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Research Article

Coupling Shape Optimization and Topological Derivative for Maxwell Equations

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The paper deals with a coupling algorithm using shape and topological derivatives of a given cost functional and a problem governed by nonstationary Maxwell's equations in 3D. To establish the shape and topological derivatives, an adjoint method is used. For the topological asymptotic expansion, two examples of cost functionals are considered with the perturbation of the electric permittivity and magnetic permeability. We combine the shape derivative and topological one to propose an algorithm. The proposed algorithm allows to insert a small inhomogeneity (electric or magnetic) in a given shape.

1. Introduction

Shape optimization is a minimization problem where the unknown variables run over a class of admissible domains; then, every shape optimization problem can be written in the form

$$\min_{\Omega\in\Theta} j(\Omega, u_\Omega), \tag{1}$$

where Θ is the class of admissible domains and j is the cost functional; generally, u_Ω is the solution of a given PDE. Generally, the existence and uniqueness of the solution of such problems are not guaranteed even for simple cases. However, the literature about the subject is abundant, and the techniques used are numerous (homogenization, domain parametrization, geometric shape derivative, topological optimization, …), see [1] and references therein.

In this paper, we focus on geometric shape optimization and topological asymptotic expansion methods, which we briefly recall the principles to propose an algorithm coupling them that we apply to Maxwell's equations.

Shape optimization problem is a minimization problem where the unknown variables run over a class of domains; then, every shape optimization problem can be written in the form

$$\min \{j(\Omega): \Omega \in \mathscr{A}\}, \tag{2}$$

where $\mathscr{A}$ is the class of admissible domains and j is the cost functional.

The underlying principle is the following. Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain whose boundary $\partial\Omega$ is a $\mathscr{C}^k$ manifold oriented by the normal vector field ν outgoing to Ω, a family $\mathscr{O}_k$ of open bounded domains of D and $t = (0, T)$, the time interval. We note that $Q = \Omega \times (0, T)$ is the cylinder evolution domain, and $\Sigma = \Gamma \times (0, T)$ is the lateral boundary associated to any $\Omega \subset \mathscr{O}_k$. Let ξ_k be the set of $V \in \mathscr{C}([0, T], \mathscr{C}^k(D, \mathbb{R}^3))$ with $\langle V, \nu_{\partial D}\rangle = 0$ for all $V \in \xi_k$; we consider the follow mapping $T_s(V)$, such that, for each x, V as the form

$$V(s)(x) = \left(\frac{\partial}{\partial s} T_s\right) \circ T_s(x). \tag{3}$$

For each $s \in [0, S[$, T_s is a one-to-one mapping from D to D such that

(i) $T_0 = I$

(ii) $(s, x) \longrightarrow T_s$ belongs to $\mathscr{C}([0, S]), \mathscr{C}^k(D, D))$ with $T_s(\partial D) = \partial D$

(iii) $(s, x) \longrightarrow T_s^{-1}$ belongs to $\mathscr{C}^1([0, S]), \mathscr{C}^k(D, D))$

Such family $\mathscr{O}_k$ is stable under the perturbation $\Omega \mapsto \Omega_s = T_s(V)(\Omega)$. We denote by Q_s, the perturbed cylinder $\Omega_s \times (0, T)$, $\Gamma_s = \partial\Omega_s$ and $\Sigma_s = \Gamma_s \times (0, T)$, the perturbed lateral boundary.

To each element $\Omega \in \mathscr{O}_k$, we associate $u_\Omega = (E_\Omega, H_\Omega)$, the solution of (11). For any $V \in \xi_k$, and $s \in [0, S[$, we set that $u_s = u(\Omega_s) \in L^2(Q_s)$ is said to be shape differentiable in $L^2(I, H^m(D))$ if there exists $U \in \mathscr{C}^1([0, S], L^2(I, H^m(D)))$ such that

$$U(s,.,.)|_{Q_s} = u(\Omega_s),$$
$$\frac{U(s) - U(0)}{s} - \partial_s U(0) \longrightarrow 0, \text{in } L^2(I, H^m(D)), s \longrightarrow 0. \tag{4}$$

Definition 1. The shape derivative of $u(\Omega, V)$ to the direction V in the unique element $u'(\Omega, V)$ verifying

$$u'(\Omega, V) = \left(\frac{\partial}{\partial s} U\right)\bigg|_{s=0,(t,x)\in Q}. \tag{5}$$

In the same way, we define the boundary shape derivative and the material derivative.

Definition 2. The element

$$g'_\Gamma(\Gamma, V) = \left(\frac{\partial}{\partial s} G\right)\bigg|_{s=0,(t,x)\in\Sigma} \tag{6}$$

is the boundary shape derivative of $g \in L^1(I, H^p)$, where $G \in \mathscr{C}^1([0, S], L^1(H^{p+1/2}(D))$ such that

$$\frac{\partial}{\partial \nu_s} G(s) = 0 \text{ on } \Sigma,$$
$$G(s,.,.)|_{\Sigma_s} = g(\Gamma_s) \text{ on } \Sigma_s. \tag{7}$$

Definition 3. The material derivative of $u(\Omega, V)$ in the direction V noted $\dot{u}(\Omega, V) \in L^2(I, H^m(D))$ is defined as the limit in $L^2(I, H^m(D))$ when s goes to zeros of

$$\frac{1}{s}(u(\Omega_s) \circ T_s - u(\Omega)). \tag{8}$$

For more details on this approach, see [2].

Remark 4. To compute the shape derivatives, we can also use the approach of Simon and Murat in [3], recalled by Henrot and Pierre in [1] and Allaire and Jouve in [4].

We consider a perturbation of the domain Ω in the following sense, for $\theta \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $\Omega_\theta = (Id + \theta)(\Omega)$. It is well known that for θ, sufficiently small $(Id + \theta)$ is a diffeomorphism from $\mathbb{R}^n$.

Definition 5. The shape derivative of $u(\Omega)$ at Ω is defined as the Frechet derivative in $W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ at 0 of the application $\theta \longrightarrow u((\mathrm{Id} + \theta)(\Omega))$; that is,

$$u((Id + \theta)(\Omega)) = u(\Omega) + u'(\Omega)(\theta) + o(\theta), \tag{9}$$

where $u'(\Omega)$ is a continuous and linear form on $W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$.

In classical shape optimization, it is the boundary of the initial domain (or a part of the boundary) which moves for reaching the optimal shape. Thus, the optimal shape has the same topology as the initial one (for example, if the initial domain is simply connected, the optimal one will be also connected). Unlike the case of classical shape optimization, the topology of the design may change during the optimization process, for example, the inclusion of holes. The physical meaning of holes depends on the nature of the design. In the case of structural optimization, the insertion of the holes means simply removing some material, see [4]. In the case of fluid dynamics, creating a hole means inserting a small obstacle, see [5].

Topological sensitivity analysis aims at providing an asymptotic expansion of a shape functional acting on the neighborhood of a small hole created inside the domain. The underlying principle is the following, see also [6]: For a criterion $j(\Omega) = J_\Omega(u_\Omega)$, $\Omega \subset \mathbb{R}^n$ and u_Ω is the solution of a boundary value problem defined over Ω (generally a PDE), the asymptotic expansion of the cost function $j(\Omega)$ can be generally written in the form:

$$j(\Omega \setminus \overline{x_0 + \varepsilon\omega}) - j(\Omega) = \rho(\varepsilon) g(x_0) + o(\rho(\varepsilon)),$$
$$\lim_{\varepsilon \longrightarrow 0} \rho(\varepsilon) = 0, \rho(\varepsilon) > 0. \tag{10}$$

$g(x_0)$ is called topological derivative (or topological sensibility) and provides an information for creating small hole located at x_0. Hence, the function g can be used like a descent direction in the optimization process.

The algorithm that we propose combines the creation of holes (insertion of a dielectric object) using the topological gradient on the one hand and on the other hand the update of the edges by making evolve using the derivative of shape.

In Section 2, we present Maxwell's equations and an existence and uniqueness result and partial regularity under some conditions. The calculous of shape derivatives and topological asymptotic expansion (under small perturbations of the magnetic or electric fields) of the adjoint state is presented in Section 3. Two examples of shape functionals are considered. The main contribution of this paper is the proposition of the algorithm coupling between shape and topological derivative, which is presented in Section 4. Section 5 gives a conclusion and some possible extensions.

2. Maxwell's Equations and the Preliminary Results

Let D be a fixed domain of $\mathbb{R}^3$ and $\Omega \subset D$ be an open bounded regular domain with Lipshitz boundary $\partial\Omega = \Gamma$. Let $\varepsilon(x,t)$ be the electric permittivity and $\mu(x,t)$ be the magnetic permeability, which are both positive definite Hermitian 3×3 matrices.

Let f_1 and f_2 be the electric density through Ω and g be the electric density through Γ.

The evolution of the electric field $E(x,t)$ and the magnetic field $H(x,t)$ in the space-time cylinder $Q = \Omega \times (0,T)$ is given by the Maxwell equation:

$$\partial_t(\varepsilon E) - \nabla \times H = f_1 \text{ in } Q, \tag{11}$$

$$\partial_t(\mu E) + \nabla \times E = f_2 \text{ in } Q, \tag{12}$$

$$\nu \times E - \alpha H_\tau = g \text{ in } \Sigma. \tag{13}$$

Here, ν denotes the normal exterior vector along Γ, H_τ is the tangential component of H, and $\alpha = \alpha(x,t)$ is a positive function. For $t = 0$, the initial conditions are given by $E(x,0) = E_0(x)$ and $H(x,0) = H_0(x)$.

Let

$$A^0 = \begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix}, A^j\partial_j = \begin{bmatrix} 0 & -\nabla\times \\ \nabla\times & 0 \end{bmatrix}, u = \begin{bmatrix} E \\ H \end{bmatrix}, f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{14}$$

The Maxwell system (15) can be written as

$$\partial_j\left(A^0 u\right) + A^j \partial_j u = f. \tag{15}$$

Furthermore, let

$$\mathscr{H}(\Omega) = \left\{(E,H) \in \left(L^2(\Omega)\right)^6 : \nabla.(\varepsilon E) = \nabla.(\mu H) = 0\right\}, \tag{16}$$

where $\nabla.(\varepsilon E) = \partial_j(\varepsilon^{jk}E_k)$. If A is a Hermitian positive definite matrix, its transpose is denoted by A^T.

The well posedness of the Maxwell equations with given boundary and initial data is widely studies in the literature. We refer the literature to [7] for modeling and for the existence and unicity and regularity of the solution, see [8]. However, we recall here on a result which established the existence, unicity, and partial regularity of the Maxwell equations.

Theorem 6. *Le ε, μ, $\partial_t E$, $\partial_t \mu$, and $\in L^\infty(Q)$ be such that ε, $\mu \geq C > 0$ almost everywhere on Q and $\alpha \geq C > 0$ almost everywhere on Σ. Given $f, (E_0, H_0) \in (L^2(\Omega))^6$ and $g \in (L^2(\Sigma))^3$ with $\nu.g = 0$, then there exists a unique weak solution $(E,H) \in C([0,T], L^2(\Omega)^6)$. Moreover, there exists a constant β_0, such that*

$$\begin{aligned} &\left\|e^{-\beta T}(E,H)(T)\right\|_\Omega^2 + \left\|e^{-\beta T}(E,H)(t)\right\|_Q^2 + \left\|e^{-\beta T}(E_\tau, H_\tau)\right\|_\Sigma^2 \\ &\quad \leq \left\|\left(E^0, H^0\right)\right\|_\Omega^2 + \frac{1}{\beta}\left\|e^{-\beta T} f\right\|_Q^2 + \left\|e^{-\beta T} g\right\|_\Sigma^2, \end{aligned} \tag{17}$$

for $\beta \geq \beta_0$.

For the proof, see [2].

3. Shape Derivatives and Topological Asymptotic Expansion for Maxwell's Equations

3.1. The Adjoint Method. To compute the shape derivative, we use the adjoint method. The fundamental propriety of the adjoint method is to provide the variation of a shape functional J with respect to a parameter using the solution u_Ω and an adjoint state v_Ω which do not depend on the chosen parameter. Numerically, it means that the two systems must be solved for obtaining an approximation of the shape derivative $\mathrm{DJ}(u_\Omega), \forall x \in \Omega$. The principle is as follows. We consider the minimization problem

$$\min_\alpha j(\alpha) = J(\alpha, u(\alpha)) \in \mathbb{R}^n, \tag{18}$$

with constraints $A(\alpha)u(\alpha) = B(\alpha)$, $\alpha \in \mathbb{R}^p, u(\alpha) \in \mathbb{R}^n, A(\alpha) \in \mathscr{M}_{n\times n}(\mathbb{R}), B \in \mathbb{R}^n$. $\forall u, v$, the Lagrangian of the system, is defined by

$$\mathscr{L}(\alpha, u(\alpha), v) = J(\alpha, u(\alpha)) + A(\alpha)u(\alpha)v - B(\alpha)v. \tag{19}$$

Its derivative with respect to α is given by

$$\begin{aligned} D\mathscr{L}(\alpha, u(\alpha), v)h &= D_\alpha \mathscr{L}(\alpha, u(\alpha), v)h \\ &\quad + D_u \mathscr{L}(\alpha, u(\alpha), v)hDu(\alpha) \\ &= D_\alpha J(\alpha, u(\alpha), v)h + D_u J(\alpha, u(\alpha))hDu(\alpha) \\ &\quad + D_\alpha A(\alpha)u(\alpha)vh \\ &= A(\alpha)D_\alpha u(\alpha)vh - D_\alpha B(\alpha)v. \end{aligned} \tag{20}$$

We set $v = v(\alpha)$ as the adjoint state to obtain

$$A(\alpha)v(\alpha)\omega = -D_u J(\alpha, u(\alpha)), \forall \omega \in \mathbb{R}^n, \tag{21}$$

as an adjoint equation. It follows

$$\begin{aligned} D\mathscr{L}(\alpha, u(\alpha), v(\alpha)) &= D_\alpha \mathscr{L}(\alpha, u(\alpha), v(\alpha)) + D_\alpha A(\alpha)v(\alpha)u(\alpha) \\ &\quad - D_\alpha B(\alpha)v(\alpha). \end{aligned} \tag{22}$$

3.2. Shape Derivative of the Cost Function. In the sequel, we turn our attention to the minimization of the cost functional defined by

$$J(\Omega) = \frac{1}{2}\int_Q \Big[(E_\Omega - E_d)^T \varepsilon (E_\Omega - E_D) + (H_\Omega - H_d)^T \mu (H_\Omega - H_d)\Big]\, dxdt \quad (23)$$

over a collection of open bounded sets $\Omega \in \mathbb{R}^3$, with fixed boundaries. (E_Ω, H_Ω) is the solution of the initial boundary value problem (11). The following result gives the shape derivative of (23) under constraints (11).

Theorem 7. *Assume that ε, μ are constants of Hermitian definite positive matrices, let α be a positive constant, $f = 0$, $g = 0$ and $(E_0, H_0) \in H^1(D)^6 \cap \mathscr{H}(D)$ and $(E_\Omega, H_\Omega) \in H^1(D \times (0,T)) \cap \mathscr{H}(D \times (0,T))$. The shape functional (23) is Frechet differentiable at Ω in the direction V with the Frechet derivative*

$$\begin{aligned} DJ(\Omega, V) = \mathscr{R}\int_\Sigma &\big[q.\partial_\nu E_\Omega \times \nu + \alpha(\partial_\nu H_\Omega)_\tau + Curl_\Gamma\big((E_\Omega)_\nu.q\big) \\ &- \alpha div_\Gamma\big((H_\Omega)_\nu.q\big)\big] V_\nu dxdt \\ &+ \frac{1}{2}\int_\Sigma (E_\Omega - E_d)^T \varepsilon (E_\Omega - E_D) \\ &+ (H_\Omega - H_d)^T \mu (H_\Omega - H_d)\Big] V_\nu dxdt, \end{aligned} \quad (24)$$

where $V_\nu = V(0).\nu$, $E_\nu = E.\nu$, $H_\nu = H.\nu$, and div_Γ are the surface divergences, $Curl_\Gamma$ is the surface curl, $\mathscr{R}(.)$ is the real part of a complex number, and (p, q) is the solution of the adjoint initial boundary value problem:

$$\begin{aligned} &\varepsilon\partial_t p - \nabla \times q = 0 \text{ in } Q, \\ &\mu\partial_t q + \nabla \times q = 0 \text{ in } Q, \\ &p_{|t=0} = 0 q_{|t=0} = 0\,; \text{ in } \Omega, \\ &\nu \times p + \alpha q_\tau = 0 \text{ in } \Sigma. \end{aligned} \quad (25)$$

As V, equation (23) is true for all V, and we have

$$\begin{aligned} DJ(\Omega) = \mathscr{R}&\big[q\big(.\partial_\nu E_\Omega \times \nu + \alpha(\partial_\nu H_\Omega)_\tau\big) + Curl_\Gamma\big((E_\Omega)_\nu.q\big) \\ &- \alpha div_\Gamma\big((H_\Omega)_\nu q\big)\big] + \frac{1}{2}(E_\Omega - E_d)^T \varepsilon (E_\Omega - E_D) \\ &+ (H_\Omega - H_d)^T \mu (H_\Omega - H_d)\Big]. \end{aligned} \quad (26)$$

Sketch of the proof. The Gâteaux derivation of $J(\Omega)$ in the direction V is defined by

$$\begin{aligned} DJ(\Omega, V) &= \lim_{s \to 0} \frac{1}{s}\left(J(\Omega_s) - J(\Omega)\right) \\ &= \mathscr{R}\int_Q (E_\Omega - E_d)^T \varepsilon E_{\Omega'} dt d\Gamma \\ &\quad + \mathscr{R}\int_Q (H_\Omega - H_d)^T \mu H'_\Omega d\Gamma dt \\ &\quad + \frac{1}{2}\int_\Sigma (E_\Omega - E_d)^T \varepsilon (E_\Omega - E_0) \\ &\quad + (H_\Omega - H_d)^T \mu H_\Omega - H_0\big)\Big] V_\nu d\Gamma dt, \end{aligned} \quad (27)$$

where (E'_Ω, H'_Ω) is the shape derivative of (15). Let (p, q) be the adjoint state, and then we have

$$\begin{aligned} DJ(\Omega, V) &= \mathscr{R}\int_Q (\varepsilon\partial_t p - \nabla q) E'_\Omega dxdt \\ &\quad + \mathscr{R}\int_Q (\mu\partial_t q - \nabla p) H'_\Omega dxdt \\ &\quad + \frac{1}{2}\int_\Sigma \Big[(E_\Omega - E_d)^T \varepsilon (E_\Omega - E_0) \\ &\quad + (H_\Omega - H_d)^T \mu H_\Omega - H_0)\Big] V_\nu d\Gamma dt. \\ &= \mathscr{R}\int_\Sigma \Big[(\nu \times p) H'_{\Omega,\tau} + q.\big(\nu \times E'_\Omega\big)\Big] dt d\Gamma. \\ &\quad + \frac{1}{2}\int_\Sigma \Big[(E_\Omega - E_d)^T \varepsilon (E_\Omega - E_0) \\ &\quad + (H_\Omega - H_d)^T \mu H_\Omega - H_0)\Big] V_\nu d\Gamma dt. \\ &= \mathscr{R}\int_\Sigma q.[(V_\nu \partial_\nu E_{\Omega,\tau} + E_{\Omega,\nu} \nabla_\tau V_\nu) \times + \alpha(V_\nu \partial_\nu H_{\Omega,\tau} \\ &\quad + H_{\Omega,\nu} \nabla_\tau V_\nu)] d\Gamma dt. \frac{1}{2}\int_\Sigma \Big[(E_\Omega - E_d)^T \varepsilon (E_\Omega - E_0) \\ &\quad + (H_\Omega - H_d)^T \mu H_\Omega - H_0)\Big] V_\nu d\Gamma dt. \end{aligned} \quad (28)$$

To achieve the proof, we have to remove the derivative V_ν, by integration by parts.

$$\begin{aligned} \mathrm{DJ}(\Omega, V) &= \mathscr{R}\int_\Sigma [q.V_\nu(\partial_\nu E_{\Omega,\tau} + \alpha \partial_\nu H_{\Omega,\tau}) \\ &\quad + \nabla_\tau V_\nu.(\nu \times E_{\Omega,\nu} q) + \alpha H_{\Omega,\nu} q \nabla_\tau V_\nu)] d\Gamma dt \\ &\quad + \frac{1}{2}\int_\Sigma \Big[(E_\Omega - E_d)^T \varepsilon (E_\Omega - E_0) \\ &\quad + (H_\Omega - H_d)^T \mu H_\Omega - H_0)\Big] V_\nu d\Gamma dt. \\ &= \mathscr{R}\int_\Sigma [q.(\partial_\nu E_{\Omega,\tau} + \alpha \partial_\nu H_{\Omega,\tau}) - \mathrm{div}_\Gamma(\nu \times E_{\Omega,\nu} q) \\ &\quad - \alpha \mathrm{div}_\Gamma(H_{\Omega,\nu} q)] V_\nu d\Gamma dt \\ &\quad + \frac{1}{2}\int_\Sigma \Big[(E_\Omega - E_d)^T \varepsilon (E_\Omega - E_0) \\ &\quad + (H_\Omega - H_d)^T \mu H_\Omega - H_0)\Big] V_\nu d\Gamma dt. \end{aligned}$$

$$\begin{aligned} &= \mathscr{R}\int_{\Sigma}[q.(\partial_{\nu}E_{\Omega,\tau}+\alpha\partial_{\nu}H_{\Omega,\tau})+\mathrm{Curl}_{\Gamma}(\nu\times E_{\Omega,\nu}q) \\ &\quad -\alpha\mathrm{div}_{\Gamma}(H_{\Omega,\nu}q)]V_{\nu}d\Gamma dt \\ &\quad +\frac{1}{2}\int_{\Sigma}\Big[(E_{\Omega}-E_{d})^{T}\varepsilon(E_{\Omega}-E_{0}) \\ &\quad +(H_{\Omega}-H_{d})^{T}\mu H_{\Omega}-H_{0})\Big]V_{\nu}d\Gamma dt, \end{aligned} \tag{29}$$

which finishes the proof.

For the complete proof, we refer the reader to [8].

3.3. Generalized Adjoint Method. Let $\mathscr{V}$ be a complex Hilbert space. For all $\varepsilon > 0$, let a_ε be a sesquilinear and continuous form on $\mathscr{V}$ and l_ε be a semilinear and continuous form on $\mathscr{V}$, such that the following problem has one and only one solution:

$$\begin{cases} u_\varepsilon \in \mathscr{V}, \\ a_\varepsilon(u_\varepsilon, v) = l_\varepsilon(v), \forall v \in \mathscr{V}. \end{cases} \tag{30}$$

Hypothesis 1. Assume that for $\varepsilon > 0$, there exists $\rho(\varepsilon) > 0$, with $\lim_{\varepsilon\longrightarrow 0^+}\rho(\varepsilon) = 0$, δa and δl two complex numbers such that

$$\begin{aligned} \|u_\varepsilon - u_0\|_{\mathscr{V}} &= O(\rho(\varepsilon)), \\ (a_\varepsilon - a_0)(u_\varepsilon, v) &= \rho(\varepsilon)\delta a + O(\rho(\varepsilon)), \\ (l_\varepsilon - l_0)(u_\varepsilon, v) &= \rho(\varepsilon)\delta l + O(\rho(\varepsilon)). \end{aligned} \tag{31}$$

Let $J(\Omega, u_\varepsilon) = J_\Omega(u_\varepsilon)$ be the cost functional. We denote by $j(\varepsilon) = J_\Omega(u_\varepsilon)$, $\forall \varepsilon > 0$.

Hypothesis 2. For all $u \in \mathscr{V}$, there exists a linear and continuous form, note L_u, such that

$$J_\Omega(u+h) = J_\Omega(u) + \delta_{J_\Omega}(u) + \mathscr{R}(L_u(h)) + O(\|h\|_{\mathscr{V}}). \tag{32}$$

$\|.\|$ designs the norm on $\mathscr{V}$.

For all $\varepsilon > 0$, let v_ε the solution of the so-called adjoint problem

$$\begin{cases} v_\varepsilon \in \mathscr{V}, \\ a_\varepsilon(v_\varepsilon, w) = L_{u_0}(w), \forall w \in \mathscr{V}. \end{cases} \tag{33}$$

u_0 is the solution of (30) for $\varepsilon = 0$. Under Hypotheses 1 and 2, we have the following result, which gives the asymptotic expansion for $j(\varepsilon)$.

Theorem 8. *If Hypotheses 1 and 2 are satisfied, the asymptotic expansion to the cost function $j(\varepsilon) = J_\Omega(u_\varepsilon)$ is given by*

$$j(\varepsilon) = j(0) + \rho(\varepsilon)\delta_j + O(\rho(\varepsilon)), \tag{34}$$

where $\delta_j = \delta_a + \delta_l + \delta_J$.

The proof of this theorem is standard in topological optimization, see, for example, [6, 9].

The function $\delta_j(x_0)$ is called topological derivative (or topological sensitivity) and provides an information for creating a small hole located at x_0. Hence, the function δ_j can be used like a descent direction in the optimization process.

Remark 9. In the following, for all approximations, we use Laplace exterior problem

$$\begin{aligned} \Delta E_\Omega &= 0 \text{ in } \frac{\mathbb{R}^3}{\Omega}, \\ E_\Omega(x) &\longrightarrow 0 \text{ at} +\infty, \\ \frac{\partial E_\Omega}{\partial \nu} &= -E^0_\Omega(0) \text{ on } \Gamma. \end{aligned} \tag{35}$$

The function E_Ω can be explicited by the help of a single layer potential

$$\begin{aligned} &\frac{E(x)}{2} + \int_{\partial\omega} \nabla_y(U(y-x)E(x).\nu(y)ds(y) \\ &\quad = \phi(y), \forall y \in \partial\omega \text{ and } E \in H^{1/2}(\partial\Omega), \end{aligned} \tag{36}$$

with U being the fundamental solution of the Laplace operator, which is given in 3D by

$$U(x) = \frac{1}{4\pi|x|}. \tag{37}$$

3.4. Problem Formulation and Topological Asymptotic Expansion. The domain perturbation corresponds to the perturbation of electric permittivity and the magnetic permeability. The corresponding perturbed problem writes (for the electric field)

$$\partial_t E^\varepsilon_\Omega - \nabla(\varepsilon_\varepsilon \nabla E^\varepsilon_\Omega) + \mu_\varepsilon E^\varepsilon_\Omega = 0 \text{ in } Q, \tag{38}$$

$$E^\varepsilon_\Omega(x, 0) = E_0(x) \text{ in } \Omega, \tag{39}$$

$$\frac{\partial E^\varepsilon_\Omega}{\partial \nu} = \sigma \text{ on } \Sigma, \tag{40}$$

where

$$\varepsilon_\varepsilon = \begin{cases} \varepsilon_0 & \text{in } Q/D_\varepsilon \\ \varepsilon_1 & \text{in } D_\varepsilon \end{cases} \tag{41}$$

and

$$\mu_\varepsilon = \begin{cases} \mu_0 & \text{in } Q/D_\varepsilon \\ \mu_1 & \text{in } D_\varepsilon \end{cases}, \tag{42}$$

For $\varepsilon = 0$, the corresponding problem is to find E_{Ω}^{0} such that

$$\partial_t E_{\Omega}^{0} - \nabla\left(\varepsilon_0 \nabla E_{\Omega}^{0}\right) + \mu_0 E_{\Omega}^{0} = 0 \text{ in } Q, \quad (43)$$

$$E_{\Omega}^{0}(x,0) = E_0(x) \text{ in } \Omega, \quad (44)$$

$$\frac{\partial E_{\Omega}^{0}}{\partial \nu} = \sigma \; on \; \Sigma. \quad (45)$$

The variational problem assisted with (38) writes the following: find E_{Ω}^{ε} such that

$$a_{\varepsilon}(E_{\Omega}^{\varepsilon}, F) = l_{\varepsilon}(F), \quad (46)$$

where

$$a_{\varepsilon}(E_{\Omega}^{\varepsilon}, F) = \int_Q \left\langle \frac{\partial E_{\Omega}^{\varepsilon}{}^{\varepsilon}}{\partial t}, F \right\rangle dxdt + \int_Q \varepsilon_{\varepsilon} \nabla . E_{\Omega}^{\varepsilon} . \mathscr{R}(\nabla F) dxdt + \int_Q \mu_{\varepsilon} E_{\Omega}^{\varepsilon} \mathscr{R}(F) dxdt, \quad (47)$$

$$l_{\varepsilon}(F) = \int_{\Sigma} \sigma F dsdt. \quad (48)$$

And the variational problem associated to (43) is to find E_{Ω}^{0} such that

$$a_{\varepsilon}(E_{\Omega}^{\varepsilon}, F) = l_{\varepsilon}(F), \quad (49)$$

where

$$a_0(E_{\Omega}, F) = \int_Q \left\langle \frac{\partial E_{\Omega}^{0}}{\partial t}, F \right\rangle dxdt + \int_Q \varepsilon 0 \nabla E_{\Omega}^{0} . \mathscr{R}(\nabla F) dxdt + \int_Q \mu_0 E_{\Omega}^{0} \mathscr{R}(F) dxdt. \quad (50)$$

Lemma 10. *Problems (47) and (50) admit one and only one solution. Moreover, there exists a function $\rho(\varepsilon) > 0$ which goes to zeros with ε, such that*

$$\left\| E_{\Omega}^{\varepsilon} - E_{\Omega}^{0} \right\|_{\mathscr{V}} = O(\rho(\varepsilon)). \quad (51)$$

Proof. For the proof, we refer the reader to [6]. □

3.4.1. Variation of the Sesquilinear Form

Proposition 11. *Let E_{Ω}^{ε} (resp E_{Ω}^{0}) solution of (47) (resp. (50)), and then there exists a real number δ_a, and a function $\rho(\varepsilon) > 0$ tending to zero as ε tends to zero such that*

$$a_{\varepsilon}(E_{\Omega}^{\varepsilon}, F) - a_0\left(E_{\Omega}^{0}\right) = \rho(\varepsilon)\delta_a + O(\rho(\varepsilon)). \quad (52)$$

Proof.

$$\begin{aligned}
a_{\varepsilon}(E_{\Omega}^{\varepsilon}, F) - a_0(E_{\Omega}, F)
&= \int_Q \left\langle \frac{\partial E_{\Omega}^{\varepsilon}{}^{\varepsilon}}{\partial t}, F \right\rangle dxdt + \int_Q \varepsilon_{\varepsilon} \nabla . E_{\Omega}^{\varepsilon} . \mathscr{R}(\nabla F) dxdt \\
&\quad + \int_Q \mu_{\varepsilon} E_{\Omega}^{\varepsilon} \mathscr{R}(F) dxdt - \int_Q \left\langle \frac{\partial E_{\Omega}^{0}}{\partial t}, F \right\rangle dxdt \\
&\quad + \int_Q \varepsilon_0 \nabla . E_{\Omega}^{0} . \mathscr{R}(\nabla F) dxdt + \int_Q \mu_0 E_{\Omega}^{0} \mathscr{R}(F) dxdt \\
&= \int_Q \left\langle \frac{\partial E_{\Omega}^{\varepsilon}}{\partial t}, F \right\rangle dxdt + \int_{Q_{\varepsilon}} \varepsilon_0 \nabla . E_{\Omega}^{\varepsilon} . \mathscr{R}(\nabla F) dxdt \\
&\quad + \int_{Q_{\varepsilon}} \mu_0 E_{\Omega}^{\varepsilon} \mathscr{R}(F) dxdt + \int_{D_{\varepsilon}} \varepsilon_1 \nabla . E_{\Omega}^{\varepsilon} . \mathscr{R}(\nabla F) dxdt \\
&\quad + \int_{D_{\varepsilon}} \mu_1 E_{\Omega}^{\varepsilon} \mathscr{R}(F) dxdt - \int_Q \left\langle \frac{\partial E_{\Omega}^{0}}{\partial t}, F \right\rangle dxdt \\
&\quad + \int_{Q_{\varepsilon}} \varepsilon_0 \nabla . E_{\Omega}^{0} . \mathscr{R}(\nabla F) dxdt + \int_{Q_{\varepsilon}} \mu_0 E_{\Omega}^{0} \mathscr{R}(F) dxdt \\
&\quad + \int_{D_{\varepsilon}} \varepsilon_0 \nabla . E_{\Omega}^{0} . \mathscr{R}(\nabla F) dxdt + \int_{D_{\varepsilon}} \mu_0 E_{\Omega}^{0} \mathscr{R}(F) dxdt \\
&= \int_Q \left\langle \frac{\partial E_{\Omega}^{\varepsilon}}{\partial t} - \frac{\partial E_{\Omega}^{0}}{\partial t}, F \right\rangle dxdt \\
&\quad + \int_{Q_{\varepsilon}} \varepsilon_0 \nabla . \left(E_{\Omega}^{\varepsilon} - E_{\Omega}^{0}\right) . \mathscr{R}(\nabla F) dxdt \\
&\quad + \int_{Q_{\varepsilon}} \mu_0 \left(E_{\Omega}^{\varepsilon} - E_{\Omega}^{0}\right) \mathscr{R}(F) dxdt + \int_{D_{\varepsilon}} \varepsilon_1 \nabla . E_{\Omega}^{\varepsilon} \mathscr{R}(\nabla F) dxdt \\
&\quad + \int_{D_{\varepsilon}} \mu_1 E_{\Omega}^{\varepsilon} \mathscr{R}(F) dxdt - \int_{D_{\varepsilon}} \varepsilon_0 \nabla . E_{\Omega}^{0} \mathscr{R}(\nabla F) dxdt \\
&\quad + \int_{D_{\varepsilon}} \mu_0 E_{\Omega}^{0} \mathscr{R}(F) dxdt.
\end{aligned} \quad (53)$$

Let $X_{\Omega}^{\varepsilon} = E_{\Omega}^{\varepsilon} - E_{\Omega}^{0}$, and then X_{Ω}^{ε} is the solution of

$$\begin{gathered}
\partial_t X_{\Omega}^{\varepsilon} - \nabla . (\varepsilon_{\varepsilon} \nabla X_{\Omega}^{\varepsilon}) + \mu_{\varepsilon} X_{\Omega}^{\varepsilon} = 0 \text{ in } Q, \\
X_{\Omega}^{\varepsilon}(x,0) = 0 \text{ in } \Omega, \\
\frac{\partial x_{\Omega}^{\varepsilon}}{\partial \nu} = 0 \text{ on } \Sigma,
\end{gathered}$$

$$\begin{aligned}
a_{\varepsilon}(E_{\Omega}^{\varepsilon}, F) - a_0(E_{\Omega}, F) &= \int_{D_{\varepsilon}} \varepsilon_0 \nabla . X_{\Omega}^{\varepsilon} . \mathscr{R}(\nabla F) dxdt \\
&\quad + \int_{D_{\varepsilon}} (\varepsilon_1 - \varepsilon_0) \nabla . E_{\Omega}^{0} \mathscr{R}(\nabla F) dxdt \\
&\quad + \int_{D_{\varepsilon}} \mu_0 X_{\Omega}^{\varepsilon} \mathscr{R}(F) dxdt \\
&\quad + \int_{D_{\varepsilon}} (\mu_1 + \mu_0) E_{\Omega}^{0} \mathscr{R}(F) dxdt.
\end{aligned} \quad (54)$$

The rest of the proof follows from the following lemmas. □

Lemma 12.

$$\int_{D_\varepsilon} \varepsilon_0 \nabla . X_\Omega^\varepsilon \mathscr{R}(\nabla F) dxdt = O(|D_\varepsilon|),$$
$$\int_{D_\varepsilon} \mu_0 X_\Omega^\varepsilon \mathscr{R}(F) dxdt = O(|D_\varepsilon|). \quad (55)$$

Proof. For the proof, see [6] for the example. □

Lemma 13.

$$\int_{D_\varepsilon} (\varepsilon_1 - \varepsilon_0) \nabla . E_\Omega^0 \mathscr{R}(\nabla F) dxdt = (\varepsilon_1 - \varepsilon_0) \nabla . E_\Omega^0(0) \mathscr{R}(\nabla F)(0) |D_\varepsilon|,$$
$$\int_{D_\varepsilon} (\mu_1 + \mu_0) E_\Omega^0 \mathscr{R}(F) dxdt = (\mu_1 - \mu_0) E_\Omega^0(0) \mathscr{R}(\nabla F(0) |D_\varepsilon|. \quad (56)$$

Proof. For the proof, see [6, 9] for the examples. □

Remark 14. According to the definition of l_ε,

$$\delta_l = 0. \quad (57)$$

3.4.2. Variation of the Cost Function. We consider two examples of cost functions. Each one corresponds to a specific analysis.

Example 1. The cost function J defined by

$$J_\Omega(E_\Omega) = \int_Q \varepsilon_\varepsilon \left(|E_\Omega - E_d|^2\right) dxdt, \varepsilon_\varepsilon = \begin{cases} \varepsilon_0 & \text{in } \dfrac{Q}{D_\varepsilon}, \\ \varepsilon_1 & \text{in } D_\varepsilon. \end{cases} \quad (58)$$

Proposition 15. *Let J_Ω^ε be the functional associated to the perturbed problem, and then there exists a real δ_J and a linear operator L_ε such that*

$$J_\Omega^\varepsilon(E_\Omega^\varepsilon) - J_\Omega^0(E_\Omega^0) = \rho(\varepsilon)\delta_J + L_\varepsilon(E) + O(\rho(\varepsilon)). \quad (59)$$

Proof.

$$\begin{aligned} & J_\Omega^\varepsilon(E_\Omega^\varepsilon) - J_\Omega^0(E_\Omega^0) \\ & = \int_Q \varepsilon_\varepsilon \left(|E_\Omega^\varepsilon - E_d|^2\right) dxdt - \int_Q \varepsilon_0 \left(|E_\Omega^0 - E_d|^2\right) dxdt \\ & = \int_{Q\backslash D_\varepsilon} \varepsilon_0 \left(|E_\Omega^\varepsilon - E_d|^2 - |E_\Omega^0 - E_d|^2\right) \\ & \quad + \int_{D_\varepsilon} (\varepsilon_0 - \varepsilon_1) \left(|E_\Omega^0 - E_d|^2\right), \\ & \int_{Q\backslash D_\varepsilon} \varepsilon_0 \left(|E_\Omega^\varepsilon - E_d|^2 - |E_\Omega^0 - E_d|^2\right) dxdt \\ & = \int_{Q\backslash D_\varepsilon} \varepsilon_0 (E_\Omega^\varepsilon - E_\Omega^0)(E_\Omega^\varepsilon + E_\Omega^0 - 2E_d) dxdt \\ & = \int_{Q\backslash D_\varepsilon} \varepsilon_0 |E_\Omega^\varepsilon - E_\Omega^0|^2 dxdt \\ & \quad + 2\varepsilon_0 \int_{Q\backslash D_\varepsilon} (E_\Omega^\varepsilon - E_\Omega^0)(E_\Omega^0 - E_d) dxdt, \\ & \int_{Q\backslash D_\varepsilon} \varepsilon_0 |E_\Omega^\varepsilon - E_\Omega^0|^2 dxdt \\ & \leq \varepsilon_0 \|E_\Omega^\varepsilon - E_\Omega^0\|^2 |Q| = O(\rho(\varepsilon)), \\ & \int_{D_\varepsilon} (\varepsilon_0 - \varepsilon_1) \left(|E_\Omega^0 - E_d|^2\right) dxdt \\ & = (\varepsilon_0 - \varepsilon_1) \|E_\Omega^0(0) - E_d(0)\|^2 |D_\varepsilon| + O(\rho(\varepsilon)). \end{aligned} \quad (60)$$

We achieve the proof by setting $L_\varepsilon(E) = 2\varepsilon_0 \int_{Q\backslash D_\varepsilon} E(E_\Omega^0 - E_d) dxdt$. □

Example 2. Here, we focus on the function

$$J_\Omega(E_\Omega) = \frac{1}{2} \int_Q \alpha_\varepsilon |\nabla(E_\Omega - E_d)|^2 dxdt, \alpha_\varepsilon = \begin{cases} \alpha_0 & \text{in } \dfrac{Q}{D_\varepsilon}, \\ \alpha_1 & \text{in } D_\varepsilon. \end{cases} \quad (61)$$

Proposition 16. *Let J_Ω^ε be the functional associated to the perturbed problem, and then there exists a real δ_J and a linear operator L_ε, such that*

$$J_\Omega^\varepsilon(E_\Omega^\varepsilon) - J_\Omega^0(E_\Omega^0) = \rho(\varepsilon)\delta_J + L_\varepsilon(E) + O(\rho(\varepsilon)). \quad (62)$$

Proof.

$$\begin{aligned} & J_\Omega^\varepsilon(E_\Omega^\varepsilon) - J_\Omega^0(E_\Omega^\varepsilon) \\ & = \frac{1}{2} \int_Q \alpha_\varepsilon |\nabla(E_\Omega^\varepsilon - E_d)|^2 dxdt - \frac{1}{2} \int_Q \alpha_0 |\nabla(E_\Omega^0 - E_d)|^2 dxdt \\ & = \underbrace{\frac{1}{2} \int_{Q\backslash D_\varepsilon} \alpha_0 \left(|\nabla(E_\Omega^\varepsilon - E_d)|^2 - |\nabla(E_\Omega^0 - E_d)|^2\right) dxdt}_{I} \\ & \quad + \underbrace{\frac{1}{2} \int_{D_\varepsilon} \alpha_1 |\nabla(E_\Omega^\varepsilon - E_d)|^2 dxdt}_{II} - \frac{1}{2} \int_{D_\varepsilon} \alpha_0 |\nabla(E_\Omega^0 - E_d)|^2 dxdt, \end{aligned} \quad (63)$$

$$I = \frac{1}{2}\int_{Q\backslash D_\varepsilon} \alpha_0 |\nabla(E^\varepsilon_\Omega - E_d)|^2 - \left|\nabla\left(E^0_\Omega - E_d\right)\right|^2 \Bigg) dxdt$$
$$= \frac{1}{2}\int_{Q\backslash D_\varepsilon} \alpha_0 \left(\nabla\left(E^\varepsilon_\Omega - E^0_\Omega\right)\nabla\left(E^\varepsilon_\Omega + E^0_\Omega - 2E_d\right)\right) dxdt, \tag{64}$$

$$I = \frac{1}{2}\int_{Q\backslash D_\varepsilon} \alpha_0 \left(\nabla\left(E^\varepsilon_\Omega - E^0_\Omega\right)\mathscr{R}\left(\nabla\left(E^\varepsilon_\Omega - E^0_\Omega\right)\right)\right)$$
$$+ \int_{Q\backslash D_\varepsilon} \alpha_0 \left(\nabla\left(E^\varepsilon_\Omega - E^0_\Omega\right)\nabla\left(E^0_\Omega - E_d\right)\right). \tag{65}$$

□

In setting $X^\varepsilon_\Omega = E^\varepsilon_\Omega - E^\varepsilon_\Omega$, equation (63) writes

$$I = \frac{1}{2}\int_{Q\backslash D_\varepsilon} \alpha_0 \nabla X^\varepsilon_\Omega \mathscr{R}(\nabla X^\varepsilon_\Omega) dxdt$$
$$+ \int_{Q\backslash D_\varepsilon} \alpha_0 \nabla X^\varepsilon_\Omega \nabla\left(E^0_\Omega - E_d\right) dxdt,$$
$$II = \frac{1}{2}\int_{D_\varepsilon} \alpha_1 |\nabla(E^\varepsilon_\Omega - E_d)|^2 dxdt$$
$$= \frac{1}{2}\int_{D_\varepsilon} \alpha_1 \left|\nabla\left(E^\varepsilon_\Omega - E^0_\Omega + E^0_\Omega - H_d\right)\right|^2 dxdt,$$
$$= \frac{1}{2}\int_{D_\varepsilon} \alpha_1 \left|\nabla\left(E^\varepsilon_\Omega - E^0_\Omega\right)\right|^2 + \frac{1}{2}\int_{D_\varepsilon} \alpha_1 \left|\nabla\left(E^0_\Omega - E_d\right)\right|^2$$
$$+ \int_{D_\varepsilon} \alpha_1 \nabla\left(E^\varepsilon_\Omega - E^0_\Omega\right)\nabla\left(E^0_\Omega - E_d\right),$$
$$= \frac{1}{2}\int_{D_\varepsilon} \alpha_1 |\nabla X^\varepsilon_\Omega|^2 dxdt + \frac{1}{2}\int_{D_\varepsilon} \alpha_1 \left|\nabla\left(E^0_\Omega - E_d\right)\right|^2 dxdt$$
$$+ \int_{D_\varepsilon} \alpha_1 \nabla X^\varepsilon_\Omega \nabla\left(E^0_\Omega - E_d\right) dxdt. \tag{66}$$

Consequently,

$$J^\varepsilon_\Omega(H^\varepsilon_\Omega) - J^0_\Omega(E^\varepsilon_\Omega) = \frac{1}{2}\int_{Q\backslash D_\varepsilon} \alpha_0 \nabla X^\varepsilon_\Omega \mathscr{R}(\nabla X^\varepsilon_\Omega) dxdt$$
$$+ \frac{1}{2}\int_{D_\varepsilon} \alpha_1 \nabla X^\varepsilon_\Omega \mathscr{R}(\nabla X^\varepsilon_\Omega) dxdt$$
$$+ \int_{Q\backslash D_\varepsilon} \alpha_0 \nabla X^\varepsilon_\Omega \nabla\left(E^0_\Omega - E_d\right)$$
$$+ \int_{D_\varepsilon} \alpha_1 \nabla X^\varepsilon_\Omega \nabla\left(E^0_\Omega - E_d\right)$$
$$- \frac{1}{2}\int_{D_\varepsilon} (\alpha_0 - \alpha_1)\left|\nabla\left(E^0_\Omega - E_d\right)\right|^2 dxdt. \tag{67}$$

Setting

$$L_\varepsilon(H) = \int_Q \alpha_\varepsilon \nabla E.\mathscr{R}\left(\nabla\left(E^0_\Omega - E_d\right)dxdt \text{ and } \rho(\varepsilon)\delta_J\right.$$
$$= \frac{1}{2}\int_{D_\varepsilon} (\alpha_0 - \alpha_1)\left|\nabla\left(E^0_\Omega - E_d\right)\right|^2 dxdt. \tag{68}$$

As

$$\frac{1}{2}\int_{Q\backslash D_\varepsilon} \alpha_0 \nabla X^\varepsilon_\Omega \mathscr{R}(\nabla X^\varepsilon_\Omega) dxdt + \frac{1}{2}\int_{D_\varepsilon} \alpha_1 \nabla X^\varepsilon_\Omega \mathscr{R}(\nabla X^\varepsilon_\Omega) dxdt$$
$$= \frac{1}{2}\int_Q \alpha_\varepsilon \nabla X^\varepsilon_\Omega \mathscr{R}(\nabla X^\varepsilon_\Omega) dxdt = O(\rho(\varepsilon)), \tag{69}$$

we obtain the desired result.

The following result gives the asymptotic expansion of the cost function for the perturbation of the electric field.

Theorem 17. *Let $j(\varepsilon) = J^\varepsilon_\Omega(E^\varepsilon_\Omega)$ be the cost functional defined by (58), where E^ε_Ω is the solution of (47) and E^0_Ω is the corresponding solution for $\varepsilon = 0$. Let $D_\varepsilon = \varepsilon B(0,1) \subset \mathbb{R}^3$. j has the following asymptotic expansion:*

$$j(\varepsilon) - j(0) = -\frac{4}{3}\pi\varepsilon^3\left[(\mu_0 - \mu_1)E^0_\Omega(0)\mathscr{R}(F(0))\right.$$
$$\left. + (\varepsilon_0 - \varepsilon_1)\left[\nabla \times E^0_\Omega(0)\mathscr{R}(\nabla F)(0)\right] + \delta_J\right] + O\left(\varepsilon^3\right), \tag{70}$$

where F is the solution of the so-called adjoint problem: find $F \in \mathscr{V}$ such that

$$\partial_t F - \nabla.(\varepsilon_0 \nabla F) + \mu_0 F = -L_\varepsilon(E) \text{ in } Q,$$
$$F(x,0) = 0 \text{ in } \Omega,$$
$$\frac{\partial F}{\partial \nu} = 0 \text{ on } \Sigma. \tag{71}$$

Proof. The Lagrangian $\mathscr{L}$ of the problem $\min_{\mathscr{V}} J_\Omega(E^0_\Omega, F)$, E^0_Ω, solution of (43) is defined by

$$\mathscr{L}(E_\Omega, F) = J_\Omega(E_\Omega) + a(E_\Omega, F) - l(F). \tag{72}$$

Its variation with respect to ε

$$\mathscr{L}_\varepsilon(E^\varepsilon_\Omega, F) = J^\varepsilon_\Omega(E^\varepsilon_\Omega) + a_\varepsilon(E^\varepsilon_\Omega, F) - l_\varepsilon(F). \tag{73}$$

□

It follows form Propositions 11 and 15 (or 16) that Hypotheses 1 and 2 are satisfied. We use the fact that the variation of the Lagrangian is equal to the variation of the cost function; that is,

$$j(\varepsilon) - j(0) = \mathscr{L}_\varepsilon(E^\varepsilon_\Omega, F) - \mathscr{L}_\varepsilon\left(E^0_\Omega, F\right), \qquad (74)$$

and we use Theorem 8 to conclude.

4. A Coupling Algorithm between Shape and Topological Derivative for Maxwell Equations

In this section, we propose an algorithm coupling the shape and topological derivative for a given shape functional under Maxwell equation as constraints. For the numerical approximation of Maxwell's equation, one can use the finite differences method, the finite elements method, the Finite Element-Finite Difference Hybrid Methods, or the block pseudospectral method and XFEM method.

The optimization algorithm is summarized in Figure 1. (see also [10] in the case of linear elasticity).

Based on the boundary of the domain being an unknown of the problem, we introduce $\tilde{\Omega}$ fixed (in general rectangular or parallelepiped) domain which includes all potential domains Ω. The numerical method approximation requires the introduction of two finite element spaces $\tilde{V}^h \subset H^1(\tilde{\Omega}; \mathbb{R}^n)$ and $\tilde{W}^h \subset L^2(\tilde{\Omega}; \mathbb{R}^n)$ on the fictitious domain $\tilde{\Omega}$. As $\tilde{\Omega}$ can be a rectangular or parallelepiped domain, the ones can be defined on the same structured mesh $\mathscr{T}^h$. Next, we shall suppose that

$$\tilde{V}^h = \left\{ v^h \in C\left(\bar{\tilde{C}}, \mathbb{R}^n\right) v^h_{|\Omega} \in (P(T))^n \forall T \in \mathscr{T}^h \right\}, \qquad (75)$$

where $P(T)$ is a finite dimensional space of regular functions such that $P(T) \supseteq P_k(T)$ for some $k \geq 1$ integer. The mesh parameter h stands for $h = \max_{T \in \mathscr{T}^h} h_T$ where h_T is the diameter of T.

Since we use the topological gradient to create holes (inhomogeneities) during the optimization process, it is possible to start with a shape containing some initial holes or not. A very small penalization is used when solving the direct problem and the adjoint one to avoid the indeterminacy of the rigid motions of the eventual isolated part. Concerning step 4, a new hole of a given radius is created by the simple operation on the level-set function, which can be written on each finite element node x_i

$$\bar{\psi}(x_i) := \max\left(\psi(x_i), \frac{\left(r^2 - \|x_i - c\|^2\right)}{2r}\right), \qquad (76)$$

where $\psi(x)$ is the level-set function, $\bar{\psi}(x_i)$ is its new value, r is the radius of the created hole, and c is its center.

In step 6, the update of the level set is done directly thanks to the shape derivative applying the following evolution equation for the level-set function:

$$\frac{\partial \psi}{\partial t} = g(x), \text{ in } \tilde{\Omega}, \qquad (77)$$

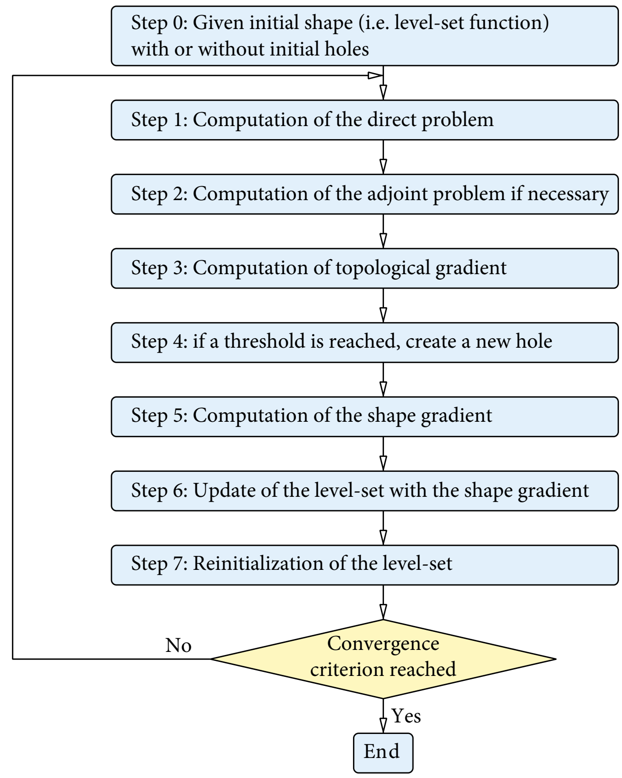

Figure 1: Proposed algorithm.

where $g(x)$ corresponds to the function in front of $\theta.\nu$ in the integral of (24). This evolution equation integrated on a small time interval. In our simulations, the gradient is extended by zero to the complementary of Ω in $\tilde{\Omega}$. However, a smoother extension could be considered. This method is simpler than the classical way which is to integrate a Hamilton-Jacobi equation (see [4]). It seems also to be numerically more robust.

Note that it is convenient to apply a threshold on the gradient to avoid some incoherent values where the shape gradient may have a singularity (corners, transition from Dirichlet to Neumann condition).

To regularize the level-set function, the reinitialization step 7 is considered. It consists classically in solving

$$\begin{cases} \dfrac{\partial \psi}{\partial t} + \text{sign}\ (\psi_0)(|\nabla \psi| - 1) = 0 \text{ in } \tilde{\Omega} \times \mathbb{R}+, \\ \psi(0, x) = \psi_0(x) \text{in } \tilde{\Omega}, \end{cases} \qquad (78)$$

whose stationary solution is a signed distance. This Hamilton-Jacobi equation is known to admit multiple nonsmooth solutions. Classically, a smooth solution is computed thanks to an upwind scheme. Since the fictitious domain $\tilde{\Omega}$ can be a rectangular/parallelepiped domain, it is possible to use a classical upwind scheme on a Cartesian grid. However, to keep the possibility of having a nonstructured mesh, for instance, to proceed to a local refinement, we use a different strategy. Equation (78) is solved on a small time interval

$[0, \Delta t]$ integrating the following equation where the nonlinearity is made explicit:

$$\begin{cases} \dfrac{\partial \bar{\psi}}{\partial t} + \text{sign}\,(\psi_0)\dfrac{\nabla \psi^n}{|\nabla \psi^n|}\nabla \bar{\psi} = \text{sign}\,(\psi_0)\text{in}\,\tilde{\Omega} \times \left]0, \Delta t\right], \\ \bar{\psi}(0, x) = \psi^n(x). \end{cases} \quad (79)$$

Here ψ^n is the level-set function at the previous time step, and ψ^{n+1} is given by $\bar{\psi}(\Delta t, \cdot)$. The problem (79) is a pure convection one. This problem can be solved, for instance, with the simple Galerkin-Characteristic scheme proposed in [11]. This scheme is unconditionally stable but rather dissipative. The effect is that the level sets are a little smoothed.

5. Conclusion and Extensions

In this paper, we used the adjoint method (respectively, generalized adjoint method) to compute shape derivatives (respectively, topological asymptotic expansions) associated with a given shape functional and Maxwell's equations. The obtained derivatives (shape and topological) allow us to construct an algorithm, which can permit simultaneously to insert a small dielectric objects (holes) with topological derivative and to control its boundary by using the shape derivative and level set method.

In the forthcoming work, we will intent to apply this algorithm for some applications, for example, for the reconstruction of metallic buried objects.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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