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Research Article

New Subclass of K-Uniformly Univalent Analytic Functions with Negative Coefficients Defined by Multiplier Transformation

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In this work, we shall derive a new subclass of univalent analytic functions denoted by $H(\alpha, \beta, \gamma, k, \lambda, m)$ in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ which is defined by multiplier transformation. Coefficient inequalities, growth and distortion theorem, extreme points, and radius of starlikeness and convexity of functions belonging to the subclass are obtained.

1. Introduction and Definition

Let A denote the class of all analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \text{ is complex number}, \tag{1}$$

defined on the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let S denote the subclass of A consisting of functions that are univalent in D. Further, let $S^*(\alpha)$ and $C(\alpha)$ be the classes of functions, respectively, starlike of order α and convex of order α, for $0 \le \alpha < 1$. In particular, the classes $S^*(0) = S^*$ and $C(0) = C$ are the familiar classes of starlike and convex functions in D, respectively.

The functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n > 0, \tag{2}$$

defined on the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$, is called a functions with negative coefficients. Let T be the subclass of S, consisting of functions of the form (2). The class T was introduced and studied by Silverman [1]. In [1] Silverman investigated the subclasses of T denoted by $S_T^*(\alpha)$, and $C_T(\alpha)$, for $0 \le \alpha < 1$ that are, respectively, starlike of order α and convex of order α.

Let $M(\alpha, \beta, \gamma)$ be the subclass of A consisting of functions $f(z)$ which satisfy the inequality

$$\left| \frac{z f'(z) - f(z)}{\alpha z f'(z) + (1-\gamma) f(z)} \right| < \beta, \tag{3}$$

where $0 \le \alpha \le 1, 0 < \beta \le 1$, and $0 \le \gamma < 1$ for all $z \in D$. This class of functions was studied by Darus [2].

Cho and Srivastava [3] (see also [4]) introduced the operator I_λ^m as the following:

Definition 1. For $f \in A$ the multiplier transformation I_λ^m is defined by $I_\lambda^m : A \longrightarrow A$

$$I_\lambda^m f(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda} \right)^m a_n z^n, (z \in D), \tag{4}$$

where $-1 < \lambda \le 1$ and $m \in \mathbb{N}_0 = \{0, 1, 2. \cdots\}$.

Special cases of this operator include the Uralegaddi and Somanatha operator in the case $\lambda = 1$ [5], and for $\lambda = 0$, the operator I_λ^m reduces to well-known Salagean operator introduced by Salagean [6].

By making use of the multiplier transformation I_λ^m, the authors derive the new subclass $H(\alpha, \beta, \gamma, k, \lambda, m)$ of functions as the following:

Definition 2. For $0 \le \alpha \le 1, 0 \le \beta < 1, 0 \le \gamma < 1, k \ge 0, -1 < \lambda \le 1$, and $m \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$, we let $H(\alpha, \beta, \gamma, k, \lambda, m)$ consist of functions $f \in T$ satisfying the condition

$$\operatorname{Re}\left(\frac{z(I_\lambda^m f(z))' - (I_\lambda^m f(z))}{\alpha z(I_\lambda^m f(z))' + (1-\gamma)(I_\lambda^m f(z))}\right) > k\left|\frac{z(I_\lambda^m f(z))' - (I_\lambda^m f(z))}{\alpha z(I_\lambda^m f(z))' + (1-\gamma)(I_\lambda^m f(z))} - 1\right| + \beta, \tag{5}$$

where $I_\lambda^m f(z) = z - \sum_{n=2}^{\infty}(n+\lambda/1+\lambda)^m a_n z^n, z \in D$ and $a_n > 0$.

Our first result is the coefficient estimate for functions $f \in H(\alpha, \beta, \gamma, k, \lambda, m)$, and the others include the growth and distortion theorem; further, we obtain the extreme points. Finally we determine the radius of starlikeness and convexity, for the function belonging to the class $H(\alpha, \beta, \gamma, k, \lambda, m)$.

First of all, let us look at the coefficient estimates.

2. Coefficient Estimates

In this section, we shall obtain the coefficient estimates for the function f belonging to the class $H(\alpha, \beta, \gamma, k, \lambda, m)$. Our first result is the following:

Theorem 3. *Let $0 \le \alpha \le 1, 0 \le \beta < 1, 0 \le \gamma < 1, k \ge 0, -1 < \lambda \le 1$, and $m \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$. A function f given by (2) is in the class $H(\alpha, \beta, \gamma, k, \lambda, m)$ if and only if*

$$\sum_{n=2}^{\infty} T_n(\alpha, \beta, \gamma, k, \lambda, m) a_n \le (2+k-\beta)(1+\alpha-\gamma), \tag{6}$$

where

$$T_n(\alpha, \beta, \gamma, k, \lambda, m) = [(1+k)|2-\gamma-n(1-\alpha)| + (1-\beta)(1+\alpha n-\gamma)]\left(\frac{n+\lambda}{1+\lambda}\right)^m. \tag{7}$$

Proof. We have $f \in H(\alpha, \beta, \gamma, k, \lambda, m)$ if and only if the condition (5) is satisfied.

Let

$$w = \frac{z(I_\lambda^m f(z))' - (I_\lambda^m f(z))}{\alpha z(I_\lambda^m f(z))' + (1-\gamma)(I_\lambda^m f(z))}, \tag{8}$$

subject to the condition that,

$$\operatorname{Re}(w) \ge k|w-1| + \beta \text{ if and only if } (k+1)|w-1| \le 1-\beta. \tag{9}$$

Now

$$(k+1)|w-1| = (k+1)\left|\frac{\sum_{n=2}^{\infty}(1-n)(n+\lambda/1+\lambda)^m a_n z^n}{(1+\alpha-\gamma)z - \sum_{n=2}^{\infty}(\alpha n+1-\gamma)(n+\lambda/1+\lambda)^m a_n z^n} - 1\right| \le 1-\beta \tag{10}$$

is equivalent to

$$(k+1)\left|\frac{\sum_{n=2}^{\infty}(1-n)(n+\lambda/1+\lambda)^m a_n z^{n-1}}{(1+\alpha-\gamma) - \sum_{n=2}^{\infty}(\alpha n+1-\gamma)(n+\lambda/1+\lambda)^m a_n z^{n-1}} - 1\right| \le 1-\beta. \tag{11}$$

So

$$(k+1)\left|\frac{\sum_{n=2}^{\infty}(2-\gamma-n(1-\alpha))(n+\lambda/1+\lambda)^m a_n z^{n-1} - (1+\alpha-\gamma)}{(1+\alpha-\gamma) - \sum_{n=2}^{\infty}(\alpha n+1-\gamma)(n+\lambda/1+\lambda)^m a_n z^{n-1}}\right| \le 1-\beta. \tag{12}$$

The above inequality reduces to

$$(k+1)\frac{\sum_{n=2}^{\infty}|2-\gamma-n(1-\alpha)|(n+\lambda/1+\lambda)^m a_n - (1+\alpha-\gamma)}{(1+\alpha-\gamma) - \sum_{n=2}^{\infty}(\alpha n+1-\gamma)(n+\lambda/1+\lambda)^m a_n} \le 1-\beta. \tag{13}$$

Then

$$\begin{aligned}(k+1)&\left[\sum_{n=2}^{\infty}|2-\gamma-n(1-\alpha)|\left(\frac{n+\lambda}{1+\lambda}\right)^m a_n - (1+\alpha-\gamma)\right] \\ &\le (1+\alpha-\gamma)(1-\beta) - (1-\beta)\sum_{n=2}^{\infty}(\alpha n+1-\gamma)\left(\frac{n+\lambda}{1+\lambda}\right)^m a_n.\end{aligned} \tag{14}$$

Thus

$$\begin{aligned}\sum_{n=2}^{\infty}&\left[(k+1)|2-\gamma-n(1-\alpha)| + (1-\beta)\sum_{n=2}^{\infty}(\alpha n+1-\gamma)\right]\left(\frac{n+\lambda}{1+\lambda}\right)^m a_n \\ &\le (1+\alpha-\gamma)(1-\beta) + (1+\alpha-\gamma)(k+1),\end{aligned} \tag{15}$$

which yield to (6).

Conversely suppose that (6) holds and we have to show that (12) holds. Here the inequality (6) is equivalent to (13). So it suffices to show that,

$$\begin{aligned}&\left|\frac{\sum_{n=2}^{\infty}(2-\gamma-n(1-\alpha))((n+\lambda)/(1+\lambda))^m a_n z^{n-1} - (1+\alpha-\gamma)}{1+\alpha-\gamma - \sum_{n=2}^{\infty}(\alpha n+1-\gamma)((n+\lambda)/(1+\lambda))^m a_n z^{n-1}}\right| \\ &\quad \le \frac{\sum_{n=2}^{\infty}|2-\gamma-n(1-\alpha)|((n+\lambda)/(1+\lambda))^m a_n - (1+\alpha-\gamma)}{1+\alpha-\gamma - \sum_{n=2}^{\infty}(\alpha n+1-\gamma)((n+\lambda)/(1+\lambda))^m a_n}.\end{aligned} \tag{16}$$

Since

$$\left|1+\alpha-\gamma-\sum_{n=2}^{\infty}(\alpha n+1-\gamma)\left(\frac{n+\lambda}{1+\lambda}\right)^{m}a_n z^{n-1}\right| \geq |1+\alpha-\gamma| - \left|\sum_{n=2}^{\infty}(\alpha n+1-\gamma)\left(\frac{n+\lambda}{1+\lambda}\right)^{m}a_n z^{n-1}\right|, \tag{17}$$

we have

$$\left|1+\alpha-\gamma-\sum_{n=2}^{\infty}(\alpha n+1-\gamma)\left(\frac{n+\lambda}{1+\lambda}\right)^{m}a_n z^{n-1}\right| \geq 1+\alpha-\gamma-\sum_{n=2}^{\infty}(\alpha n+1-\gamma)\left(\frac{n+\lambda}{1+\lambda}\right)^{m}a_n, \text{ where } |z|<1, \tag{18}$$

and hence, we obtain (16). □

Theorem 4. *Let $0 \leq \alpha \leq 1, 0 \leq \beta < 1, 0 \leq \gamma < 1, k \geq 0, -1 < \lambda \leq 1$, and $m \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$. If the function f given by (2) be in the class $H(\alpha, \beta, \gamma, k, \lambda, m)$, then*

$$a_n \leq \frac{(2+k-\beta)(1+\alpha-\gamma)}{T_n(\alpha, \beta, \gamma, k, \lambda, m)]}, n = 2, 3, 4, \cdots, \tag{19}$$

where $T_n(\alpha, \beta, \gamma, k, \lambda, m)$ is given by (7).

Equality holds for the functions given by,

$$f(z) = z - \frac{(2+k-\beta)(1+\alpha-\gamma)z^n}{T_n(\alpha, \beta, \gamma, k, \lambda, m)}. \tag{20}$$

Proof. Since $f \in H(\alpha, \beta, \gamma, k, \lambda, m)$, therefore, Theorem 3 holds.

Now

$$\sum_{n=2}^{\infty} T_n(\alpha, \beta, \gamma, k, \lambda, m)\, a_n \leq (2+k-\beta)(1+\alpha-\gamma), \tag{21}$$

we have,

$$a_n \leq \frac{(2+k-\beta)(1+\alpha-\gamma)}{T_n(\alpha, \beta, \gamma, k, \lambda, m)}. \tag{22}$$

Clearly the function given by (20) satisfies (19), and, therefore, f given by (20) is in $H(\alpha, \beta, \gamma, k, \lambda, m)$ for this function, and the result is clearly sharp. □

3. Growth and Distortion Theorems for the Subclass $H(\alpha, \beta, \gamma, k, \lambda, m)$

In this section, growth and distortion theorem will be considered, and the covering property for function in the class $H(\alpha, \beta, \gamma, k, \lambda, m)$ will also be given.

Theorem 5. *Let $0 \leq \alpha \leq 1, 0 \leq \beta < 1, 0 \leq \gamma < 1, k \geq 0, -1 < \lambda \leq 1$, and $m \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$. If the function f given by (2) is in the class $H(\alpha, \beta, \gamma, k, \lambda, m)$, then for $0 < |z| = r < 1$, we have*

$$r - \frac{(2+k-\beta)(1+\alpha-\gamma)r^2}{T_2(\alpha, \beta, \gamma, k, \lambda, m)} \leq |f(z)| \leq r + \frac{(2+k-\beta)(1+\alpha-\gamma)r^2}{T_2(\alpha, \beta, \gamma, k, \lambda, m)}, \tag{23}$$

where $T_2(\alpha, \beta, \gamma, k, \lambda, m)$ is given by (7).

Equality holds for the function,

$$f(z) = z - \frac{(2+k-\beta)(1+\alpha-\gamma)z^2}{T_2(\alpha, \beta, \gamma, k, \lambda, m)}, (z = \pm r, \pm ir). \tag{24}$$

Proof. We only prove the right hand side inequality in (23), since the other inequality can be justified using similar arguments.

Since $f \in H(\alpha, \beta, \gamma, k, \lambda, m)$, then by Theorem 3, we have

$$\sum_{n=2}^{\infty} T_n(\alpha, \beta, \gamma, k, \lambda, m) a_n \leq (2+k-\beta)(1+\alpha-\gamma). \tag{25}$$

Now

$$\begin{aligned} T_2(\alpha, \beta, \gamma, k, \lambda, m) \sum_{n=2}^{\infty} a_n &= \sum_{n=2}^{\infty} T_2(\alpha, \beta, \gamma, k, \lambda, m) a_n \\ &\leq \sum_{n=2}^{\infty} T_n(\alpha, \beta, \gamma, k, \lambda, m) a_n \\ &\leq (2+k-\beta)(1+\alpha-\gamma). \end{aligned} \tag{26}$$

And, therefore,

$$\sum_{n=2}^{\infty} a_n \leq \frac{(2+k-\beta)(1+\alpha-\gamma)}{T_2(\alpha, \beta, \gamma, k, \lambda, m)}. \tag{27}$$

Since

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \tag{28}$$

we have,

$$|f(z)| = \left|z - \sum_{n=2}^{\infty} a_n z^n\right| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n |z|^{n-2} \leq r + r^2 \sum_{n=2}^{\infty} a_n. \tag{29}$$

By aid of inequality (27), yields the right hand side inequality of (23). □

Theorem 6. *Let $0 \leq \alpha \leq 1, 0 \leq \beta < 1, 0 \leq \gamma < 1, k \geq 0, -1 < \lambda \leq 1$, and $m \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$. If the function f given by (2) is in the class $H(\alpha, \beta, \gamma, k, \lambda, m)$ for $0 < |z| = r < 1$, then* *we have*

$$1-\frac{2(2+k-\beta)(1+\alpha-\gamma)r}{T_2(\alpha,\beta,\gamma,k,\lambda,m)}\le\left|f'(z)\right|\le 1+\frac{2(2+k-\beta)(1+\alpha-\gamma)r}{T_2(\alpha,\beta,\gamma,k,\lambda,m)},\tag{30}$$

where $T_2(\alpha,\beta,\gamma,k,\lambda,m)$ is given by (7).

Equality holds for the function f given by

$$f(z)=z-\frac{(2+k-\beta)(1+\alpha-\gamma)z^2}{T_2(\alpha,\beta,\gamma,k,\lambda,m)},\ (z=\pm r,\pm ir).\tag{31}$$

Proof. Since $f\in H(\alpha,\beta,\gamma,k,\lambda,m)$, therefore, in Theorem 3, we have

$$\sum_{n=2}^{\infty}T_n(\alpha,\beta,\gamma,k,\lambda,m)a_n\le(2+k-\beta)(1+\alpha-\gamma).\tag{32}$$

Now,

$$\begin{aligned}T_2(\alpha,\beta,\gamma,k,\lambda,m)\sum_{n=2}^{\infty}na_n&\le 2\sum_{n=2}^{\infty}T_n(\alpha,\beta,\gamma,k,\lambda,m)a_n\\&\le 2(2+k-\beta)(1+\alpha-\gamma).\end{aligned}\tag{33}$$

Hence

$$\sum_{n=2}^{\infty}na_n\le\frac{2(2+k-\beta)(1+\alpha-\gamma)}{T_2(\alpha,\beta,\gamma,k,\lambda,m)}.\tag{34}$$

Since

$$f'(z)=1-\sum_{n=2}^{\infty}n\,a_nz^{n-1}.\tag{35}$$

Therefore, we have

$$1-|z|\sum_{n=2}^{\infty}na_n\,|z|^{n-2}\le\left|f'(z)\right|\le 1+|z|\sum_{n=2}^{\infty}na_n|z|^{n-2},\ \text{where}|z|<1.\tag{36}$$

By using inequality (34), we get Theorem 6, and this completes the proof. □

Theorem 7. *Let $0\le\alpha\le 1, 0\le\beta<1, 0\le\gamma<1, k\ge 0, -1<\lambda\le 1$, and $m\in\mathbb{N}_0=\{0,1,2,\cdots\}$. If the function f given by (2) is in the class $H(\alpha,\beta,\gamma,k,\lambda,m)$, then f is starlike of order δ, where*

$$\delta=1-\frac{(2+k-\beta)(1+\alpha-\gamma)}{-(2+k-\beta)(1+\alpha-\gamma)+T_2(\alpha,\beta,\gamma,k,\lambda,m)}.\tag{37}$$

The result is sharp with

$$f(z)=z-\frac{(2+k-\beta)(1+\alpha-\gamma)z^2}{T_2(\alpha,\beta,\gamma,k,\lambda,m)},\tag{38}$$

where $T_2(\alpha,\beta,\gamma,k,\lambda,m)$ is given by (7).

Proof. It is sufficient to show that (6) implies

$$\sum_{n=2}^{\infty}a_n(n-\delta)\le 1-\delta.\tag{39}$$

That is,

$$\frac{n-\delta}{1-\delta}\le\frac{T_n(\alpha,\beta,\gamma,k,\lambda,m)}{(2+k-\beta)(1+\alpha-\gamma)},\ n\ge 2.\tag{40}$$

The above inequality is equivalent to

$$\delta\le 1-\frac{(2+k-\beta)(1+\alpha-\gamma)(n-1)}{-(2+k-\beta)(1+\alpha-\gamma)+T_n(\alpha,\beta,\gamma,k,\lambda,m)}=\psi(n),\tag{41}$$

where $n=2,3,4,\cdots$. And $\psi(n)\ge\psi(2)$, (40) holds true for any $0\le\alpha\le 1, 0\le\beta<1, 0\le\gamma<1, k\ge 0, -1<\lambda\le 1$, and $m\in\mathbb{N}_0=\{0,1,2,\cdots\}$. This completes the proof of Theorem 7. □

4. Extreme Points of the Subclass $H(\alpha,\beta,\gamma,k,\lambda,m)$

The main purpose of this section is to characterize the set of linear homeomorphisms of the extreme points of the closed convex hulls of the subclass $H(\alpha,\beta,\gamma,k,\lambda,m)$, and the extreme points are given by the following theorem.

Theorem 8. *Let $f_1(z)=z$,*

$$f_n(z)=z-\frac{(2+k-\beta)(1+\alpha-\gamma)z^n}{T_n(\alpha,\beta,\gamma,k,\lambda,m)},\ n=2,3,4,\cdots,\tag{42}$$

where $T_n(\alpha,\beta,\gamma,k,\lambda,m)$ is given by (7).

Then $f\in H(\alpha,\beta,\gamma,k,\lambda,m)$ if and only if it can be represented in the form

$$f(z)=\sum_{n=1}^{\infty}\gamma_nf_n(z),\tag{43}$$

where $\gamma_n\ge 0$ and $\sum_{n=1}^{\infty}\gamma_n=1$.

Proof. Suppose f can be expressed as in (43). Our goal is to show that $f\in H(\alpha,\beta,\gamma,k,\lambda,m)$.

By (43) we have

$$f(z)=\sum_{n=1}^{\infty}\gamma_n\left\{z-\frac{(2+k-\beta)(1+\alpha-\gamma)z^n}{T_n(\alpha,\beta,\gamma,k,\lambda,m)}\right\}.\tag{44}$$

Then

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n = z - \sum_{n=2}^{\infty} \frac{(2+k-\beta)(1+\alpha-\gamma)y_n z^n}{T_n(\alpha,\beta,\gamma,k,\lambda,m)}. \tag{45}$$

So that

$$a_n = \frac{(2+k-\beta)(1+\alpha-\gamma)y_n}{T_n(\alpha,\beta,\gamma,k,\lambda,m)}, n \geq 2. \tag{46}$$

Now, we have

$$\sum_{n=2}^{\infty} y_n = 1 - y_1 \leq 1. \tag{47}$$

Setting

$$\sum_{n=2}^{\infty} y_n \frac{(2+k-\beta)(1+\alpha-\gamma)}{T_n(\alpha,\beta,\gamma,k,\lambda,m)} \times \frac{T_n(\alpha,\beta,\gamma,k,\lambda,m)}{(2+k-\beta)(1+\alpha-\gamma)} = \sum_{n=2}^{\infty} y_n = 1 - y_1 \leq 1. \tag{48}$$

It follows from Theorem 3 that the function $f \in H(\alpha, \beta, \gamma, k, \lambda, m)$.

Conversely, it suffices to show that

$$a_n = \frac{(2+k-\beta)(1+\alpha-\gamma)y_n}{T_n(\alpha,\beta,\gamma,k,\lambda,m)}. \tag{49}$$

Now we have $f \in H(\alpha, \beta, \gamma, k, \lambda, m)$, and then by Theorem 4, we have

$$a_n \leq \frac{(2+k-\beta)(1+\alpha-\gamma)}{T_n(\alpha,\beta,\gamma,k,\lambda,m)}, n \geq 2. \tag{50}$$

That is,

$$\frac{T_n(\alpha,\beta,\gamma,k,\lambda,m)a_n}{(2+k-\beta)(1+\alpha-\gamma)} \leq 1, \tag{51}$$

but $y_n \leq 1$. Setting,

$$y_n = \frac{T_n(\alpha,\beta,\gamma,k,\lambda,m)a_n}{(2+k-\beta)(1+\alpha-\gamma)}, n \geq 2. \tag{52}$$

Yields the desired result. This completes the proof of the theorem. □

Corollary 9. *The extreme points of the class $H(\alpha, \beta, \gamma, k, \lambda, m)$ are the function*

$$f_1(z) = z,$$
$$f_n(z) = z - \frac{(2+k-\beta)(1+\alpha-\gamma)z^n}{T_n(\alpha,\beta,\gamma,k,\lambda,m)}, n = 2, 3, 4, \cdots, \tag{53}$$

where $T_n(\alpha, \beta, \gamma, k, \lambda, m)$ is given by (7).

Finally, in this paper, we consider the radius of starlikeness and convexity for the class $H(\alpha, \beta, \gamma, k, \lambda, m)$.

5. Radius of Starlikeness and Convexity

The radius of starlikeness and convexity for the class $H(\alpha, \beta, \gamma, k, \lambda, m)$ is given by the following theorems.

Theorem 10. *If the function f given by (2) is in the class $H(\alpha, \beta, \gamma, k, \lambda, m)$, then f is starlike of order $\delta (0 \leq \delta < 1)$, in the disk $|z| < R$ where*

$$R = \inf \left[\frac{T_n(\alpha,\beta,\gamma,k,\lambda,m)}{(2+k-\beta)(1+\alpha-\gamma)} \left(\frac{1-\delta}{n-\delta} \right) \right]^{1/(n-1)}, n = 2, 3, 4, \cdots, \tag{54}$$

where $T_n(\alpha, \beta, \gamma, k, \lambda, m)$ is given by (7).

Proof. Here (54) implies

$$(2+k-\beta)(1+\alpha-\gamma)(n-\delta)|z|^{n-1} \leq T_n(\alpha,\beta,\gamma,k,\lambda,m)(1-\delta). \tag{55}$$

It suffices to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \delta, \tag{56}$$

for $|z| < R$, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} (n-1) a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}. \tag{57}$$

By aid of (19), we have

$$\left| \frac{zf'}{f} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} \left(\left((n-1)(2+k-\beta)(1+\alpha-\gamma)|z|^{n-1} \right) / \left(T_n(\alpha,\beta,\gamma,k,\lambda,m) \right) \right)}{1 - \sum_{n=2}^{\infty} \left(\left((2+k-\beta)(1+\alpha-\gamma)|z|^{n-1} \right) / \left(T_n(\alpha,\beta,\gamma,k,\lambda,m) \right) \right)}, \tag{58}$$

the last expression is bounded above by $1 - \delta$ if,

$$\sum_{n=2}^{\infty} \frac{(n-1)(2+k-\beta)(1+\alpha-\gamma)|z|^{n-1}}{T_n(\alpha,\beta,\gamma,k,\lambda,m)} \leq \left[1 - \sum_{n=2}^{\infty} \frac{(2+k-\beta)(1+\alpha-\gamma)|z|^{n-1}}{T_n(\alpha,\beta,\gamma,k,\lambda,m)} \right] \times (1-\delta), \tag{59}$$

and it follows that

$$|z|^{n-1} \leq \left[\frac{T_n(\alpha,\beta,\gamma,k,\lambda,m)}{(2+k-\beta)(1+\alpha-\gamma)} \left(\frac{1-\delta}{n-\delta} \right) \right], n \geq 2, \tag{60}$$

which is equivalent to our condition (54) of the theorem. □

Theorem 11. *If the function f given by (2) is in the class $H(\alpha, \beta, \gamma, k, \lambda, m)$, then f is convex of order $\varepsilon (0 \leq \varepsilon < 1)$, in the disk $|z| < w$ where*

$$w = \inf \left[\frac{T_n(\alpha, \beta, \gamma, k, \lambda, m)}{(2 + k - \beta)(1 + \alpha - \gamma)} \left(\frac{1 - \varepsilon}{n(n - \varepsilon)} \right) \right]^{1/n-1}, n = 2, 3, 4, \cdots, \tag{61}$$

where $T_n(\alpha, \beta, \gamma, k, \lambda, m)$ is given by (7).

Proof. Here (61) implies

$$\begin{aligned} &(2 + k - \beta)(1 + \alpha - \gamma)n(n - \varepsilon)|z|^{n-1} \\ &\quad \leq T_n(\alpha, \beta, \gamma, k, \lambda, m)(1 - \varepsilon). \end{aligned} \tag{62}$$

It suffices to show that

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - \varepsilon, \tag{63}$$

for $|z| < w$, we have

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} n a_n |z|^{n-1}}. \tag{64}$$

Then by using the same technique in the proof of Theorem 10, we can show that

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - \varepsilon, \text{for } |z| \leq w, \tag{65}$$

with the aid of (19). Thus we have the assertion of Theorem 11. □

6. Conclusion

The aim object of this article is to introduce a novel new subclass of univalent analytic functions on the open unit disc defined by multiplier transformation. We study their properties, begin by coefficients' characterization. The functions in this class are acquired via this approach; for example, it can provide a number of fascinating features. The coefficient estimates, growth and distortion theorem, extreme points, starlikeness radius, and convexity of functions in the subclass are all introduced.

We remark that several for a wider subclasses of univalent analytic functions can be introduced by using multiplier transformation and studied their coefficients' characterization.

Data Availability

No data were used, available upon request, or included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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