_0^\infty \mu_3(x)e^{-\gamma x}dx = \lambda_3 Q_{3,0} + N_1 V_0 + N_0 V_1 + \int_0^\infty \mu_3(x)e^{-\gamma x}\int_0^x y_{3,2}(\tau)e^{\gamma\tau}d\tau dx, \tag{6c}$$

$$b_n - b_{n+1}\int_0^\infty \mu_3(x)e^{-\gamma x}dx = \sum_{k=0}^{n} N_k V_{n-k} + \int_0^\infty \mu_3(x)e^{-\gamma x}\int_0^x y_{3,n+1}(\tau)e^{\gamma\tau}d\tau dx,\ n \geq 2. \tag{6d}$$

If we define

$$C_j := \begin{pmatrix} 1 & -\alpha_j & 0 & \cdots \\ 0 & 1 & -\alpha_j & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},\ \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix}, \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \end{pmatrix}, \quad j = 1,3, \tag{7}$$

where $\alpha_j = \int_0^\infty \mu_j(x)e^{-\gamma x}dx$, then system (6a)–(7) can be written as

$$C_1\vec{a} = \begin{pmatrix} \lambda_1 Q_{1,0} + \int_0^\infty \mu_1(x)e^{-\gamma x}\int_0^x y_{1,2}(\tau)e^{\gamma\tau}d\tau dx \\ \int_0^\infty \mu_1(x)e^{-\gamma x}\int_0^x y_{1,3}(\tau)e^{\gamma\tau}d\tau dx \\ \int_0^\infty \mu_1(x)e^{-\gamma x}\int_0^x y_{1,4}(\tau)e^{\gamma\tau}d\tau dx \\ \vdots \end{pmatrix}, \tag{8a}$$

$$C_3\vec{b} = \begin{pmatrix} \lambda_3 Q_{3,0} + \sum_{k=0}^{1} N_k V_{1-k} + \int_0^\infty \mu_3(x)e^{-\gamma x}\int_0^x y_{3,2}(\tau)e^{\gamma\tau}d\tau dx \\ \sum_{k=0}^{2} N_k V_{2-k} + \int_0^\infty \mu_3(x)e^{-\gamma x}\int_0^x y_{3,3}(\tau)e^{\gamma\tau}d\tau dx \\ \sum_{k=0}^{3} N_k V_{3-k} + \int_0^\infty \mu_3(x)e^{-\gamma x}\int_0^x y_{3,4}(\tau)e^{\gamma\tau}d\tau dx \\ \vdots \end{pmatrix}. \tag{8b}$$

The inverse of C_j can be directly calculated as

$$C_j^{-1} = \begin{pmatrix} 1 & \alpha_j & \alpha_j^2 & \alpha_j^3 & \cdots \\ 0 & 1 & \alpha_j & \alpha_j^2 & \cdots \\ 0 & 0 & 1 & \alpha_j & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ j = 1, 3. \tag{9}$$

By Equations (8a)–(9), it is easy to see that

$$a_1 = \lambda_1 Q_{1,0} + \sum_{m=1}^{\infty} \left(\int_0^\infty \mu_1(x) e^{-\gamma x} dx \right)^{m-1} \int_0^\infty \mu_1(x) e^{-\gamma x} \int_0^x y_{1,m+1}(\tau) e^{\gamma \tau} d\tau dx, \tag{10a}$$

$$a_n = \sum_{m=1}^{\infty} \left(\int_0^\infty \mu_1(x) e^{-\gamma x} dx \right)^{m-1} \int_0^\infty \mu_1(x) e^{-\gamma x} \int_0^x y_{1,m+n}(\tau) e^{\gamma \tau} d\tau dx, \quad n \geq 2, \tag{10b}$$

$$\begin{aligned} b_1 &= \lambda_3 Q_{3,0} + \sum_{m=1}^{\infty} \left(\int_0^\infty \mu_3(x) e^{-\gamma x} dx \right)^{m-1} \\ &\quad \times \left(\sum_{k=0}^{m} N_k V_{m-k} + \int_0^\infty \mu_3(x) e^{-\gamma x} \int_0^x y_{3,m+1}(\tau) e^{\gamma \tau} d\tau dx \right), \end{aligned} \tag{10c}$$

$$\begin{aligned} b_n &= \sum_{m=1}^{\infty} \left(\int_0^\infty \mu_3(x) e^{-\gamma x} dx \right)^{m-1} \\ &\quad \times \left(\sum_{k=0}^{m+n-1} N_k V_{m+n-1-k} + \int_0^\infty \mu_3(x) e^{-\gamma x} \int_0^x y_{3,m+n}(\tau) e^{\gamma \tau} d\tau dx \right), \quad n \geq 2. \end{aligned} \tag{10d}$$

For any $\gamma > 0$, from system (10a)–(10d) and the Fubini theorem, it is easy to calculate that

$$|a_1| \leq \lambda_1 |Q_{1,0}| + \sum_{m=1}^{\infty} \left(\frac{\overline{\mu}_1}{\gamma} \right)^m \|y_{1,m+1}\|_{L^1[0,\infty)}, \tag{11a}$$

$$|a_n| \leq \sum_{m=1}^{\infty} \left(\frac{\overline{\mu}_1}{\gamma} \right)^m \|y_{1,m+n}\|_{L^1[0,\infty)}, \quad n \geq 2, \tag{11b}$$

$$|b_1| \leq \lambda_3 |Q_{3,0}| + \sum_{m=1}^{\infty} \left(\frac{\overline{\mu}_3}{\gamma} \right)^{m-1} \sum_{k=0}^{m} N_k |V_{m-k}| + \sum_{m=1}^{\infty} \left(\frac{\overline{\mu}_3}{\gamma} \right)^m \|y_{3,m+1}\|_{L^1[0,\infty)}, \tag{11c}$$

$$|b_n| \leq \sum_{m=1}^{\infty} \left(\frac{\overline{\mu}_3}{\gamma} \right)^{m-1} \sum_{k=0}^{m+n-1} N_k |V_{m+n-1-k}| + \sum_{m=1}^{\infty} \left(\frac{\overline{\mu}_3}{\gamma} \right)^m \|y_{3,m+n}\|_{L^1[0,\infty)}, n \geq 2. \tag{11d}$$

Thus, by using system (5a) and (5b) and inequalities system (11a)–(11d), we compute for all $\gamma > 0$ that

$$\begin{aligned} \sum_{n=1}^{\infty} \|p_{1,n}\|_{L^1[0,\infty)} &= \left(\int_0^\infty \left| a_n e^{-\gamma x} + e^{-\gamma x} \int_0^x y_{1,n}(\tau) e^{\gamma \tau} d\tau \right| dx \right) \\ &\leq \sum_{n=1}^{\infty} \left(\frac{1}{\gamma} |a_n| + \frac{1}{\gamma} \|y_{1,n}\|_{L^1[0,\infty)} \right) \\ &\leq \frac{1}{\gamma} \left[\lambda_1 |Q_{1,0}| + \sum_{m=1}^{\infty} \left(\frac{\overline{\mu}_1}{\gamma} \right)^m \|y_{1,m+1}\|_{L^1[0,\infty)} \right] \\ &\quad + \frac{1}{\gamma} \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \left(\frac{\overline{\mu}_1}{\gamma} \right)^m \|y_{1,m+n}\|_{L^1[0,\infty)} + \frac{1}{\gamma} \sum_{n=1}^{\infty} \|y_{1,n}\|_{L^1[0,\infty)} \end{aligned}$$

$$\leq \frac{\lambda_1}{\gamma}|Q_{1,0}| + \frac{1}{\gamma - \overline{\mu}_1} \sum_{n=1}^{\infty} \|y_{1,n}\|_{L^1[0,\infty)}, \tag{12a}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \|p_{3,n}\|_{L^1[0,\infty)} &= \left(\int_0^{\infty} \left| b_n e^{-\gamma x} + e^{-\gamma x} \int_0^x y_{3,n}(\tau) e^{\gamma \tau} d\tau \right| dx \right) \\
&\leq \sum_{n=1}^{\infty} \left(\frac{1}{\gamma}|b_n| + \frac{1}{\gamma}\|y_{3,n}\|_{L^1[0,\infty)} \right) \\
&\leq \frac{1}{\gamma}\left[\lambda_3|Q_{3,0}| + \sum_{m=1}^{\infty} \left(\frac{\overline{\mu}_3}{\gamma}\right)^{m-1} \sum_{k=0}^{m} N_k|V_{m-k}| + \sum_{m=1}^{\infty} \left(\frac{\overline{\mu}_3}{\gamma}\right)^{m} \|y_{3,m+1}\|_{L^1[0,\infty)} \right] \\
&\quad + \frac{1}{\gamma} \sum_{n=2}^{\infty} \left[\sum_{m=1}^{\infty} \left(\frac{\overline{\mu}_3}{\gamma}\right)^{m-1} \sum_{k=0}^{m+n-1} N_k |V_{m+n-1-k}| \right. \\
&\quad \left. + \sum_{m=1}^{\infty} \left(\frac{\overline{\mu}_3}{\gamma}\right)^{m} \|y_{3,m+n}\|_{L^1[0,\infty)} \right] + \frac{1}{\gamma} \sum_{n=1}^{\infty} \|y_{3,n}\|_{L^1[0,\infty)} \\
&\leq \frac{\lambda_3}{\gamma}|Q_{3,0}| + \frac{1}{\gamma - \overline{\mu}_3} \sum_{n=0}^{\infty} |V_n| + \frac{1}{\gamma - \overline{\mu}_3} \sum_{n=1}^{\infty} \|y_{3,n}\|_{L^1[0,\infty)}.
\end{aligned} \tag{12b}$$

When $\gamma > 0$, through simple calculations from system (4a)–(4d), we can conclude that

$$|Q_{1,0}| \leq \frac{(\gamma+1)(\gamma+\lambda_3+\theta_3)|z_{1,0}| + \theta_3(\gamma+1)|z_{3,0}| + \theta_3 N_0 |w_0|}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3) - \theta_1\theta_3 N_0}, \tag{13a}$$

$$|Q_{3,0}| \leq \frac{\theta_1 N_0 |z_{1,0}| + (\gamma+1)(\gamma+\lambda_1+\theta_1)|z_{3,0}| + N_0(\gamma+\lambda_1+\theta_1)|w_0|}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3) - \theta_1\theta_3 N_0}, \tag{13b}$$

$$|V_0| \leq \frac{\theta_1(\gamma+\lambda_3+\theta_3)|z_{1,0}| + \theta_1\theta_3|z_{3,0}| + (\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)|w_0|}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3) - \theta_1\theta_3 N_0}, \tag{13c}$$

$$|V_n| = \frac{1}{\gamma+1}|w_n|, \quad n \geq 1. \tag{13d}$$

Hence, for all $\gamma > \mathbb{M} = \max\{\overline{\mu}_1, \overline{\mu}_3\}$ and $\overline{\mu}_3 \geq N_0$, using the inequalities system (12a)–(13d), we conclude that

$$\begin{aligned}
\|(V, p_1, p_3)\| &\leq |Q_{1,0}| + |Q_{3,0}| + |V_0| + \sum_{n=1}^{\infty} |V_n| + \frac{\lambda_1}{\gamma}|Q_{1,0}| + \frac{1}{\gamma - \overline{\mu}_1} \sum_{n=1}^{\infty} \|y_{1,n}\|_{L^1[0,\infty)} \\
&\quad + \frac{\lambda_3}{\gamma}|Q_{3,0}| + \frac{1}{\gamma - \overline{\mu}_3} \sum_{n=0}^{\infty} |V_n| + \frac{1}{\gamma - \overline{\mu}_3} \sum_{n=1}^{\infty} \|y_{3,n}\|_{L^1[0,\infty)} \\
&\leq \frac{(\gamma+\lambda_1)[(\gamma+1)(\gamma+\lambda_3+\theta_3)|z_{1,0}| + \theta_3(\gamma+1)|z_{3,0}| + \theta_3 N_0|w_0|]}{\gamma[(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3) - \theta_1\theta_3 N_0]} \\
&\quad + \frac{(\gamma+\lambda_3)[\theta_1 N_0|z_{1,0}| + (\gamma+1)(\gamma+\lambda_1+\theta_1)|z_{3,0}| + N_0(\gamma+\lambda_1+\theta_1)|w_0|]}{\gamma[(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3) - \theta_1\theta_3 N_0]} \\
&\quad + \frac{(\gamma-\overline{\mu}_3+1)[\theta_1(\gamma+\lambda_3+\theta_3)|z_{1,0}| + \theta_1\theta_3|z_{3,0}| + (\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)|w_0|]}{\gamma[(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3) - \theta_1\theta_3 N_0]} \\
&\quad + \frac{\gamma - \overline{\mu}_3 + 1}{(\gamma+1)(\gamma-\overline{\mu}_3)} \sum_{n=1}^{\infty} |w_n| + \frac{1}{\gamma - \overline{\mu}_1} \sum_{n=1}^{\infty} \|y_{1,n}\|_{L^1[0,\infty)} + \frac{1}{\gamma - \overline{\mu}_3} \sum_{n=1}^{\infty} \|y_{3,n}\|_{L^1[0,\infty)} \\
&\leq \frac{1}{\gamma}|z_{1,0}| + \frac{1}{\gamma}|z_{3,0}| + \frac{1}{\gamma}|w_0| + \frac{1}{\gamma - \overline{\mu}_3} \sum_{n=1}^{\infty} |w_n| + \sum_{j=1,3} \frac{1}{\gamma - \overline{\mu}_j} \sum_{n=1}^{\infty} \|y_{j,n}\|_{L^1[0,\infty)}
\end{aligned}$$

$$< \frac{1}{\gamma - \mathbb{M}} \|(w, y_1, y_3)\|. \tag{14}$$

In the third inequality of the above equation, the following inequalities are utilized

$$0 < \frac{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3) - \theta_1(\gamma+\lambda_3)(\overline{\mu}_3 - N_0) - \theta_1\theta_3\overline{\mu}_3}{\gamma[(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3) - \theta_1\theta_3 N_0]} \leq \frac{1}{\gamma}, \tag{15a}$$

$$0 < \frac{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3) - \theta_1\theta_3\overline{\mu}_3}{\gamma[(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3) - \theta_1\theta_3 N_0]} \leq \frac{1}{\gamma}, \tag{15b}$$

$$\begin{aligned} 0 \quad < & \frac{1}{\gamma[(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3) - \theta_1\theta_3 N_0]} \\ & \times\{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3) \\ & -\theta_1\theta_3 N_0 - (\overline{\mu}_3 - N_0)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)\} \leq \frac{1}{\gamma}. \end{aligned} \tag{15c}$$

Therefore, for any $\gamma > \mathbb{M} = \max\{\overline{\mu}_1, \overline{\mu}_3\}$ and $\overline{\mu}_3 \geq N_0$, the inequality system (14) implies that

$$(\gamma I - A)^{-1} : X \to D(A), \quad \|(\gamma I - A)^{-1}\| < \frac{1}{\gamma - \mathbb{M}}.$$

Step 2. $D(A)$ is dense in X. Since $(V, p_1, p_3) \in X$, for any $\varepsilon > 0$, $\exists\, K \in \mathbb{N}_+$, if $n > K$, then

$$\sum_{n=K+1}^{\infty} |V_n| < \varepsilon, \quad \sum_{n=K+1}^{\infty} \|p_{j,n}\|_{L^1[0,\infty)} < \varepsilon, \quad j = 1, 3.$$

Due to the set

$$Y_1 = \left\{ (V, p_1, p_3) \middle| \begin{array}{l} V = (Q_{1,0}, Q_{3,0}, V_0, V_1, \cdots, V_K, 0, \cdots), Q_{j,0}, V_n \in \mathbb{R}, 0 \leq n \leq K, \\ p_j = (p_{j,1}, p_{j,2}, \cdots, p_{j,K}, 0, \cdots), p_{j,n} \in L^1[0,\infty), 1 \leq n \leq K, K \in \mathbb{N}_+ \end{array} \right\}$$

being dense in X, by the corollary 2.30 of [18], we know that the set

$$Y_2 = \left\{ (V, p_1, p_3) \middle| \begin{array}{l} V = (Q_{1,0}, Q_{3,0}, V_0, V_1, \cdots, V_{K_0}, 0, \cdots), Q_{j,0}, V_n \in \mathbb{R}, 0 \leq n \leq K_0, \\ p_j = (p_{j,1}, p_{j,2}, \cdots, p_{j,K_0}, 0, \cdots), p_{j,n} \in C_0^\infty[0,\infty), \exists\, c_{j,n} \text{ such that} \\ p_{j,n}(x) = 0, x \in [0, c_{j,n}],\ j = 1,3;\ n = 1, 2, \cdots, K_0,\ K_0 \in \mathbb{N}_+ \end{array} \right\}$$

is dense in Y_1. Hence, to prove the denseness of $D(A)$, it is necessary to prove $Y_2 \subset \overline{D(A)}$. Actually, if $Y_2 \subset \overline{D(A)}$, then $\overline{Y_2} \subset \overline{\overline{D(A)}} = \overline{D(A)}$. Then, using $\overline{Y_2} = \overline{\overline{Y_2}} = \overline{Y_1} = X$ and $D(A) \subset X$, we know that $\overline{D(A)} \subset \overline{X} = X$. Therefore, $\overline{D(A)} = X$.

To this aim, we take $(V, p_1, p_3) \in Y_2$. Then, for any $x \in [0, 2s]$, we have

$$p_{j,n}(x) = 0,\ j = 1,3;\ n = 1, 2, \cdots, K_0,$$

where $0 < 2s < \min\{c_{1,1}, c_{1,2}, \cdots, c_{1,K_0}, c_{3,1}, c_{3,2}, \cdots, c_{3,K_0}\}$. If we define (V', f_1, f_3) by

$$\begin{aligned} & V' = (Q'_{1,0}, Q'_{3,0}, V'_0, V'_1, \cdots, V'_{K_0}, 0, \cdots) = (Q_{1,0}, Q_{3,0}, V_0, V_1, \cdots, V_{K_0}, 0, \cdots), \\ & f_j(x) = (f_{j,1}(x), f_{j,2}(x), \cdots, f_{j,K_0}(x), 0, \cdots) = (p_{j,1}(x), p_{j,2}(x), \cdots, p_{j,K_0}(x), 0, \cdots), \\ & f_{1,1}(0) = \lambda_1 Q_{1,0} + \int_0^\infty p_{1,2}(x)\mu_1(x)dx, \\ & f_{1,n}(0) = \int_0^\infty p_{1,n+1}(x)\mu_1(x)dx,\ f_{1,K_0}(0) = 0, \\ & f_{3,1}(0) = \lambda_3 Q_{3,0} + N_1 V_0 + N_0 V_1 + \int_0^\infty p_{3,2}(x)\mu_3(x)dx, \end{aligned}$$

$$f_{3,n}(0)=\sum_{k=0}^{n}N_kV_{n-k}+\int_0^\infty p_{3,n+1}(x)\mu_3(x)dx,\ f_{3,K_0}(0)=\sum_{k=0}^{K_0}N_kV_{K_0-k}, \tag{16}$$

where $j=1,3$, $2\le n\le K_0-1$, and

$$h_{j,n}=\frac{\int_0^s f_{j,n}(0)\left(1-\frac{x}{s}\right)^2\mu_j(x)dx}{\int_s^{2s}(x-s)^2(x-2s)^2\mu_j(x)dx},\quad f_{j,n}(x)=\begin{cases} f_{j,n}(0)\left(1-\frac{x}{s}\right)^2 & x\in[0,s],\\ -h_{j,n}(x-s)^2(x-2s)^2 & x\in[s,2s],\\ p_{j,n}(x) & x\in[2s,\infty)\end{cases}$$

for any $n=1,2,\cdots,K_0$, then, it is not difficult to prove that $(V',f_1,f_3)\in D(A)$. Therefore,

$$\begin{aligned}\|(V,p_1,p_3)-(V',f_1,f_3)\|&=\sum_{j=1,3}\sum_{n=1}^{K_0}\int_0^\infty|p_{j,n}(x)-f_{j,n}(x)|dx\\ &=\sum_{j=1,3}\sum_{n=1}^{K_0}|f_{j,n}(0)|\frac{s}{3}+\sum_{j=1,3}\sum_{n=1}^{K_0}|h_{j,n}|\frac{s^5}{30}\to 0\text{ as } s\to 0.\end{aligned} \tag{17}$$

That is, $Y_2\subset\overline{D(A)}$. Hence, we have $\overline{D(A)}=X$.

Therefore, using Step 1, Step 2 and the Hille–Yosida theorem (see, e.g., [14], Thm 1.68), we obtain that the operator A generates a C_0-semigroup.

Step 3. The operators U and E are linear bounded. According to definitions of U and E, for any $(V,p_1,p_3)\in X$, through simple calculations, it can be concluded that

$$\begin{aligned}\|U(V,p_1,p_3)\|\quad&\le\sum_{j=1,3}\sum_{n=1}^{\infty}\int_0^\infty[2\lambda_j+\theta_j+\mu_j(x)]|p_{j,n}(x)|dx\\ &\le\max\{2\lambda_1+\theta_1+\overline{\mu}_1,2\lambda_3+\theta_3+\overline{\mu}_3\}\|(V,p_1,p_3)\|,\end{aligned} \tag{18a}$$

$$\begin{aligned}\|E(V,p_1,p_3)\|\quad&\le\sum_{j=1,3}\theta_j\sum_{n=1}^{\infty}\int_0^\infty|p_{j,n}(x)|dx+\sum_{j=1,3}\int_0^\infty\mu_j(x)|p_{j,1}(x)|dx\\ &\le\max\{\theta_1+\overline{\mu}_1,\theta_3+\overline{\mu}_3\}\|(V,p_1,p_3)\|.\end{aligned} \tag{18b}$$

The inequalities system (18a) and (18b) mean that U and E are bounded. Obviously, U and E are linear operators. Therefore, the proof of this step is complete.

Thus, by the perturbation theorem of a strongly continuous semigroup (see, e.g., [14], Thm 1.80), we deduce that the operator $A+U+E$ generates a C_0-semigroup $e^{(A+U+E)t}$.

Step 4. The operator $A+U+E$ is dispersive. For any $(V,p_1,p_3)\in X$, we define (B,B_1,B_3) by

$$B=\left(\frac{[Q_{1,0}]^+}{Q_{1,0}},\frac{[Q_{3,0}]^+}{Q_{3,0}},\frac{[V_0]^+}{V_0},\frac{[V_1]^+}{V_1},\cdots\right),\ B_j=\left(\frac{[p_{j,1}(x)]^+}{p_{j,1}(x)},\frac{[p_{j,2}(x)]^+}{p_{j,2}(x)},\cdots\right),\ j=1,3,$$

where

$$[Q_{j,0}]^+=\begin{cases}Q_{j,0}, & Q_{j,0}>0,\\ 0, & Q_{j,0}\le 0,\end{cases}\qquad [V_n]^+=\begin{cases}V_n, & V_n>0,\\ 0, & V_n\le 0,\end{cases}\ n=0,1,2,\cdots$$

$$[p_{j,n}(x)]^+=\begin{cases}p_{j,n}(x), & p_{j,n}(x)>0,\\ 0, & p_{j,n}(x)\le 0,\end{cases}\quad n=1,2,\cdots.$$

By simple calculations (for the detailed calculations see the Equation (2.12) of [19]), it can be concluded that

$$\int_0^\infty \frac{dp_{j,n}(x)}{dx}\frac{[p_{j,n}(x)]^+}{p_{j,n}(x)}dx = -[p_{j,n}(0)]^+, \quad j=1,3;\ n=1,2,\cdots.$$

Therefore, for any $\lambda_j, \theta_j > 0, j = 1,3$ and $0 < N_0 < 1$, using the boundary conditions system (4f)–(4i), it is computed that

$$\begin{aligned}
&\langle (A+U+E)(V,p_1,p_3),(B,B_1,B_3)\rangle \\
&= \left[-(\lambda_1+\theta_1)Q_{1,0}+\theta_3 Q_{3,0}+\int_0^\infty p_{1,1}(x)\mu_1(x)dx+\theta_3\sum_{n=1}^\infty\int_0^\infty p_{3,n}(x)dx\right]\frac{[Q_{1,0}]^+}{Q_{1,0}} \\
&\quad+\left[-(\lambda_3+\theta_3)Q_{3,0}+N_0V_0+\int_0^\infty p_{3,1}(x)\mu_3(x)dx\right]\frac{[Q_{3,0}]^+}{Q_{3,0}} \\
&\quad+(-V_0+\theta_1 Q_{1,0})\frac{[V_0]^+}{V_0}+\sum_{n=1}^\infty\left[-V_n+\theta_1\int_0^\infty p_{1,n}(x)dx\right]\frac{[V_n]^+}{V_n} \\
&\quad+\sum_{n=1}^\infty\int_0^\infty\left[-\frac{dp_{1,n}(x)}{dx}-(\lambda_1+\theta_1+\mu_1(x))p_{1,n}(x)\right]\frac{[p_{1,n}(x)]^+}{p_{1,n}}dx \\
&\quad+\sum_{n=2}^\infty\int_0^\infty \lambda_1 p_{1,n-1}(x)\frac{[p_{1,n}(x)]^+}{p_{1,n}}dx \\
&\quad+\sum_{n=1}^\infty\int_0^\infty\left[-\frac{dp_{3,n}(x)}{dx}-(\lambda_3+\theta_3+\mu_3(x))p_{3,n}(x)\right]\frac{[p_{3,n}(x)]^+}{p_{3,n}}dx \\
&\quad+\sum_{n=2}^\infty\int_0^\infty \lambda_3 p_{3,n-1}(x)\frac{[p_{3,n}(x)]^+}{p_{3,n}}dx \\
&\le \theta_1\left(\frac{[V_0]^+}{V_0}-1\right)[Q_{1,0}]^+ + \theta_3\left(\frac{[Q_{1,0}]^+}{Q_{1,0}}-1\right)\sum_{n=1}^\infty\int_0^\infty[p_{3,n}(x)]^+dx \\
&\quad+\theta_3\left(\frac{[Q_{1,0}]^+}{Q_{1,0}}-1\right)[Q_{3,0}]^+ + \left(\frac{[Q_{1,0}]^+}{Q_{1,0}}-1\right)\int_0^\infty[p_{1,1}(x)]^+\mu_1(x)dx \\
&\quad+\left(\frac{[Q_{3,0}]^+}{Q_{3,0}}N_0-1\right)[V_0]^+ + \left(\frac{[Q_{3,0}]^+}{Q_{3,0}}-1\right)\int_0^\infty[p_{3,1}(x)]^+\mu_3(x)dx \\
&\quad+\theta_1\sum_{n=1}^\infty\left(\frac{[V_n]^+}{V_n}-1\right)\int_0^\infty[p_{1,n}(x)]^+dx \le 0.
\end{aligned} \tag{19}$$

The inequality system (19) shows that the operator $A+U+E$ is dispersive.

Hence, by the above discussions and the Phillips theorem (see, e.g., [14] Thm 1.77, we see that the result of this Theorem holds true. □

Next, we consider the isometry of the semigroup $e^{(A+U+E)t}$. Clearly, the dual space X^* of X is

$$X^* = \left\{(V^*,p_1^*,p_3^*)\;\middle|\; \begin{array}{l} V^* = (Q_{1,0}^*, Q_{3,0}^*, V_0^*, V_1^*, \cdots) \in l^\infty, p_j^* = (p_{j,1}^*, p_{j,2}^*, \cdots), \\ p_{j,n}^* \in L^\infty[0,\infty), \|(V^*,p_1^*,p_3^*)\|_* \\ = \sup\left\{|Q_{j,0}^*|, \sup_{n\ge 0}|V_n^*|, \sup_{n\ge 1}\|p_{j,n}^*\|_{L^\infty[0,\infty)}\right\} < \infty \end{array}\right\}.$$

If we define

$$X_+ = \left\{(V,p_1,p_3)\in X \;\middle|\; \begin{array}{l} V = (Q_{1,0}, Q_{3,0}, V_0, V_1, \cdots) \in l^1, Q_{j,0} \ge 0, V_n \ge 0, n \ge 0, \\ p_j = (p_{j,1}, p_{j,2}, \cdots),\ 0 \le p_{j,n} \in L^1[0,\infty),\ \ j = 1,3; n \ge 1 \end{array}\right\},$$

then, by Theorem 1, we have $e^{(A+U+E)t}X_+ \subset X_+$. For any $(V, p_1, p_3) \in D(A) \cap X_+$, we choose

$$(V^*, p_1^*, p_3^*) = \|(V, p_1, p_3)\| \left(\begin{pmatrix} 1 \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 \\ \vdots \end{pmatrix} \right).$$

Then, it is easy to show that $(V^*, p_1^*, p_3^*) \in X^*$ and

$$\begin{aligned} &\langle (V, p_1, p_3), (V^*, p_1^*, p_3^*) \rangle \\ &= \left(Q_{1,0} + Q_{3,0} + \sum_{n=0}^{\infty} V_n + \sum_{j=1,3} \sum_{n=1}^{\infty} \int_0^\infty p_{j,n}(x)dx \right) \|(V, p_1, p_3)\| \\ &= \|(V, p_1, p_3)\|^2. \end{aligned} \tag{20}$$

This means that $(V^*, p_1^*, p_3^*) \in Q(V, p_1, p_3)$, where

$$Q(V, p_1, p_3) = \left\{ (V^*, p_1^*, p_3^*) \in X^* \middle| \ \langle (V, p_1, p_3), (V^*, p_1^*, p_3^*) \rangle = \|(V, p_1, p_3)\|^2 \ \right\}.$$

In addition, for any $(V, p_1, p_3) \in D(A)$ and $(V^*, p_1^*, p_3^*) \in Q(V, p_1, p_3)$, using the boundary conditions system (4f)–(4i) and $\sum_{n=0}^{\infty} N_n = 1$, we have

$$\begin{aligned} &\langle A(V, p_1, p_3), (V^*, p_1^*, p_3^*) \rangle \\ &= \left[-(\lambda_1 + \theta_1)Q_{1,0} + \theta_3 Q_{3,0} + \int_0^\infty p_{1,1}(x)\mu_1(x)dx + \theta_3 \sum_{n=1}^{\infty} \int_0^\infty p_{3,n}(x)dx \right] \\ &\times \|(V, p_1, p_3)\| \\ &\quad + \left[-(\lambda_3 + \theta_3)Q_{3,0} + N_0 V_0 + \int_0^\infty p_{3,1}(x)\mu_3(x)dx \right] \|(V, p_1, p_3)\| \\ &\quad + (-V_0 + \theta_1 Q_{1,0})\|(V, p_1, p_3)\| + \sum_{n=1}^{\infty} \left[-V_n + \theta_1 \int_0^\infty p_{1,n}(x)dx \right] \|(V, p_1, p_3)\| \\ &\quad + \sum_{n=1}^{\infty} \int_0^\infty \left[-\frac{dp_{1,n}(x)}{dx} - (\lambda_1 + \theta_1 + \mu_1(x))p_{1,n}(x) \right] \|(V, p_1, p_3)\| dx \\ &\quad + \sum_{n=2}^{\infty} \int_0^\infty \lambda_1 p_{1,n-1}(x) \|(V, p_1, p_3)\| dx + \sum_{n=2}^{\infty} \int_0^\infty \lambda_3 p_{3,n-1}(x) \|(V, p_1, p_3)\| dx \\ &\quad + \sum_{n=1}^{\infty} \int_0^\infty \left[-\frac{dp_{3,n}(x)}{dx} - (\lambda_3 + \theta_3 + \mu_3(x))p_{3,n}(x) \right] \|(V, p_1, p_3)\| dx \\ &= \left[-\lambda_1 Q_{1,0} + \int_0^\infty p_{1,1}(x)\mu_1(x)dx - \lambda_3 Q_{3,0} + N_0 V_0 + \int_0^\infty p_{3,1}(x)\mu_3(x)dx \right. \\ &\quad - \sum_{n=0}^{\infty} V_n + \sum_{n=1}^{\infty} \left(p_{1,n}(0) - \int_0^\infty \mu_1(x)p_{1,n}(x) \right) dx \\ &\quad \left. + \sum_{n=1}^{\infty} \left(p_{3,n}(0) - \int_0^\infty \mu_3(x)p_{3,n}(x) \right) dx \right] \|(V, p_1, p_3)\| = 0. \end{aligned} \tag{21}$$

system (21) means that $A + U + E$ is a conservative operator with respect to the set $Q(\cdot)$.

By Theorem 3.6.1 of [20] (p. 155), we have the following Theorem 2.

Theorem 2. *If $\lambda_j,\ \theta_j > 0, 0 < \overline{\mu}_j < \infty,\ j = 1,3$ and the initial value $(V(0), p_1(\cdot,0), p_3(\cdot,0))$ of the system (3) satisfies $(V(0), p_1(\cdot,0), p_3(\cdot,0)) \in D(A^2)$, then*

$$\|e^{(A+U+E)t}(V(0), p_1(\cdot,0), p_3(\cdot,0))\| = \|(V(0), p_1(\cdot,0), p_3(\cdot,0))\|, \quad t \in [0, \infty).$$

Now, using Theorems 1 and 2, we obtain the main result in this section.

Theorem 3. *If $\lambda_j,\ \theta_j > 0, 0 < \overline{\mu}_j < \infty,\ j = 1, 3$ and the initial value $(V(0), p_1(\cdot, 0), p_3(\cdot, 0))$ of the system (3) satisfies $(V(0), p_1(\cdot, 0), p_3(\cdot, 0)) \in D(A^2)$, then the system (3) admits a unique positive TDS $(V(t), p_1(\cdot, t), p_3(\cdot, t))$, which satisfies*

$$\|(V(t), p_1(\cdot, t), p_3(\cdot, t))\| = 1,\ \forall t \in [0, \infty).$$

Proof. It is not difficult to see that $(V(0), p_1(\cdot, 0), p_3(\cdot, 0)) \in D(A^2) \cap X_+$. Theorem 1 and Theorem 1.81 of [14] mean that system (3) admits a unique positive TDS $(V(t), p_1(\cdot, t), p_3(\cdot, t))$, which can be described by

$$(V(t), p_1(\cdot, t), p_3(\cdot, t)) = e^{(A+U+E)t}(V(0), p_1(\cdot, 0), p_3(\cdot, 0)), \quad t \in [0, \infty).$$

Due to Theorem 2, it is easy to see that

$$\|(V(t), p_1(\cdot, t), p_3(\cdot, t))\| = \|(V(0), p_1(\cdot, 0), p_3(\cdot, 0))\| = 1, \quad t \in [0, \infty).$$

This shows that the physics reflected by $(V(\cdot), p_1(\cdot, \cdot), p_3(\cdot, \cdot))$ is reasonable. □

Theorem 3 means that (**H1**) holds under some conditions.

3. Asymptotic Behavior of the TDS of the System

When the service rates $\mu_1(\cdot) = \mu_1$ and $\mu_3(\cdot) = \mu_3$ are constants, the M/G/1 stochastic clearing queueing model in a three-phase environment is called the M/M/1 stochastic clearing queueing model in a three-phase environment. In this section, we discuss the strong convergence of the TDS of system (3) in the above special case. For this aim, first of all, we determine the adjoint operator $(A + U + E)^*$ of the system operator $A + U + E$. Then, we find the spectrum of $(A + U + E)^*$ on the imaginary axis. Hence, we provide the spectrum of $A + U + E$ on the imaginary axis by using the relationship between the spectrum of the operator and its adjoint operator. Finally, we derive that zero is a point spectrum of $A + U + E$ and $(A + U + E)^*$, and the geometric multiplicity of zero is one. Therefore, we obtain that the TDS of system (3) strongly converges to its SSS. That is, (**H2**) holds true under strong convergence.

The following Theorem 4 is the main result of this section.

Theorem 4. *If $\lambda_j, \theta_j, \mu_j > 0$ and $\lambda_j < \theta_j + \mu_j, j = 1, 3$, then the TDS of the system (3) strongly converges to its SSS; that is*

$$\lim_{t\to\infty} \|(V(t), p_1(\cdot, t), p_3(\cdot, t)) - \langle (V^*, p_1^*(\cdot), p_3^*(\cdot)), (u_2, u_1(\cdot), u_3(\cdot))\rangle (V, p_1(\cdot), p_3(\cdot))\| = 0,$$

where $(V, p_1(\cdot), p_3(\cdot))$ and $(V^*, p_1^*(\cdot), p_3^*(\cdot))$ are the eigenvectors in Lemmas 3 and 4, respectively.

To prove the above Theorem 4, first we prove the following four lemmas.

Lemma 1. *The adjoint operator $(A + U + E)^*$ of $(A + U + E)$ is given by*

$$(A + U + E)^*(V^*, p_1^*, p_3^*) = (L + J + G + H)(V^*, p_1^*, p_3^*), \quad (V^*, p_1^*, p_3^*) \in D(L),$$

where the operators L, J, G, H and its domain are defined by

$$L(V^*, p_1^*, p_3^*) = \left(\begin{pmatrix} -(\lambda_1+\theta_1)Q_{1,0}^* + \theta_1 V_0^* \\ \theta_3 Q_{1,0}^* - (\lambda_3+\theta_3)Q_{3,0}^* \\ N_0 Q_{3,0}^* - V_0^* \\ -V_1^* \\ -V_2^* \\ \vdots \end{pmatrix}, \begin{pmatrix} \left(\frac{d}{dx} - \eta_1\right) p_{1,1}^*(x) \\ \left(\frac{d}{dx} - \eta_1\right) p_{1,2}^*(x) \\ \left(\frac{d}{dx} - \eta_1\right) p_{1,3}^*(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} \left(\frac{d}{dx} - \eta_3\right) p_{3,1}^*(x) \\ \left(\frac{d}{dx} - \eta_3\right) p_{3,2}^*(x) \\ \left(\frac{d}{dx} - \eta_3\right) p_{3,3}^*(x) \\ \vdots \end{pmatrix} \right),$$

$$D(L) = \left\{ (V^*, p_1^*, p_3^*) \,\middle|\, \frac{dp_{j,n}^*}{dx} \text{ exist and } p_{j,n}^*(\infty) = \alpha, j = 1, 3;\ n \geq 1 \right\},$$

$$J(V^*, p_1^*, p_3^*) = \left(\begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} \lambda_1 p_{1,2}^*(x) \\ \lambda_1 p_{1,3}^*(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} \lambda_3 p_{3,2}^*(x) \\ \lambda_3 p_{3,3}^*(x) \\ \vdots \end{pmatrix} \right),$$

$$G(V^*, p_1^*, p_3^*) = \left(\begin{pmatrix} \lambda_1 p_{1,1}^*(0) \\ \lambda_3 p_{3,1}^*(0) \\ \sum_{k=1}^{\infty} N_k p_{3,k}^*(0) \\ \sum_{k=0}^{\infty} N_k p_{3,k+1}^*(0) \\ \sum_{k=0}^{\infty} N_k p_{3,k+2}^*(0) \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ \mu_1 p_{1,1}^*(0) \\ \mu_1 p_{1,2}^*(0) \\ \mu_1 p_{1,3}^*(0) \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ \mu_3 p_{3,1}^*(0) \\ \mu_3 p_{3,2}^*(0) \\ \mu_3 p_{3,3}^*(0) \\ \vdots \end{pmatrix} \right),$$

$$H(V^*, p_1^*, p_3^*) = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} \mu_1 Q_{1,0}^* + \theta_1 V_1^* \\ \theta_1 V_2^* \\ \theta_1 V_3^* \\ \vdots \end{pmatrix}, \begin{pmatrix} \theta_3 Q_{1,0}^* + \mu_3 Q_{3,0}^* \\ \theta_3 Q_{1,0}^* \\ \theta_3 Q_{1,0}^* \\ \vdots \end{pmatrix} \right),$$

$$D(J) = X^*, \quad D(G) = X^*, \quad D(H) = X^*;$$

here, $\eta_j = \lambda_j + \theta_j + \mu_j$, and α is a positive constant irrelevant of j and n.

Proof. For every $(V, p_1, p_3) \in D(A)$ and $(V^*, p_1^*, p_3^*) \in D(L)$, using the boundary conditions system (4f)–(4i) and integration by parts, we have

$$\begin{aligned}
&\langle (A+U+E)(V, p_1, p_3), (V^*, p_1^*, p_3^*) \rangle \\
&= \left[-(\lambda_1+\theta_1)Q_{1,0} + \theta_3 \sum_{n=1}^{\infty} \int_0^{\infty} p_{3,n}(x)dx + \theta_3 Q_{3,0} + \mu_1 \int_0^{\infty} p_{1,1}(x)dx \right] Q_{1,0}^* \\
&\quad + \left[-(\lambda_3+\theta_3)Q_{3,0} + N_0 V_0 + \mu_3 \int_0^{\infty} p_{3,1}(x)dx \right] Q_{3,0}^* \\
&\quad + (-V_0 + \theta_1 Q_{1,0})V_0^* + \sum_{n=1}^{\infty} \left[-V_n + \theta_1 \int_0^{\infty} p_{1,n}(x)dx \right] V_n^* \\
&\quad + \int_0^{\infty} \left[-\frac{dp_{1,1}(x)}{dx} - (\lambda_1+\theta_1+\mu_1)p_{1,1}(x) \right] p_{1,1}^*(x)dx \\
&\quad + \sum_{n=2}^{\infty} \int_0^{\infty} \left[-\frac{dp_{1,n}(x)}{dx} - (\lambda_1+\theta_1+\mu_1)p_{1,n}(x) + \lambda_1 p_{1,n-1}(x) \right] p_{1,n}^*(x)dx \\
&\quad + \int_0^{\infty} \left[-\frac{dp_{3,1}(x)}{dx} - (\lambda_3+\theta_3+\mu_3)p_{3,1}(x) \right] p_{3,1}^*(x)dx \\
&\quad + \sum_{n=2}^{\infty} \int_0^{\infty} \left[-\frac{dp_{3,n}(x)}{dx} - (\lambda_3+\theta_3+\mu_3)p_{3,n}(x) + \lambda_1 p_{3,n-1}(x) \right] p_{3,n}^*(x)dx
\end{aligned}$$

$$
\begin{aligned}
&= -(\lambda_1+\theta_1)Q_{1,0}Q_{1,0}^* + \theta_3 Q_{1,0}^* \sum_{n=1}^{\infty}\int_0^{\infty} p_{3,n}(x)dx + \theta_3 Q_{3,0}Q_{1,0}^* \\
&\quad + \mu_1 Q_{1,0}^* \int_0^{\infty} p_{1,1}(x)dx - (\lambda_3+\theta_3)Q_{3,0}Q_{3,0}^* + N_0 V_0 Q_{3,0}^* \\
&\quad + \mu_3 Q_{3,0}^* \int_0^{\infty} p_{3,1}(x)dx + \theta_1 Q_{1,0} V_0^* - \sum_{n=0}^{\infty} V_n V_n^* + \theta_1 \sum_{n=1}^{\infty} V_n^* \int_0^{\infty} p_{1,n}(x)dx \\
&\quad + \lambda_1 Q_{1,0} p_{1,1}^*(0) + \mu_1 \sum_{n=1}^{\infty} p_{1,n}^*(0) \int_0^{\infty} p_{1,n+1}(x)dx \\
&\quad + \sum_{n=1}^{\infty}\int_0^{\infty} p_{1,n}(x)\left[\frac{dp_{1,n}^*(x)}{dx} - (\lambda_1+\theta_1+\mu_1)p_{1,n}^*(x)\right]dx \\
&\quad + \lambda_1 \sum_{n=1}^{\infty}\int_0^{\infty} p_{1,n}(x)p_{1,n+1}^*(x)dx + \lambda_3 Q_{3,0} p_{3,1}^*(0) + V_0 \sum_{k=1}^{\infty} N_k p_{3,k}^*(0) \\
&\quad + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} N_k V_n p_{3,k+n}^*(0) + \mu_3 \sum_{n=1}^{\infty} p_{3,n}^*(0)\int_0^{\infty} p_{3,n+1}(x)dx \\
&\quad + \sum_{n=1}^{\infty}\int_0^{\infty} p_{3,n}(x)\left[\frac{dp_{3,n}^*(x)}{dx} - (\lambda_3+\theta_3+\mu_3)p_{3,n}^*(x)\right]dx \\
&\quad + \lambda_3 \sum_{n=1}^{\infty}\int_0^{\infty} p_{3,n}(x)p_{3,n+1}^*(x)dx \\
&= \langle (V,p_1,p_3),(L+J+G+H)(V^*,p_1^*,p_3^*)\rangle. \qquad (22)
\end{aligned}
$$

system (22) means that this Lemma holds true. □

Lemma 2. *If $\lambda_j,\theta_j,\mu_j > 0,\ j=1,3$, then all points in the set*

$$
\Lambda := \left\{ \gamma \in \mathbb{C} \;\middle|\;
\begin{array}{l}
\sup\Big\{ \frac{1}{|(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3 N_0|} \sup\Big\{ \frac{\lambda_1(\theta_1+\mu_1)|(\gamma+1)(\gamma+\lambda_3+\theta_3)|}{|\sigma_1|}, \\
\frac{\lambda_1^2|(\gamma+1)(\gamma+\lambda_3+\theta_3)|}{\Re\sigma_1}, \frac{\lambda_3\theta_1 N_0(\theta_3+\mu_3)}{|\sigma_3|}, \frac{\lambda_3^2\theta_1 N_0}{\Re\sigma_3}, \frac{\theta_1\theta_3(1-N_0)|\gamma+\lambda_3+\theta_3|}{|\sigma_3|-\mu_3}, \\
\frac{\lambda_3\mu_3\theta_1(1-N_0)|\gamma+\lambda_3+\theta_3|}{\Re\sigma_3(|\sigma_3|-\mu_3)}, \frac{\lambda_1\theta_3(\theta_1+\mu_1)|\gamma+1|}{|\sigma_1|}, \frac{\lambda_1^2\theta_3|\gamma+1|}{\Re\sigma_1}, \frac{\theta_1\theta_3^2(1-N_0)}{|\sigma_3|-\mu_3}, \\
\frac{\lambda_3(\theta_3+\mu_3)|(\gamma+1)(\gamma+\lambda_1+\theta_1)|}{|\sigma_3|}, \frac{\lambda_3^2|(\gamma+1)(\gamma+\lambda_1+\theta_1)|}{\Re\sigma_3}, \frac{\lambda_3\mu_3\theta_1\theta_3(1-N_0)}{\Re\sigma_3(|\sigma_3|-\mu_3)}, \\
\frac{\lambda_3\theta_1 N_0(\theta_1+\mu_1)}{|\sigma_1|}, \frac{\lambda_1\lambda_3\theta_1 N_0}{\Re\sigma_1}, \frac{\lambda_3 N_0(\theta_3+\mu_3)|\gamma+\lambda_1+\theta_1|}{|\sigma_3|}, \frac{\lambda_3^2 N_0|\gamma+\lambda_1+\theta_1|}{\Re\sigma_3}, \\
\frac{\theta_3(1-N_0)|(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)|}{|\sigma_3|-\mu_3}, \frac{\lambda_3\mu_3(1-N_0)|(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)|}{\Re\sigma_3(|\sigma_3|-\mu_3)} \Big\}, \\
\frac{\theta_3}{|\gamma+1||\sigma_3|-\mu_3}, \frac{\lambda_3\mu_3}{|\gamma+1|\Re\sigma_3(|\sigma_3|-\mu_3)}, \frac{\theta_1+\mu_1}{|\sigma_1|}, \frac{\lambda_1}{\Re\sigma_1}, \frac{\theta_1}{|\sigma_1|-\mu_1}, \frac{\lambda_1\mu_1}{\Re\sigma_1(|\sigma_1|-\mu_1)}, \\
\frac{\theta_3+\mu_3}{|\sigma_3|}, \frac{\lambda_3}{\Re\sigma_3}, \frac{\theta_3}{|\sigma_3|-\mu_3}, \frac{\lambda_3\mu_3}{\Re\sigma_3(|\sigma_3|-\mu_3)} \Big\} < 1, \Re\sigma_j > 0,\ |\sigma_j| > \mu_j
\end{array}
\right\}
$$

belong to the resolvent set $\rho((A+U+E)^*)$. In particular, $i\mathbb{R}\setminus\{0\} \subset \rho((A+U+E)^*), i^2=-1$. Hence, $i\mathbb{R}\setminus\{0\} \subset \rho(A+U+E)$, where $\sigma_j = \gamma+\lambda_j+\theta_j+\mu_j, 0 < N_0 < 1$, and $\Re\gamma$ is the real part of γ.

Proof. For every $(w^*, y_1^*, y_3^*) \in X^*$, we consider

$$
(\gamma I - L - G)(V^*, p_1^*, p_3^*) = (J+H)(w^*, y_1^*, y_3^*),
$$

where $w^* = (z_{1,0}^*, z_{3,0}^*, w_0^*, w_1^*, \cdots),\ y_j^* = (y_{j,1}^*, y_{j,2}^*, y_{j,3}^*, \cdots),\ j=1,3$. That is,

$$
(\gamma+\lambda_1+\theta_1)Q_{1,0}^* = \theta_1 V_0^* + \lambda_1 p_{1,1}^*(0), \qquad (23a)
$$

$$
(\gamma+\lambda_3+\theta_3)Q_{3,0}^* = \theta_3 Q_{1,0}^* + \lambda_3 p_{3,1}^*(0), \qquad (23b)
$$

$$(\gamma+1)V_0^* = N_0 Q_{3,0}^* + \sum_{k=1}^{\infty} N_k p_{3,k}^*(0), \tag{23c}$$

$$(\gamma+1)V_n^* = \sum_{k=0}^{\infty} N_k p_{3,k+n}^*(0), \quad n \geq 1, \tag{23d}$$

$$\frac{dp_{1,1}^*(x)}{dx} = (\gamma+\lambda_1+\theta_1+\mu_1)p_{1,1}^*(x) - \mu_1 z_{1,0}^* - \theta_1 w_1^* - \lambda_1 y_{1,2}^*(x), \tag{23e}$$

$$\frac{dp_{1,n}^*(x)}{dx} = (\gamma+\lambda_1+\theta_1+\mu_1)p_{1,n}^*(x) - \theta_1 w_n^* - \mu_1 p_{1,n-1}^*(0) - \lambda_1 y_{1,n+1}^*(x),\ n \geq 2, \tag{23f}$$

$$\frac{dp_{3,1}^*(x)}{dx} = (\gamma+\lambda_3+\theta_3+\mu_3)p_{3,1}^*(x) - \theta_3 z_{1,0}^* - \mu_3 z_{3,0}^* - \lambda_3 y_{3,2}^*(x), \tag{23g}$$

$$\frac{dp_{3,n}^*(x)}{dx} = (\gamma+\lambda_3+\theta_3+\mu_3)p_{3,n}^*(x) - \theta_3 z_{1,0}^* - \mu_3 p_{3,n-1}^*(0) - \lambda_3 y_{3,n+1}^*(x),\ n \geq 2, \tag{23h}$$

$$p_{1,n}^*(\infty) = p_{3,n}^*(\infty) = \alpha, \quad n \geq 1. \tag{23i}$$

Solve the Equations (23e)–(23h) to obtain

$$p_{1,1}^*(x) = p_{1,1}^*(0)e^{\sigma_1 x} - e^{\sigma_1 x}\int_0^x [\mu_1 z_{1,0}^* + \theta_1 w_1^* + \lambda_1 y_{1,2}^*(\tau)]e^{-\sigma_1 \tau}d\tau \tag{24a}$$

$$p_{1,n}^*(x) = p_{1,n}^*(0)e^{\sigma_1 x} - e^{\sigma_1 x}\int_0^x [\theta_1 w_n^* + \mu_1 p_{1,n-1}^*(0) + \lambda_1 y_{1,n+1}^*(\tau)]e^{-\sigma_1 \tau}d\tau,\ n \geq 2, \tag{24b}$$

$$p_{3,1}^*(x) = p_{3,1}^*(0)e^{\sigma_3 x} - e^{\sigma_3 x}\int_0^x [\theta_3 z_{1,0}^* + \mu_3 z_{3,0}^* + \lambda_3 y_{3,2}^*(\tau)]e^{-\sigma_3 \tau}d\tau \tag{24c}$$

$$p_{3,n}^*(x) = p_{3,n}^*(0)e^{\sigma_3 x} - e^{\sigma_3 x}\int_0^x [\theta_3 z_{1,0}^* + \mu_3 p_{3,n-1}^*(0) + \lambda_3 y_{3,n+1}^*(\tau)]e^{-\sigma_3 \tau}d\tau,\ n \geq 2, \tag{24d}$$

where $\sigma_j = \gamma + \lambda_j + \theta_j + \mu_j,\ j = 1,3$. Through multiplying $e^{-\sigma_1 x}$ and $e^{-\sigma_3 x}$ to two sides of system (24a)–(24d), respectively, then taking a limit for these new equations when $x \to \infty$, and using boundary condition system (23i) and $\Re\gamma + \lambda_j + \theta_j + \mu_j = \Re\sigma_j > 0,\ j = 1,3$, we have

$$p_{1,1}^*(0) = \frac{\mu_1}{\sigma_1} z_{1,0}^* + \frac{\theta_1}{\sigma_1} w_1^* + \lambda_1 \int_0^\infty y_{1,2}^*(x)e^{-\sigma_1 x}dx, \tag{25a}$$

$$p_{1,n}^*(0) = \frac{\mu_1}{\sigma_1} p_{1,n-1}^*(0) + \frac{\theta_1}{\sigma_1} w_n^* + \lambda_1 \int_0^\infty y_{1,n+1}^*(x)e^{-\sigma_1 x}dx,\ n \geq 2, \tag{25b}$$

$$p_{3,1}^*(0) = \frac{\theta_3}{\sigma_3} z_{1,0}^* + \frac{\mu_3}{\sigma_3} z_{3,0}^* + \lambda_3 \int_0^\infty y_{3,2}^*(x)e^{-\sigma_3 x}dx, \tag{25c}$$

$$p_{3,n}^*(0) = \frac{\theta_3}{\sigma_3} z_{1,0}^* + \frac{\mu_3}{\sigma_3} p_{3,n-1}^*(0) + \lambda_3 \int_0^\infty y_{3,n+1}^*(x)e^{-\sigma_3 x}dx,\ n \geq 2. \tag{25d}$$

Repeatedly using system (25a)–(25d), it is not difficult to find that

$$p_{1,n}^*(0) = \left(\frac{\mu_1}{\sigma_1}\right)^n z_{1,0}^* + \frac{\theta_1}{\mu_1}\sum_{k=1}^{n}\left(\frac{\mu_1}{\sigma_1}\right)^{n+1-k} w_k^* + \lambda_1 \sum_{k=1}^{n}\left(\frac{\mu_1}{\sigma_1}\right)^{n-k} \int_0^\infty y_{1,k+1}^*(x)e^{-\sigma_1 x}dx, \tag{26a}$$

$$p_{3,n}^*(0) = \left(\frac{\mu_3}{\sigma_3}\right)^n z_{3,0}^* + \frac{\theta_3}{\mu_3}\sum_{k=1}^{n}\left(\frac{\mu_3}{\sigma_3}\right)^{k} z_{1,0}^* + \lambda_3 \sum_{k=1}^{n}\left(\frac{\mu_3}{\sigma_3}\right)^{n-k} \int_0^\infty y_{3,k+1}^*(x)e^{-\sigma_3 x}dx. \tag{26b}$$

Thus, for $\Re\sigma_j > 0,\ j = 1,3$, by Equations (24a)–(24d) and system (26a) and (26b), we obtain the following expressions of $p_{1,n}^*(x)$ and $p_{3,n}^*(x)$ without boundary conditions

$$\begin{aligned} p_{1,1}^*(x) &= e^{\sigma_1 x}\int_x^\infty [\mu_1 z_{1,0}^* + \theta_1 w_1^* + \lambda_1 y_{1,2}^*(\tau)]e^{-\sigma_1 \tau}d\tau \\ &= \frac{\mu_1}{\sigma_1} z_{1,0}^* + \frac{\theta_1}{\sigma_1} w_1^* + \lambda_1 e^{\sigma_1 x}\int_x^\infty y_{1,2}^*(\tau)e^{-\sigma_1 \tau}d\tau, \end{aligned} \tag{27a}$$

$$\begin{aligned}
p^*_{1,n}(x) &= e^{\sigma_1 x}\int_x^\infty [\theta_1 w^*_n + \mu_1 p^*_{1,n-1}(0) + \lambda_1 y^*_{1,n+1}(\tau)]e^{-\sigma_1\tau}d\tau \\
&= \frac{\mu_1}{\sigma_1}p^*_{1,n-1}(0) + \frac{\theta_1}{\sigma_1}w^*_n + \lambda_1 e^{\sigma_1 x}\int_x^\infty y^*_{1,n+1}(\tau)e^{-\sigma_1\tau}d\tau \\
&= \left(\frac{\mu_1}{\sigma_1}\right)^n z^*_{1,0} + \frac{\theta_1}{\mu_1}\sum_{k=1}^{n}\left(\frac{\mu_1}{\sigma_1}\right)^{n+1-k} w^*_k + \lambda_1\sum_{k=1}^{n-1}\left(\frac{\mu_1}{\sigma_1}\right)^{n-k}\int_0^\infty y^*_{1,k+1}(x)e^{-\sigma_1 x}dx \\
&\quad + \lambda_1 e^{\sigma_1 x}\int_x^\infty y^*_{1,n+1}(\tau)e^{-\sigma_1\tau}d\tau,\ n\geq 2,
\end{aligned} \tag{27b}$$

$$\begin{aligned}
p^*_{3,1}(x) &= e^{\sigma_3 x}\int_x^\infty [\theta_3 z^*_{1,0} + \mu_3 z^*_{3,0} + \lambda_3 y^*_{3,2}(\tau)]e^{-\sigma_3\tau}d\tau \\
&= \frac{\mu_3}{\sigma_3}z^*_{3,0} + \frac{\theta_3}{\sigma_3}z^*_{1,0} + \lambda_3 e^{\sigma_3 x}\int_x^\infty y^*_{3,2}(\tau)e^{-\sigma_3\tau}d\tau,
\end{aligned} \tag{27c}$$

$$\begin{aligned}
p^*_{3,n}(x) &= \int_x^\infty [\theta_3 z^*_{1,0} + \mu_3 p^*_{3,n-1}(0) + \lambda_3 y^*_{3,n+1}(\tau)]e^{-\sigma_3\tau}d\tau \\
&= \frac{\mu_3}{\sigma_3}p^*_{3,n-1}(0) + \frac{\theta_3}{\sigma_3}z^*_{1,0} + \lambda_3 e^{\sigma_3 x}\int_x^\infty y^*_{3,n+1}(\tau)e^{-\sigma_3\tau}d\tau \\
&= \left(\frac{\mu_3}{\sigma_3}\right)^n z^*_{3,0} + \frac{\theta_3}{\mu_3}\sum_{k=1}^{n}\left(\frac{\mu_3}{\sigma_3}\right)^{k} z^*_{1,0} + \lambda_3\sum_{k=1}^{n-1}\left(\frac{\mu_3}{\sigma_3}\right)^{n-k}\int_0^\infty y^*_{3,k+1}(x)e^{-\sigma_3 x}dx \\
&\quad + \lambda_3 e^{\sigma_3 x}\int_x^\infty y^*_{3,n+1}(\tau)e^{-\sigma_3\tau}d\tau,\ n\geq 2.
\end{aligned} \tag{27d}$$

Hence , when $|\sigma_j| > \mu_j$ and $\Re\sigma_j > 0,\ j = 1,3$, from system (27a)–(27d), we conclude that the following inequalities hold true

$$\begin{aligned}
\|p^*_{1,1}\|_{L^\infty[0,\infty)} &\leq \frac{\mu_1}{|\sigma_1|}|z^*_{1,0}| + \frac{\theta_1}{|\sigma_1|}|w^*_1| + \frac{\lambda_1}{\Re\sigma_1}\|y^*_{1,2}\|_{L^\infty[0,\infty)} \\
&\leq \sup\left\{\frac{\theta_1+\mu_1}{|\sigma_1|}, \frac{\lambda_1}{\Re\sigma_1}\right\}\|(w^*, y^*_1, y^*_3)\|,
\end{aligned} \tag{28a}$$

$$\begin{aligned}
\sup_{n\geq 2}\|p^*_{1,n}\|_{L^\infty[0,\infty)} &\leq \sup_{n\geq 2}\left\{\left(\frac{\mu_1}{|\sigma_1|}\right)^n |z^*_{1,0}| + \frac{\theta_1}{\mu_1}\sum_{k=1}^{n}\left(\frac{\mu_1}{|\sigma_1|}\right)^{n+1-k}|w^*_k|\right. \\
&\quad \left. + \frac{\lambda_1}{\Re\sigma_1}\sum_{k=1}^{n-1}\left(\frac{\mu_1}{|\sigma_1|}\right)^{n-k}\|y^*_{1,k+1}\|_{L^\infty[0,\infty)} + \frac{\lambda_1}{\Re\sigma_1}\|y^*_{1,n+1}\|_{L^\infty[0,\infty)}\right\} \\
&\leq \sup\left\{\frac{\theta_1}{|\sigma_1|-\mu_1}, \frac{\lambda_1\mu_1}{(\Re\sigma_1)(|\sigma_1|-\mu_1)}\right\}\|(w^*, y^*_1, y^*_3)\|,
\end{aligned} \tag{28b}$$

$$\begin{aligned}
\|p^*_{3,1}\|_{L^\infty[0,\infty)} &\leq \frac{\mu_3}{|\sigma_3|}|z^*_{3,0}| + \frac{\theta_3}{|\sigma_3|}|z^*_{1,0}| + \frac{\lambda_3}{\Re\sigma_3}\|y^*_{3,2}\|_{L^\infty[0,\infty)} \\
&\leq \sup\left\{\frac{\theta_3+\mu_3}{|\sigma_3|}, \frac{\lambda_3}{\Re\sigma_3}\right\}\|(w^*, y^*_1, y^*_3)\|,
\end{aligned} \tag{28c}$$

$$\sup_{n\geq 2}\|p^*_{3,n}\|_{L^\infty[0,\infty)} \leq \sup_{n\geq 2}\left\{\left(\frac{\mu_3}{|\sigma_3|}\right)^n |z^*_{3,0}| + \frac{\theta_3}{\mu_3}\sum_{k=1}^{n}\left(\frac{\mu_3}{|\sigma_3|}\right)^{k}|z^*_{1,0}|\right.$$

$$+\frac{\lambda_3}{\Re\sigma_3}\sum_{k=1}^{n-1}\left(\frac{\mu_3}{|\sigma_3|}\right)^{n-k}\|y_{3,k+1}^*\|_{L^\infty[0,\infty)}+\frac{\lambda_3}{\Re\sigma_3}\|y_{3,n+1}^*\|_{L^\infty[0,\infty)}\Bigg\}$$

$$\leq \sup\left\{\frac{\theta_3}{|\sigma_3|-\mu_3},\frac{\lambda_3\mu_3}{(\Re\sigma_3)(|\sigma_3|-\mu_3)}\right\}\|(w^*,y_1^*,y_3^*)\|. \tag{28d}$$

On the other hand, if $|\sigma_j|>\mu_j$ and $\Re\sigma_j>0,\ j=1,3$, then, by Equations (23a)–(23d), through simple but tedious calculations, we obtain

$$Q_{1,0}^* = \frac{\lambda_1(\gamma+1)(\gamma+\lambda_3+\theta_3)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}p_{1,1}^*(0)$$
$$+\frac{\lambda_3\theta_1N_0}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}p_{3,1}^*(0)$$
$$+\frac{\theta_1(\gamma+\lambda_3+\theta_3)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}\sum_{k=1}^{\infty}N_kp_{3,k}^*(0), \tag{29a}$$

$$Q_{3,0}^* = \frac{\lambda_1\theta_3(\gamma+1)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}p_{1,1}^*(0)$$
$$+\frac{\lambda_3(\gamma+1)(\gamma+\lambda_1+\theta_1)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}p_{3,1}^*(0)$$
$$+\frac{\theta_1\theta_3}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}\sum_{k=1}^{\infty}N_kp_{3,k}^*(0), \tag{29b}$$

$$V_0^* = \frac{\lambda_1\theta_3N_0}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}p_{1,1}^*(0)$$
$$+\frac{\lambda_3N_0(\gamma+\lambda_1+\theta_1)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}p_{3,1}^*(0)$$
$$+\frac{(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)}{(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0}\sum_{k=1}^{\infty}N_kp_{3,k}^*(0), \tag{29c}$$

$$V_n^* = \frac{1}{\gamma+1}\sum_{k=0}^{\infty}N_kp_{3,k+n}^*(0). \tag{29d}$$

Then, by system (29a)–(29d) and system (26a) and (26b), it is not difficult to see that $Q_{1,0}^*, Q_{3,0}^*$ and $V_n, n\geq 0$ satisfy the following inequalities

$$|Q_{1,0}^*| \leq \frac{1}{|(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0|}$$
$$\times\sup\Bigg\{\frac{\lambda_1(\theta_1+\mu_1)|(\gamma+1)(\gamma+\lambda_3+\theta_3)|}{|\sigma_1|},\frac{\lambda_1^2|(\gamma+1)(\gamma+\lambda_3+\theta_3)|}{\Re\sigma_1},$$
$$\frac{\lambda_3\theta_1N_0(\theta_3+\mu_3)}{|\sigma_3|},\frac{\lambda_3^2\theta_1N_0}{\Re\sigma_3},\frac{\theta_1\theta_3(1-N_0)|\gamma+\lambda_3+\theta_3|}{|\sigma_3|-\mu_3},$$
$$\frac{\lambda_3\mu_3\theta_1(1-N_0)|\gamma+\lambda_3+\theta_3|}{(\Re\sigma_3)(|\sigma_3|-\mu_3)}\Bigg\}\|(w^*,y_1^*,y_3^*)\|, \tag{30a}$$

$$|Q_{3,0}^*| \leq \frac{1}{|(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0|}\sup\Bigg\{\frac{\lambda_1\theta_3(\theta_1+\mu_1)|\gamma+1|}{|\sigma_1|},$$

$$\frac{\lambda_1^2\theta_3|\gamma+1|}{\Re\sigma_1}, \frac{\lambda_3(\theta_3+\mu_3)|(\gamma+1)(\gamma+\lambda_1+\theta_1)|}{|\sigma_3|}, \frac{\theta_1\theta_3^2(1-N_0)}{|\sigma_3|-\mu_3},$$
$$\left.\frac{\lambda_3^2|(\gamma+1)(\gamma+\lambda_1+\theta_1)|}{\Re\sigma_3}, \frac{\lambda_3\mu_3\theta_1\theta_3(1-N_0)}{\Re\sigma_3(|\sigma_3|-\mu_3)}\right\}\|(w^*,y_1^*,y_3^*)\|, \tag{30b}$$

$$\begin{aligned} |V_0^*| \quad &\le \frac{1}{|(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0|} \\ &\times \sup\left\{\frac{\lambda_3\theta_1N_0(\theta_1+\mu_1)}{|\sigma_1|}, \frac{\lambda_1\lambda_3\theta_1N_0}{\Re\sigma_1},\right. \\ &\frac{\lambda_3N_0(\theta_3+\mu_3)|\gamma+\lambda_1+\theta_1|}{|\sigma_3|}, \frac{\lambda_3^2N_0|\gamma+\lambda_1+\theta_1|}{\Re\sigma_3}, \\ &\frac{\theta_3(1-N_0)|(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)|}{|\sigma_3|-\mu_3}, \\ &\left.\frac{\lambda_3\mu_3(1-N_0)|(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)|}{\Re\sigma_3(|\sigma_3|-\mu_3)}\right\}\|(w^*,y_1^*,y_3^*)\|, \end{aligned} \tag{30c}$$

$$\sup_{n\ge1}|V_n^*| \quad \le \sup\left\{\frac{\theta_3}{|\gamma+1||\sigma_3|-\mu_3}, \frac{\lambda_3\mu_3}{|\gamma+1|\Re\sigma_3(|\sigma_3|-\mu_3)}\right\}\|(w^*,y_1^*,y_3^*)\|. \tag{30d}$$

Therefore, when $|\sigma_j| > \mu_j$ and $\Re\sigma_j > 0$, $j = 1,3$, from the inequalities system (28a)–(28d) and system (30a)–(30d), it can be computed that

$$\|(V^*,p_1^*,p_3^*)\|_* = \sup\{|Q_{1,0}^*|, |Q_{3,0}^*|, \sup_{k\ge0}|V_k^*|, \sup_{k\ge1}\|p_{1,k}\|_{L^\infty[0,\infty)}, \sup_{k\ge1}\|p_{3,k}\|_{L^\infty[0,\infty)}\}$$

$$\begin{aligned} &\le \sup\left\{\frac{1}{|(\gamma+1)(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)-\theta_1\theta_3N_0|}\right. \\ &\quad \times \sup\left\{\frac{\lambda_1(\theta_1+\mu_1)|(\gamma+1)(\gamma+\lambda_3+\theta_3)|}{|\sigma_1|}, \frac{\lambda_1^2|(\gamma+1)(\gamma+\lambda_3+\theta_3)|}{\Re\sigma_1}, \frac{\lambda_3^2\theta_1N_0}{\Re\sigma_3},\right. \\ &\quad \frac{\lambda_3\theta_1N_0(\theta_3+\mu_3)}{|\sigma_3|}, \frac{\theta_1\theta_3(1-N_0)|\gamma+\lambda_3+\theta_3|}{|\sigma_3|-\mu_3}, \frac{\lambda_3\mu_3\theta_1(1-N_0)|\gamma+\lambda_3+\theta_3|}{\Re\sigma_3(|\sigma_3|-\mu_3)}, \\ &\quad \frac{\lambda_1\theta_3(\theta_1+\mu_1)|\gamma+1|}{|\sigma_1|}, \frac{\lambda_1^2\theta_3|\gamma+1|}{\Re\sigma_1}, \frac{\lambda_3(\theta_3+\mu_3)|(\gamma+1)(\gamma+\lambda_1+\theta_1)|}{|\sigma_3|}, \\ &\quad \frac{\lambda_3^2|(\gamma+1)(\gamma+\lambda_1+\theta_1)|}{\Re\sigma_3}, \frac{\theta_1\theta_3^2(1-N_0)}{|\sigma_3|-\mu_3}, \frac{\lambda_3\mu_3\theta_1\theta_3(1-N_0)}{\Re\sigma_3(|\sigma_3|-\mu_3)}, \frac{\lambda_3\theta_1N_0(\theta_1+\mu_1)}{|\sigma_1|}, \\ &\quad \frac{\lambda_1\lambda_3\theta_1N_0}{\Re\sigma_1}, \frac{\lambda_3N_0(\theta_3+\mu_3)|\gamma+\lambda_1+\theta_1|}{|\sigma_3|}, \frac{\lambda_3^2N_0|\gamma+\lambda_1+\theta_1|}{\Re\sigma_3}, \\ &\quad \frac{\theta_3(1-N_0)|(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)|}{|\sigma_3|-\mu_3}, \\ &\quad \left.\frac{\lambda_3\mu_3(1-N_0)|(\gamma+\lambda_1+\theta_1)(\gamma+\lambda_3+\theta_3)|}{\Re\sigma_3(|\sigma_3|-\mu_3)}\right\}, \\ &\quad \frac{\theta_3}{|\gamma+1||\sigma_3|-\mu_3}, \frac{\lambda_3\mu_3}{|\gamma+1|\Re\sigma_3(|\sigma_3|-\mu_3)}, \frac{\theta_1+\mu_1}{|\sigma_1|}, \frac{\lambda_1}{\Re\sigma_1}, \frac{\theta_1}{|\sigma_1|-\mu_1}, \\ &\quad \left.\frac{\lambda_1\mu_1}{\Re\sigma_1(|\sigma_1|-\mu_1)}, \frac{\theta_3+\mu_3}{|\sigma_3|}, \frac{\lambda_3}{\Re\sigma_3}, \frac{\theta_3}{|\sigma_3|-\mu_3}, \frac{\lambda_3\mu_3}{\Re\sigma_3(|\sigma_3|-\mu_3)}\right\}\|(w^*,y_1^*,y_3^*)\|. \end{aligned} \tag{31}$$

The inequality system (31) means that if $\gamma \in \Lambda$, then the operator $[I-(\gamma I-L-G)^{-1}(J+H)]^{-1}$ exists and is bounded. Now, we consider $(\gamma I-L-G)(V^*,p_1^*,p_3^*) = (w^*,y_1^*,y_3^*), (w^*,y_1^*,y_3^*) \in X^*$. Then, it is not difficult to show that the operator $(\gamma I-L-G)^{-1}$ also exists and is bounded. By the relationship

$$\begin{aligned}&[\gamma I-(L+G+J+H)]^{-1} = \{(\gamma I-L-G)[I-(\gamma I-L-G)^{-1}(J+H)]\}^{-1}\\ &=[I-(\gamma I-L-G)^{-1}(J+H)]^{-1}(\gamma I-L-G)^{-1},\end{aligned} \tag{32}$$

we obtain that if $\gamma \in \Lambda$, then the operator $(\gamma I-L-G-J-H)^{-1}$ also exists and is bounded. Therefore, all points in the set Λ belong to the resolvent set $\rho(L+J+G+H)$.

Additionally, if we take $\gamma = ib$, $i^2=-1$, $b \in \mathbb{R}\setminus\{0\}$, then, for any $\lambda_j,\theta_j,\mu_j>0$, $j=1,3$ and $0<N_0<1$, through simple but tedious calculations, we obtain the following inequalities:

$$\frac{\lambda_j}{|ib+\lambda_j+\theta_j+\mu_j|-\mu_j} = \frac{\lambda_j}{\sqrt{b^2+(\lambda_j+\theta_3+\mu_j)^2}-\mu_j} < 1, \tag{33a}$$

$$\frac{\theta_j}{|ib+\lambda_j+\theta_j+\mu_j|-\mu_j} = \frac{\theta_j}{\sqrt{b^2+(\lambda_j+\theta_j+\mu_j)^2}-\mu_j} < 1, \tag{33b}$$

$$\frac{\theta_j+\mu_j}{|ib+\lambda_j+\theta_j+\mu_j|} = \frac{\theta_j+\mu_j}{\sqrt{b^2+(\lambda_j+\theta_j+\mu_j)^2}} < 1, \tag{33c}$$

$$\begin{aligned}&|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3N_0|^2-\lambda_1^2|(ib+1)(ib+\lambda_3+\theta_3)|^2\\ &=b^6+b^4[2\lambda_1\theta_1+\theta_1^2+(\lambda_3+\theta_3)^2+1]+b^2\{2\lambda_1\theta_1+\theta_1^2+(\lambda_3+\theta_3)^2\\ &\quad+(2\lambda_1\theta_1+\theta_1^2)(\lambda_3+\theta_3)^2+2(\lambda_1+\theta_1+\lambda_3+\theta_3+1)\theta_1\theta_3N_0\}\\ &\quad+2\lambda_1(\lambda_3+\theta_3)[\theta_1(\lambda_3+\theta_3)-\theta_1\theta_3N_0]+[\theta_1(\lambda_3+\theta_3)-\theta_1\theta_3N_0]^2>0\\ &\Rightarrow \frac{\lambda_1|(ib+1)(ib+\lambda_3+\theta_3)|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3N_0|}<1\\ &\Rightarrow \frac{\lambda_1(\theta_1+\mu_1)|(ib+1)(ib+\lambda_3+\theta_3)|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3N_0||ib+\lambda_1+\theta_1+\mu_1|}<1,\end{aligned} \tag{33d}$$

$$\frac{\lambda_1^2|(ib+1)(ib+\lambda_3+\theta_3)|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3N_0|(\lambda_1+\theta_1+\mu_1)}<1, \tag{33e}$$

$$\begin{aligned}&\frac{\lambda_3\theta_1N_0}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3N_0|}<1\\ &\Rightarrow\\ &\frac{\lambda_3\theta_1N_0(\theta_3+\mu_3)}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3N_0||ib+\lambda_3+\theta_3+\mu_3|}<1,\end{aligned} \tag{33f}$$

$$\frac{\lambda_3^2\theta_1N_0}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3N_0|(\lambda_3+\theta_3+\mu_3)}<1, \tag{33g}$$

$$\frac{\lambda_3\theta_1N_0(\theta_1+\mu_1)}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3N_0||ib+\lambda_1+\theta_1+\mu_1|}<1, \tag{33h}$$

$$\frac{\lambda_1\lambda_3\theta_1N_0}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3N_0||ib+\lambda_1+\theta_1+\mu_1|}<1, \tag{33i}$$

$$\frac{\theta_1(1-N_0)|ib+\lambda_3+\theta_3|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3 N_0|}<1$$
$$\Rightarrow$$
$$\frac{\theta_1\theta_3(1-N_0)|ib+\lambda_3+\theta_3|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3 N_0|(|ib+\lambda_3+\theta_3+\mu_3|-\mu_3)}$$
$$<1, \tag{33j}$$

$$\frac{\lambda_3\mu_3\theta_1(1-N_0)|ib+\lambda_3+\theta_3|(\lambda_3+\theta_3+\mu_3)(|ib+\lambda_3+\theta_3+\mu_3|-\mu_3)}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3 N_0|}<1, \tag{33k}$$

$$\frac{\lambda_1\theta_3|ib+1|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3 N_0|}<1$$
$$\Rightarrow \frac{\lambda_1\theta_3(\theta_1+\mu_1)|ib+1|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3 N_0|(\lambda_1+\theta_1+\mu_1)}<1, \tag{33l}$$

$$\frac{\lambda_1^2|(ib+1)(ib+\lambda_3+\theta_3)|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3 N_0|(\lambda_1+\theta_1+\mu_1)}<1, \tag{33m}$$

$$\frac{\lambda_3(\theta_3+\mu_3)|(ib+1)(ib+\lambda_1+\theta_1)|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3 N_0||ib+\lambda_3+\theta_3+\mu_3|}<1, \tag{33n}$$

$$\frac{\lambda_3^2|(ib+1)(ib+\lambda_1+\theta_1)|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3 N_0|(\lambda_3+\theta_3+\mu_3)}<1, \tag{33o}$$

$$\frac{\theta_1\theta_3^2(1-N_0)}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3 N_0|(|ib+\lambda_3+\theta_3+\mu_3|-\mu_3)}<1, \tag{33p}$$

$$\frac{\lambda_3\mu_3\theta_1\theta_3(1-N_0)(\lambda_3+\theta_3+\mu_3)(|ib+\lambda_3+\theta_3+\mu_3|-\mu_3)}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3 N_0|}<1, \tag{33q}$$

$$\frac{\lambda_3 N_0(\theta_3+\mu_3)|ib+\lambda_1+\theta_1|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3 N_0||ib+\lambda_3+\theta_3+\mu_3|}<1, \tag{33r}$$

$$\frac{\lambda_3^2 N_0|ib+\lambda_1+\theta_1|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3 N_0|(\lambda_3+\theta_3+\mu_3)}<1, \tag{33s}$$

$$\frac{\theta_3(1-N_0)|(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3 N_0|(|ib+\lambda_3+\theta_3+\mu_3|-\mu_3)}<1, \tag{33t}$$

$$\frac{\lambda_3\mu_3(1-N_0)|(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)(\lambda_3+\theta_3+\mu_3)(|ib+\lambda_3+\theta_3+\mu_3|-\mu_3)|}{|(ib+1)(ib+\lambda_1+\theta_1)(ib+\lambda_3+\theta_3)-\theta_1\theta_3 N_0|}<1. \tag{33u}$$

The inequalities system (33a)–(33u) imply that $i\mathbb{R}\setminus\{0\}\subset\rho((A+U+E)^*)$. Then, by the relationship between the spectrum $\sigma(A+U+E)$ and $\sigma((A+U+E)^*)$, we obtain that $i\mathbb{R}\setminus\{0\}\subset\rho(A+U+E)$. $\square$

Let $\sigma_p(A+U+E)$ denote the point spectrum of $A+U+E$.

Lemma 3. *If $\lambda_j,\theta_j,\mu_j>0$ and $\lambda_j<\theta_j+\mu_j$, $j=1,3$, then $0\in\sigma_p(A+U+E)$, and the geometric multiplicity of zero is one.*

Proof. We consider $(A+U+E)(V,p_1,p_3)=0$, which is equivalent to

$$(\lambda_1+\theta_1)Q_{1,0}=\theta_3\sum_{n=1}^{\infty}\int_0^{\infty}p_{3,n}(x)dx+\theta_3 Q_{3,0}+\mu_1\int_0^{\infty}p_{1,1}(x)dx, \tag{34a}$$

$$(\lambda_3+\theta_3)Q_{3,0}=N_0V_0+\mu_3\int_0^\infty p_{3,1}(x)dx, \tag{34b}$$

$$V_0=\theta_1 Q_{1,0},\ V_n=\theta_1\int_0^\infty p_{1,n}(x)dx, \quad n\geq 1, \tag{34c}$$

$$\frac{dp_{j,1}(x)}{dx}=-(\lambda_j+\theta_j+\mu_j)p_{j,1}(x), \quad j=1,3, \tag{34d}$$

$$\frac{dp_{j,n}(x)}{dx}=-(\lambda_j+\theta_j+\mu_j)p_{j,n}(x)+\lambda_j p_{j,n-1}(x),\ n\geq 2;\ j=1,3, \tag{34e}$$

$$p_{1,1}(0)=\lambda_1 Q_{1,0}+\mu_1\int_0^\infty p_{1,2}(x)dx, \tag{34f}$$

$$p_{1,n}(0)=\mu_1\int_0^\infty p_{1,n+1}(x)dx, \quad n\geq 2, \tag{34g}$$

$$p_{3,1}(0)=\lambda_3 Q_{3,0}+N_1V_0+N_0V_1+\mu_3\int_0^\infty p_{3,2}(x)dx, \tag{34h}$$

$$p_{3,n}(0)=\sum_{k=0}^{n}N_{n-k}V_k+\mu_3\int_0^\infty p_{3,n+1}(x)dx, \quad n\geq 2. \tag{34i}$$

Solving Equations (34d) and (34e), we obtain

$$p_{j,n}(x)=e^{-(\lambda_j+\theta_j+\mu_j)x}\sum_{k=1}^{n}\frac{(\lambda_j x)^{n-k}}{(n-k)!}p_{j,k}(0),\ j=1,3;\ n\geq 1. \tag{35}$$

In the following, by introducing the probability generating functions (PGFs), see, e.g., [6], we show that $0\in\sigma_p(A+U+E)$. For any conplex number z such that $|z|<1$, we define the PGFs by

$$\overline{V}(z):=\sum_{n=0}^{\infty}V_n z^n, \quad p_j(x,z):=\sum_{n=1}^{\infty}p_{j,n}(x)z^n, \quad j=1,3.$$

Then , Theorem 3 means that $\overline{V}(z)$ and $p_j(x,z)$ are well-posed. From Equations (34b)–(34e), it is easy to compute that

$$\overline{V}(z)=V_0+\sum_{n=1}^{\infty}V_n z^n=\theta_1 Q_{1,0}+\theta_1\sum_{n=1}^{\infty}\int_0^\infty p_{1,n}(x)z^n dx, \tag{36a}$$

$$\begin{aligned}&\frac{\partial\sum_{n=1}^{\infty}p_{j,n}(x)z^n}{\partial x}=-\sum_{n=1}^{\infty}(\lambda_j+\theta_j+\mu_j)p_{j,n}(x)z^n+\lambda_j\sum_{n=1}^{\infty}p_{j,n-1}(x)z^n\\ &\Rightarrow\quad p_j(x,z)=p_j(0,z)e^{-[\theta_j+\lambda_j(1-z)+\mu_j]x}, \quad j=1,3.\end{aligned} \tag{36b}$$

From the boundary conditions system (34f) and (34g) and system (36b), we have

$$\begin{aligned}&p_1(0,z)=p_{1,1}(0)z+\sum_{n=2}^{\infty}p_{1,n}(0)z^n\\ &=z\lambda_1 Q_{1,0}+\mu_1\int_0^\infty p_{1,2}(x)zdx+\mu_1\sum_{n=2}^{\infty}\int_0^\infty p_{1,n+1}(x)z^n dx\\ &=z\lambda_1 Q_{1,0}+\frac{\mu_1}{z[\theta_1+\lambda_1(1-z)+\mu_1]}p_1(0,z)-\mu_1\int_0^\infty p_{1,1}(x)dx.\end{aligned}$$

That is,

$$\left(z-\frac{\mu_1}{\theta_1+\lambda_1(1-z)+\mu_1}\right)p_1(0,z)=z^2\lambda_1 Q_{1,0}-z\mu_1\int_0^\infty p_{1,1}(x)dx. \tag{37}$$

By Equations (34b) and (34c), (34h)–(34i), and (36b) and

$$\sum_{n=0}^{\infty} N_n z^n = \sum_{n=0}^{\infty} e^{-\lambda_2 pd} \frac{(\lambda_2 pdz)^n}{n!} = e^{-\lambda_2 pd(1-z)},$$

we compute that

$$\begin{aligned} p_3(0,z) &= p_{3,1}(0)z + \sum_{n=2}^{\infty} p_{3,n}(0)z^n = z\lambda_3 Q_{3,0} + zN_1 V_0 + zN_0 V_1 + \mu_3 \int_0^{\infty} p_{3,2}(x) z dx \\ &+ \sum_{n=2}^{\infty} \left[\sum_{k=0}^{n} N_{n-k} V_k + \mu_3 \int_0^{\infty} p_{3,n+1}(x) dx \right] z^n \\ &= -[\theta_3 + \lambda_3(1-z)]Q_{3,0} + \overline{V}(z) e^{-\lambda_2 pd(1-z)} + \frac{\mu_3}{z[\theta_3 + \lambda_3(1-z) + \mu_3]} p_3(0,z). \end{aligned}$$

This is equivalent to

$$\left(z - \frac{\mu_3}{\theta_3 + \lambda_3(1-z) + \mu_3} \right) p_3(0,z) = -z[\theta_3 + \lambda_3(1-z)]Q_{3,0} + z\overline{V}(z) e^{-\lambda_2 pd(1-z)}. \tag{38}$$

It is not difficult to show that if $\lambda_j < \theta_j + \mu_j,\ j = 1,3$, then

$$0 < \gamma_j := \frac{\lambda_j + \theta_j + \mu_j - \sqrt{(\lambda_j + \theta_j + \mu_j)^2 - 4\lambda_j \mu_j}}{2\lambda_j} < 1,$$

and γ_j is the unique solution of the function $z = \frac{\mu_j}{\theta_j + \lambda_j(1-z) + \mu_j}$. Hence, using system (37) and (38), we obtain

$$Q_{1,0} = \frac{\mu_1}{\gamma_1 \lambda_1} \int_0^{\infty} p_{1,1}(x) dx, \quad Q_{3,0} = \frac{\overline{V}(\gamma_3) e^{-\lambda_2 pd(1-\gamma_3)}}{\theta_3 + \lambda_3(1-\gamma_3)}. \tag{39}$$

In addition, from system (37) and (38), it is easy to obtain that

$$p_1(0,z) = \frac{\lambda_1 Q_{1,0} z(z-\gamma_1)}{z - \frac{\mu_1}{\theta_1 + \lambda_1(1-z) + \mu_1}} = \frac{\lambda_1 Q_{1,0} z(z-\gamma_1)[\theta_1 + \lambda_1(1-z) + \mu_1]}{z[\theta_1 + \lambda_1(1-z) + \mu_1] - \mu_1}, \tag{40a}$$

$$p_3(0,z) = \frac{-z[\theta_3 + \lambda_3(1-z)]Q_{3,0} + z\overline{V}(z) e^{\lambda_2 pd(1-z)}}{z - \frac{\mu_3}{\theta_3 + \lambda_3(1-z) + \mu_3}}. \tag{40b}$$

Taking z on both sides of system (40a) and (40b) tends towards the limit of one, and it can be obtained that

$$p_1(0,1) = \frac{\lambda_1 Q_{1,0}(1-\gamma_1)(\theta_1 + \mu_1)}{\theta_1}, \tag{41a}$$

$$p_3(0,1) = \frac{[-\theta_3 Q_{3,0} + \overline{V}(1)](\theta_3 + \mu_3)}{\theta_3}. \tag{41b}$$

Then, using system (36b), (41a), and (39), we obtain

$$\sum_{n=1}^{\infty} \int_0^{\infty} p_{1,n}(x) dx = \lim_{z \to 1} \int_0^{\infty} p_1(0,z) e^{-[\theta_1 + \lambda_1(1-z) + \mu_1]x} dx = \frac{\lambda_1 Q_{1,0}(1-\gamma_1)}{\theta_1}, \tag{42a}$$

$$\begin{aligned} &\sum_{n=1}^{\infty} \int_0^{\infty} p_{3,n}(x) dx = \lim_{z \to 1} \int_0^{\infty} p_3(0,z) e^{-[\theta_3 + \lambda_3(1-z) + \mu_3]x} dx \\ &= \frac{-\theta_3 Q_{3,0} + \theta_1 Q_{1,0} + \lambda_1 Q_{1,0}(1-\gamma_1)}{\theta_3}. \end{aligned} \tag{42b}$$

Therefore, using system (34c), (39), (42a), and (42b) and by simple calculations, it can be obtained that

$$Q_{1,0}+Q_{3,0}+\sum_{n=0}^{\infty}V_n+\sum_{n=1}^{\infty}\int_0^{\infty}p_{1,n}(x)dx+\sum_{n=1}^{\infty}\int_0^{\infty}p_{3,n}(x)dx$$
$$=\frac{(1+\theta_1)\theta_1\theta_3+\theta_1^2+\lambda_1(1-\gamma_1)(\theta_1+\theta_3)}{\theta_1\theta_3}Q_{1,0}. \tag{43}$$

system (43) means that $0\in\sigma_p(A+U+E)$. Additionally, from system (34a)–(34c) and system (34f)–(35), it is not difficult to show that the geometric multiplicity of zero is one. □

Lemma 4. *If $\lambda_j,\theta_j,\mu_j>0,\ j=1,3$, then $0\in\sigma_p((A+U+E)^*)$, and the geometric multiplicity of zero is one.*

Proof. We consider $(A+U+E)^*(V^*,p_1^*,p_3^*)=0$; that is,

$$-(\lambda_1+\theta_1)Q_{1,0}^*+\theta_1V_0^*+\lambda_1p_{1,1}^*(0)=0, \tag{44a}$$

$$\theta_3Q_{1,0}^*-(\lambda_3+\theta_3)Q_{3,0}^*+\lambda_3p_{3,1}^*(0)=0, \tag{44b}$$

$$N_0Q_{3,0}^*-V_0^*+\sum_{k=1}^{\infty}N_kp_{3,k}^*(0)=0, \tag{44c}$$

$$-V_n^*+\sum_{k=0}^{\infty}N_kp_{3,k+n}^*(0)=0,\quad n\geq 1, \tag{44d}$$

$$\frac{dp_{1,1}^*(x)}{dx}-(\lambda_1+\theta_1+\mu_1)p_{1,1}^*(x)+\lambda_1p_{1,2}^*(x)+\theta_1V_1^*+\mu_1Q_{1,0}^*=0, \tag{44e}$$

$$\frac{dp_{1,n}^*(x)}{dx}-(\lambda_1+\theta_1+\mu_1)p_{1,n}^*(x)+\lambda_1p_{1,n+1}^*(x)+\theta_1V_n^*+\mu_1p_{1,n-1}^*(0)=0,\ n\geq 2, \tag{44f}$$

$$\frac{dp_{3,1}^*(x)}{dx}-(\lambda_3+\theta_3+\mu_3)p_{3,1}^*(x)+\lambda_3p_{3,2}^*(x)+\theta_3Q_{1,0}^*+\mu_3Q_{3,0}^*=0, \tag{44g}$$

$$\frac{dp_{3,n}^*(x)}{dx}-(\lambda_3+\theta_3+\mu_3)p_{3,n}^*(x)+\lambda_3p_{3,n+1}^*(x)+\theta_3Q_{1,0}^*+\mu_3(x)p_{3,n-1}^*(0)=0,\ n\geq 2. \tag{44h}$$

It is not difficult to see that

$$(V^*,p_1^*,p_3^*)=\left(\begin{pmatrix}\alpha\\ \vdots\end{pmatrix},\begin{pmatrix}\alpha\\ \vdots\end{pmatrix},\begin{pmatrix}\alpha\\ \vdots\end{pmatrix}\right)\in D((A+U+E)^*)$$

is a positive eigenvalue of the system of Equations (44a)–(44h). Moreover, by system (44a)–(44h), it is not difficult to calculate that

$$Q_{1,0}^*=\frac{\lambda_1(\lambda_3+\theta_3)}{(\lambda_1+\theta_1)(\lambda_3+\theta_3)-N_0\theta_1\theta_3}p_{1,1}^*(0)+\frac{\lambda_3\theta_1N_0}{(\lambda_1+\theta_1)(\lambda_3+\theta_3)-N_0\theta_1\theta_3}p_{3,1}^*(0)$$
$$+\frac{\theta_1(\lambda_3+\theta_3)}{(\lambda_1+\theta_1)(\lambda_3+\theta_3)-N_0\theta_1\theta_3}\sum_{k=1}^{\infty}N_kp_{3,k}^*(0), \tag{45a}$$

$$Q_{3,0}^*=\frac{\lambda_1\theta_3}{(\lambda_1+\theta_1)(\lambda_3+\theta_3)-N_0\theta_1\theta_3}p_{1,1}^*(0)+\frac{\lambda_3(\lambda_1+\theta_1)}{(\lambda_1+\theta_1)(\lambda_3+\theta_3)-N_0\theta_1\theta_3}p_{3,1}^*(0)$$
$$+\frac{\theta_1\theta_3}{(\lambda_1+\theta_1)(\lambda_3+\theta_3)-N_0\theta_1\theta_3}\sum_{k=1}^{\infty}N_kp_{3,k}^*(0), \tag{45b}$$

$$V_0^* = \frac{\lambda_1\theta_3 N_0}{(\lambda_1+\theta_1)(\lambda_3+\theta_3)-N_0\theta_1\theta_3}p_{1,1}^*(0) + \frac{\lambda_3(\lambda_1+\theta_1)N_0}{(\lambda_1+\theta_1)(\lambda_3+\theta_3)-N_0\theta_1\theta_3}p_{3,1}^*(0) + \frac{(\lambda_1+\theta_1)(\lambda_3+\theta_3)}{(\lambda_1+\theta_1)(\lambda_3+\theta_3)-N_0\theta_1\theta_3}\sum_{k=1}^{\infty}N_k p_{3,k}^*(0), \quad (45c)$$

$$V_n^* = \sum_{k=0}^{\infty} N_k p_{3,k+n}^*(0),\ n \geq 1, \quad (45d)$$

$$p_{1,2}^*(x) = -\frac{1}{\lambda_1}\left[\frac{dp_{1,1}^*(x)}{dx} - (\lambda_1+\theta_1+\mu_1)p_{1,1}^*(x) + \theta_1 V_1^* + \mu_1 Q_{1,0}^*\right], \quad (45e)$$

$$p_{1,n+1}^*(x) = -\frac{1}{\lambda_1}\left[\frac{dp_{1,n}^*(x)}{dx} - (\lambda_1+\theta_1+\mu_1)p_{1,n}^*(x) + \theta_1 V_n^* + \mu_1 p_{1,n-1}^*(0)\right],\ n \geq 2, \quad (45f)$$

$$p_{3,2}^*(x) = -\frac{1}{\lambda_3}\left[\frac{dp_{3,1}^*(x)}{dx} - (\lambda_3+\theta_3+\mu_3)p_{3,1}^*(x) + \theta_3 Q_{1,0}^* + \mu_3 Q_{3,0}^*\right], \quad (45g)$$

$$p_{3,n+1}^*(x) = -\frac{1}{\lambda_3}\left[\frac{dp_{3,n}^*(x)}{dx} - (\lambda_3+\theta_3+\mu_3)p_{3,n}^*(x) + \theta_3 Q_{1,0}^* + \mu_3 p_{3,n-1}^*(0)\right],\ n \geq 2. \quad (45h)$$

Equations (45a)–(45h) mean that the geometric multiplicity of zero is one. □

Proof of Theorem 4. Theorem 3 means that $A+U+E$ generates a uniformly bounded C_0-semigroup on the Banach space X. Moreover, by Lemmas 2, 3, and 4 it is easy to see that $\sigma_p(A+U+E)\cap i\mathbb{R} = \sigma_p((A+U+E)^*)\cap i\mathbb{R} = \{0\}$, $\{\gamma \in \mathbb{C} \mid \gamma = ib, b \neq 0, b \in \mathbb{R}\} \subset \rho(A+U+E)$, and $0 \in \sigma_p((A+U+E)^*)$, and the geometric multiplicity of zero is one. Therefore, by theorem 1.96 of [14], we obtain that the TDS of system (3) strongly converges to its SSS; that is

$$\lim_{t\to\infty} \|(V(t), p_1(\cdot,t), p_3(\cdot,t)) - \langle (V^*, p_1^*(\cdot), p_3^*(\cdot)), (u_2, u_1(\cdot), u_3(\cdot))\rangle (V, p_1(\cdot), p_3(\cdot))\| = 0,$$

where $(V, p_1(\cdot), p_3(\cdot))$ and $(V^*, p_1^*(\cdot), p_3^*(\cdot))$ are the eigenvectors corresponding to the zero in Lemmas 3 and 4, respectively. □

4. Stability of the Semigroup

In this section, when the service rates $\mu_1(\cdot)$ and $\mu_3(\cdot)$ are constants, based on the preceding discussion, we study the stability of the C_0-semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A} := A+U+E$, from which we derive that the SDGC holds true.

Theorem 5. *The SDGC holds for the semigroup $e^{\mathcal{A}t}$, and $\omega_0(\mathcal{A}) = s(\mathcal{A}) = 0$, where $\omega_0(\mathcal{A}) = \{\omega \mid$ there exists an $\mathbb{M}$ such that $\|e^{\mathcal{A}t}\| \leq \mathbb{M}e^{\omega t}\}$ is the growth bound of the C_0-semigroup $e^{\mathcal{A}t}$, and $s(\mathcal{A}) = \{\Re\gamma \mid \gamma \in \sigma(\mathcal{A})\}$ is the spectral bound of $\mathcal{A}$. Furthermore, $e^{\mathcal{A}t}$ is not asymptotically stable.*

Proof. Obviously, X is a Banach lattice, and the semigroup $e^{\mathcal{A}}$ is positive on the Banach lattice X by Theorem 1. For $(V, p_1, p_3), (w, y_1, y_3) \in X_+$, where $w = (z_{1,0}, z_{3,0}, w_0, w_1, \cdots)$, $y_j = (y_{j,1}, y_{j,2}, y_{j,3}, \cdots), j = 1, 3$, it is easy to obtain that

$$\begin{aligned}
&\|(V, p_1, p_3) + (w, y_1, y_3)\|_+ \\
&= |Q_{1,0} + z_{1,0}| + |Q_{3,0} + z_{3,0}| + \sum_{n=0}^{\infty} |V_n + w_n| + \sum_{j=1,3} \sum_{n=1}^{\infty} \int_0^{\infty} |p_{j,n}(x) + y_{j,n}(x)| dx \\
&= Q_{1,0} + Q_{3,0} + \sum_{n=0}^{\infty} V_n + \sum_{j=1,3} \sum_{n=1}^{\infty} \int_0^{\infty} p_{j,n}(x) dx \\
&+ z_{1,0} + z_{3,0} + \sum_{n=0}^{\infty} w_n + \sum_{j=1,3} \sum_{n=1}^{\infty} \int_0^{\infty} y_{j,n}(x) dx \\
&= \|(V, p_1, p_3)\|_+ + \|(w, y_1, y_3)\|_+,
\end{aligned}$$

where X_+ is introduced in Section 2, and $\|\cdot\|_+$ is the norm on X_+. This shows that X is an AL-space. By Lemmas 2 and 3, we see that $0 \in \sigma_p(\mathcal{A})$, and $\sigma(\mathcal{A}) \subset \{\gamma \in \mathbb{C} \mid \Re\gamma < 0\} \cup \{0\}$. Then, we have $s(\mathcal{A}) \in \sigma(\mathcal{A})$ and $s(\mathcal{A}) = 0$ according to corollary 12.9 of [21] (p. 188). Thus, using Theorem 12.17 of [21] (p. 193) to obtain the SDGC holds, and $\omega_0(\mathcal{A}) = s(\mathcal{A}) = 0$. This implies that $e^{\mathcal{A}t}$ is not exponentially stable, see, e.g., [22] (Thm VI.1.14, p. 357). That is, there are no constants $K > 0$ and $\varepsilon > 0$ such that

$$\|e^{\mathcal{A}t}\| \le K e^{-\varepsilon t}, \quad \forall t \ge 0.$$

Furthermore, since $e^{\mathcal{A}t}$ is uniformly bounded (Thm 1), Lemmas 2 and 3 show that $\sigma(\mathcal{A}) \cap i\mathbb{R} \subset \sigma_p(\mathcal{A})$, and the asymptotic stability then follows from Theorem 3.26 of [23] (p. 130) . This is different from the recent result [11]. □

5. Conclusions

In this paper, we investigate the well-posedness and asymptotic behavior of the M/G/1 stochastic clearing queueing model in a three-phase environment. In the natural state space L^1, it is not easy to prove semigroup property. Based on semigroup theory and spectrum analysis on the imaginary axis, we show that this queueing model admits a unique positive TDS. When the service rates of servers are constants, we prove that the TDS of the queueing model is strongly convergent to its nonzero SSS. However, we did not answer whether the general situation holds true, and the solution exponentially converges to its SSS. Moreover, exponential convergence of the TDS of Equations (1) and (2) depends on the growth bound and essential growth bound of the corresponding semigroup, see, e.g., [7,15–17,24]. For this aim, we need to know the spectrum of the system operator on the left half of the complex plane. This is work to complete in the future.

Additionally, we show that the SDGC is equal to zero, and the semigroup generated by the system operator is not asymptotically stable. However, we did not study the compactness, reducibility, and ergodicity of the corresponding semigroup of the system. There have been many studies on these asymptotic behavior of semigroups, such as corresponding queueing models, reliable models, and population models, see, e.g., [11,14,16,17,24]. These are also future research topics.

Funding: This work was supported by the National Natural Science Foundation of China (grant number: 12301150).

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this articleplicable.

Acknowledgments: The author would like to thank the editor and referees for their valuable comments.

Conflicts of Interest: The author declares no conflicts of interests.

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