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Characterizations of Pareto-Nash Equilibria for Multiobjective Potential Population Games

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Abstract: This paper studies the characterizations of (weakly) Pareto-Nash equilibria for multiobjective potential population games with a vector-valued potential function called multiobjective potential population games, where agents synchronously maximize multiobjective functions with finite strategies via a partial order on the criteria-function set. In such games, multiobjective payoff functions are equal to the transpose of the Jacobi matrix of its potential function. For multiobjective potential population games, based on Kuhn-Tucker conditions of multiobjective optimization, a strongly (weakly) Kuhn-Tucker state is introduced for its vector-valued potential function and it is proven that each strongly (weakly) Kuhn-Tucker state is one (weakly) Pareto-Nash equilibrium. The converse is obtained for multiobjective potential population games with two strategies by utilizing Tucker's Theorem of the alternative and Motzkin's one of linear systems. Precisely, each (weakly) Pareto-Nash equilibrium is equivalent to a strongly (weakly) Kuhn-Tucker state for multiobjective potential population games with two strategies. These characterizations by a vector-valued approach are more comprehensive than an additive weighted method. Multiobjective potential population games are the extension of population potential games from a single objective to multiobjective cases. These novel results provide a theoretical basis for further computing (weakly) Pareto-Nash equilibria of multiobjective potential population games and their practical applications.

Keywords: multiobjective potential population games; Pareto-Nash equilibria; strongly Kuhn-Tucker states



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1. Introduction

In the current technological, social and economic environments, large populations of small anonymous agents are involved in strategic interactions with multiple objectives. Population games [1] provide a unified framework for describing strategy interactions within a mass of small anonymous agents with a single decision-making objective. To efficiently model interactions in large populations with multiple objectives, multiobjective population games (MPG) are further proposed [2]. However, to our best knowledge, up to now, no published results on agents' incentives to change strategies are available for MPG. Thus, MPG with agents' strategy-switching incentives are of high research and practical importance.

1.1. Related Work

Normal potential games are ones where all information on payoffs of players' incentives to switch strategies can be expressed by a real-valued function called a potential function. Potential games have aroused many researchers' interest because of their attractive properties: the existence and the convergence of Nash equilibria. In particular, on the existence, it is shown that an optimal solution of potential function is a Nash equilibrium of the resulting potential games. Hence the optimization approach is more convenient than a classical fixed-point method to find Nash equilibria for noncooperative games.

For normal form games, Monderer and Shapley define and study potential games and present the crucial existence and convergence for ordinal potential games [3]. The characterizations of Nash equilibria play an important role in the existence of Nash equilibria for potential games. From a mathematical point of view, the relevant researches have been intensively investigated in recent years [4–7]. More precisely, the authors of [4,5] propose best-response potential games and characterize Nash equilibria, respectively, and the author of [6] provides a necessary and sufficient condition of Nash equilibria for certain potential games, and [7] proves that each ordinary symmetric game with two strategies is an ordinal potential one. In practical applications, potential games are applied and widely spread in wireless networks [8–11]. In [8], the resource blocks allocation problem is precisely formulated as exact potential games and the existence of Nash equilibria is proven. Tsiropoulou et al. address the problem of joint customized price and power control in multi-service wireless networks by S-modular theory, which is essentially potential games [9]. The other related applications are referred to in [10,11].

For population games, Sandholm [1,12,13] introduces the idea of potential games to population games with finite strategies. In such a context, potential games are those where the payoff functions are equal to the gradient of a real-valued function termed a potential function. Analogous to the result in [3], Sandholm establishes a key result that optimal solutions of the potential function are Nash equilibria of population potential games. In the context of population games, Lahkar [14] lucubrates large population aggregative potential games and further extends to the case of continuous strategy sets [15]. Similarly, Lahkar shows that any maximizer of a potential function is a Nash equilibrium of the underlying potential games.

However, it is noted that all the payoffs in the existing literature are commonly considered as single-objective ones for both normal and population potential games. Meanwhile, we note that practical applications are in sore need of potential games with multiple objectives; for instance, [8] maximizes its utility function and simultaneously minimizes the total interference in the network, and [9] targets optimizing the cost of resources and satisfaction of quality of service. The analogous situations also exist in population potential games, for example, congestion games [1,15]. Population games with multiple noncommensurable criteria are therefore more suitable to meet practical needs than the scalar case. Hence, we establish the model with vector-valued payoffs called multiobjective population games (MPG) with finite strategies [2,16]. Based on the spirit of (weakly) efficient solutions to vector optimization, we define (weakly) Pareto-Nash equilibria for MPG. The characterizations of (weakly) Pareto-Nash equilibria are closely related to agents' incentives to change strategies and it is fundamental to the existence of (weakly) Pareto-Nash equilibria. To our best knowledge, no relevant results on agents' incentives to change strategies have yet been published for MPG. Consequently, the characterizations of (weakly) Pareto-Nash equilibria are of significance from both theoretical and practical points of view.

1.2. Paper Contributions

In this paper, in order to capture the information on agents' incentives to change strategies in MPG, a vector-valued potential function is introduced to MPG called multiobjective potential population games (MPPG). We focus on the characterizations of (weakly) Pareto-Nash equilibria of MPPG, where agents aim at synchronously maximizing all the objective functions, and their decision-making preference order is a partial one on the so-called criteria-function set. In essence, such a partial order is weaker than the normal real-number order. In other words, the strict optimality does not necessarily exist for the partial order. Thus, MPPG is only based on the efficiency in conventional multiobjective optimization. For this reason, the characterizations of (weakly) Pareto-Nash equilibria of MPPG is of challenge.

To achieve the characterizations of (weakly) Pareto-Nash equilibria for MPPG, an additive weighted method and a vector-valued approach are successively adopted. Firstly,

an additive weighted method essentially converts MPG to single-objective population games for a given weight vector corresponding to all multiobjective functions. In this case, it is shown that for a given weight vector each weighted Nash equilibrium is a (weakly) Pareto-Nash equilibrium, but the converse is not true [16]. This shows that an additive weighted method partly characterizes (weakly) Pareto-Nash equilibria for MPG due to missing some important information in multiobjective payoff functions.

To comprehensively characterize (weakly) Pareto-Nash equilibria for MPPG, a vector-valued approach is secondly utilized, where MPPG keep a multiobjective model. In this context, built on Kuhn-Tucker conditions of multiobjective optimization, a strongly (weakly) Kuhn-Tucker state is introduced for vector-valued potential functions and formulated as a solution of a linear system. On the one hand, it is shown that each strongly (weakly) Kuhn-Tucker state is a (weakly) Pareto-Nash equilibrium for MPPG. On the other hand, by using Tucker’s Theorem of the alternative and Motzkin’s one of linear systems, the converse of the above statement is also true for MPPG with two strategies. Precisely, (weakly) Pareto-Nash equilibria are equivalent to strongly (weakly) Kuhn-Tucker states for those MPPG with two strategies.

As a consequence, the key contribution of this paper is to supply a theoretical gap in MPPG and establish more comprehensive characterizations of (weakly) Pareto-Nash equilibria for MPPG. The results play a cornerstone role in theoretical analysis and future computing (weakly) Pareto-Nash equilibria for MPPG and its practical applications.

1.3. Outline

The outline of the paper is as follows. In Section 2, we review some necessary preliminaries. Section 3 reviews the model of MPG and a (weakly) Pareto-Nash equilibrium and a weighted Nash equilibrium and proposes the model of MPPG. Section 4 is the core of this paper, and it is devoted to the characterizations of Pareto-Nash equilibria of MPPG. Section 5 presents a conclusion and a concise discussion for this paper.

2. Preliminaries

In this whole paper, for each positive integer k , denote $\mathbb{R}_+^k = \{a = (a_1, \dots, a_k) \in \mathbb{R}^k : a_i \geq 0, j = 1, \dots, k\}$, and its interior $\text{int}\mathbb{R}_+^k = \{a = (a_1, \dots, a_k) \in \mathbb{R}^k : a_i > 0, j = 1, \dots, k\}$, respectively. The simplex of \mathbb{R}_+^k is denoted by $\mathbb{T}_+^k = \{a = (a_1, \dots, a_k) \in \mathbb{R}_+^k : \sum_{j=1}^k a_j = 1\}$ and $\text{int}\mathbb{T}_+^k$ denotes the interior of \mathbb{T}_+^k .

For $a = (a_1, \dots, a_k), b = (b_1, \dots, b_k) \in \mathbb{R}^k$, the partial orders are defined as follows
 $a \geq b$ if and only if $a_j \geq b_j, j = 1, 2, \dots, k$;
 $a = b$ if and only if $a_j = b_j, j = 1, 2, \dots, k$;
 $a \geq b$ if and only if $a_j \geq b_j, j = 1, 2, \dots, k$ and $a \neq b$;
 $a > b$ if and only if $a_j > b_j, j = 1, 2, \dots, k$.
 a^T denotes the transposition of $a \in \mathbb{R}^k, a^T b = \sum_{t=1}^k a_t b_t$ is the inner product of a and b , 0 denotes the zero vector in \mathbb{R}^k .

For the readers’ convenience, here is a list of symbols in this paper:

- (MPG): multiobjective population games;
- (MPPG): multiobjective potential population games;
- (VP): multiobjective optimization;
- $F(x)$: payoff functions for multiobjective population games, i.e., $F(x) = (F^1(x); \dots; F^p(x))^T$;
- S^p : population p ’s the strategy set, i.e., $S^p = \{1, \dots, n^p\}$;
- $F^p(x)$: population p ’s payoff functions, i.e., $F^p(x) = (F_1^p(x); \dots; F_{n^p}^p(x))^T$;
- $F_i^p(x)$: population p ’s payoff functions by strategy $i \in S^p$, i.e., $F_i^p(x) = (F_{i1}^p(x), \dots, F_{i n^p}^p(x))$;
- $F_{\cdot t}^p(x)$: population p ’s payoff functions for objective $t \in \{1, 2, \dots, k\}$, i.e., $F_{\cdot t}^p(x) = (F_{1t}^p(x), \dots, F_{n^p t}^p(x))^T$;
- $(F_{\lambda}^p)_i(x)$: population p ’s additive weighted payoff by strategy $i \in S^p$ for a given weight

- vector λ ;
- $PE(F)$: the set of Pareto-Nash equilibria;
- $PE_w(F)$: the set of weakly Pareto-Nash equilibria;
- $E_\lambda(F)$: the set of weighted Nash equilibria of F for a given weight vector λ ;
- $\nabla f_t(x) = (\frac{\partial f_t}{\partial x_1}(x), \dots, \frac{\partial f_t}{\partial x_n}(x))^T$: the gradient of $f_t(x)$;
- $\nabla f(x) = (\nabla f_1(x), \dots, \nabla f_k(x))$: the transpose of Jacobi matrix of vector-valued potential functions $f = (f_1(x), \dots, f_k(x))$;
- $DF_t(x)$: the derivative matrices of $F_t(x)$;
- $S-KT(f)$: the set of all strongly Kuhn-Tucker states;
- $W-KT(f)$: the set of all weakly Kuhn-Tucker states.

Furthermore, the following two theorems of the alternative [17] are well-known in solving linear system problems, they are essential for the characterizations of (weakly) Pareto-Nash equilibria in the context.

Lemma 1. (Tucker’s Theorem of the alternative) Assume that B, C and D are given matrices, with B being nonvacuous. Then

- either (I) $Bx \geq 0, Cx \geq 0$, and $Dx = 0$ has a solution x ;
- or (II)

$$\begin{aligned} B^T y_2 + C^T y_3 + D^T y_4 &= 0 \\ y_2 > 0, y_3 &\geq 0 \end{aligned}$$

has a solution y_2, y_3, y_4 ; but never both.

Lemma 2. (Motzkin’s Theorem of the alternative) Assume that A, B and C are given matrices, with A being nonvacuous. Then

- either (I) $Ax > 0, Bx \geq 0$, and $Cx = 0$ has a solution x ;
- or (II)

$$\begin{aligned} A^T y_1 + B^T y_2 + C^T y_3 &= 0 \\ y_1 \geq 0, y_2 &\geq 0 \end{aligned}$$

has a solution y_1, y_2, y_3 ; but never both.

3. Model of MPPG

In this section, we mainly introduce the model of MPPG, which is based on the model of MPG. Referred to [2,16], we review the model of MPG and (weakly) Pareto-Nash equilibria and weighted Nash equilibria.

3.1. Model of MPG

Let a society $\mathcal{P} = \{1, \dots, P\}$ ($P \geq 1$) is comprised of P populations of agents. For each $p \in \mathcal{P}$, the mass of agents is assumed to be one unit, they share a common set $S^p = \{1, \dots, n^p\}$ with finite pure strategies, and $X^p = \{x^p = (x_1^p, \dots, x_{n^p}^p) \in \mathbb{R}_+^{n^p} : \sum_{i=1}^{n^p} x_i^p = 1\}$ denotes the set of population states, which is an $n^p - 1$ dimensional simplex, and its component x_i^p represents the share distribution of agents choosing strategy $i \in S^p$. Let the total number of pure strategies in the society be $m = \sum_{p \in \mathcal{P}} n^p$, and the set of social states is marked as $X = \prod_{p \in \mathcal{P}} X^p = \{x = (x^1; \dots; x^P) \in \mathbb{R}^m : x^p \in X^p\}$, where the component $x = (x^1; \dots; x^P) \in X$ describes all populations’ behaviors.

Assume that all agents possess k^p objectives in each population $p \in \mathcal{P}$ when playing one strategy. The row vector $F_i^p = (F_{i1}^p, \dots, F_{ik^p}^p) : X \rightarrow \mathbb{R}^{k^p}$ defines payoff functions by a strategy $i \in S^p$, where F_{it}^p represents the t th objective payoff by the strategy $i \in S^p$; in another way, the column vector $F_t^p = (F_{1t}^p, \dots, F_{n^p t}^p)^T : X \rightarrow \mathbb{R}^{n^p}$ expresses the t th objective payoff by all the strategies in S^p ; and $F^p = (F_1^p; \dots; F_{n^p}^p)^T : X \rightarrow \mathbb{R}^{n^p \times k^p}$ totally

describes population p 's multiobjective payoff functions. Let $N = \sum_{p \in \mathcal{P}} n^p k^p$, the map $F = (F^1; \dots; F^P)^T : X \rightarrow \mathbb{R}^N$ represents the multiobjective payoff functions of the whole society. An MPG is denoted by its multiobjective payoff functions F in this paper.

When there is exactly one population, i.e., $P = 1$, the superscript p is omitted from all the notations, so the pure stratege set is $S = \{1, \dots, n\}$, the state set is $X = \{x = (x_1, \dots, x_n) \in \mathbb{R}_+^n : \sum_{i=1}^n x_i = 1\}$, the simplex in \mathbb{R}^n , and the payoff functions are $F : X \rightarrow \mathbb{R}^{n \times k}$.

Definition 1. Let F be an MPG,

(i) a social state $\bar{x} = (\bar{x}^1; \dots; \bar{x}^P) \in X$ is called a Pareto-Nash equilibrium of F if $\forall p \in \mathcal{P}, \forall i \in S^p$,

$$\bar{x}_i^p > 0 \Rightarrow F_i^p(\bar{x}) - F_i^p(\bar{x}) \notin \mathbb{R}_+^{k^p} \setminus \{0\}, \forall l \in S^p.$$

(ii) a social state $\bar{x} = (\bar{x}^1; \dots; \bar{x}^P) \in X$ is called a weakly Pareto-Nash equilibrium of F if $\forall p \in \mathcal{P}, \forall i \in S^p$,

$$\bar{x}_i^p > 0 \Rightarrow F_i^p(\bar{x}) - F_i^p(\bar{x}) \notin \text{int}\mathbb{R}_+^{k^p}, \forall l \in S^p.$$

$PE_w(F)$ and $PE(F)$ denote the set of weakly Pareto-Nash equilibria and that of Pareto-Nash equilibria, respectively.

Obviously, $PE(F) \subseteq PE_w(F)$.

Definition 2. A social state $\bar{x} = (\bar{x}^1; \dots; \bar{x}^P) \in X$ is called a weighted Nash equilibrium of F for a given weight vector $\lambda = (\lambda^1; \dots; \lambda^P)$ satisfying $\lambda^p \in \mathbb{T}_+^{k^p} (\forall p \in \mathcal{P})$ if $\forall p \in \mathcal{P}, \forall i \in S^p$,

$$\bar{x}_i^p > 0 \Rightarrow (F_\lambda^p)_i(\bar{x}) \geq (F_\lambda^p)_l(\bar{x}), \forall l \in S^p,$$

where $(F_\lambda^p)_i(x) = (\lambda^p)^T F_i^p(x) = \sum_{j=1}^{k^p} \lambda_j^p F_{ij}^p(x)$ is the additive weighted payoff by a strategy $i \in S^p$ for a given weight vector λ . $E_\lambda(F)$ denotes the set of weighted Nash equilibria of F for a given weight vector λ .

Clearly, if $k^p = 1$ for each $p \in \mathcal{P}$ for MPG, a (weakly) Pareto-Nash equilibrium or a weighted Nash equilibrium reduces to a Nash equilibrium of population games [1].

3.2. Model of MPPG

Based on the model of MPG and Theorem 5.1 of [18], we now introduce the model of MPPG.

Definition 3. Let $F : X \rightarrow \mathbb{R}^{n \times k}$ be an MPG with a single population and n strategies and k objectives, where $X = \{x = (x_1, \dots, x_n) \in \mathbb{R}_+^n : \sum_{i=1}^n x_i = 1\}$. F is called an MPPG if there is a continuously differentiable vector-valued function $f = (f_1, f_2, \dots, f_k) : X \rightarrow \mathbb{R}^k$ satisfying

$$\nabla f_t(x) = F_t(x), \quad t = 1, 2, \dots, k; \forall x \in X, \tag{1}$$

where $\nabla f_t(x) = (\frac{\partial f_t}{\partial x_1}(x), \dots, \frac{\partial f_t}{\partial x_n}(x))^T$.

Property (1) is precisely stated as

$$\frac{\partial f_t}{\partial x_i}(x) = F_{it}(x), \quad t = 1, 2, \dots, k; i = 1, 2, \dots, n; \forall x \in X.$$

f is called the vector-valued potential function for MPG F .

Apparently, if $k = 1$ in Definition 3, MPPG reduces to population potential games [1].

Remark 1. Denote $\nabla f(x) = (\nabla f_1(x), \dots, \nabla f_k(x))$, then $(\nabla f(x))^T$ is the Jacobian matrix of $f(x)$. By Definition 3 and the model of MPG,

$$\begin{aligned} \nabla f(x) &= (\nabla f_1(x), \dots, \nabla f_k(x)) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_1}(x) & \cdots & \frac{\partial f_k}{\partial x_1}(x) \\ \frac{\partial f_1}{\partial x_2}(x) & \frac{\partial f_2}{\partial x_2}(x) & \cdots & \frac{\partial f_k}{\partial x_2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(x) & \frac{\partial f_2}{\partial x_n}(x) & \cdots & \frac{\partial f_k}{\partial x_n}(x) \end{pmatrix} \\ &= \begin{pmatrix} F_{11}(x) & F_{12}(x) & \cdots & F_{1k}(x) \\ F_{21}(x) & F_{22}(x) & \cdots & F_{2k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ F_{n1}(x) & F_{n2}(x) & \cdots & F_{nk}(x) \end{pmatrix} = F(x), \end{aligned}$$

thus this indicates MPPG F is expressed more concisely as $\nabla f(x) = F(x)$.

The vector-valued potential function’s role can be interpreted as: suppose that $x \in X$ is a social state at which $F_i(x) - F_j(x) \in \text{int}\mathbb{R}_+^k$ (or $\mathbb{R}_+^k \setminus \{0\}$), then an agent prefers choosing strategy i to selecting strategy j . If a small group of agents switch to strategy i from strategy j , these switches are represented by the displacement vector $z = e_i - e_j$, where e_i, e_j are the i th and j th standard basis vectors in \mathbb{R}^n , respectively. The effects that these switches have on the value of potential are therefore

$$(\nabla f(x))^T z = F_i(x) - F_j(x) \in \text{int}\mathbb{R}_+^k \text{ (or } \mathbb{R}_+^k \setminus \{0\} \text{)}.$$

Namely, profitable strategy revisions increase vector-valued potential, which is similar to potential games with a single objective. This fact thus establishes a solid foundation for numerous attractive properties that MPPG possess.

The following characterization of MPPG is easy to obtain.

Theorem 1. Assume that $F : X \rightarrow \mathbb{R}^{n \times k}$ is continuously differentiable. Then F is an MPPG if and only if the derivative matrices $DF_{\cdot t}(x)$ are symmetric for each $t \in \{1, 2, \dots, k\}, x \in X$. More explicitly, F is an MPPG if and only if

$$\frac{\partial F_{it}}{\partial x_j}(x) = \frac{\partial F_{jt}}{\partial x_i}(x), \quad \forall t \in \{1, 2, \dots, k\}, i, j \in S, x \in X.$$

4. Characterizations of Pareto-Nash Equilibria

Section 4, including two subsections, is the core of this paper. It is devoted to the characterizations of (weakly) Pareto-Nash equilibria for MPPG by taking two methods, namely, an additive weighted method and a vector-valued approach, respectively.

4.1. Characterizations of Pareto-Nash Equilibria for MPG

In Section 4.1, by a well-known additive weighted method, MPG is converted to single-objective population games for a given weight vector. The characterizations of (weakly) Pareto-Nash equilibria are further discussed.

Theorem 2. Suppose that $F : X \rightarrow \mathbb{R}^N$ is an MPG, for a given weight vector $\lambda = (\lambda^1; \dots; \lambda^P)$ in which $\lambda^p \in \mathbb{T}_+^{k^p}$ ($p \in \mathcal{P}$),

- (i) if $\lambda^p \in \text{int}\mathbb{T}_+^{k^p}, E_\lambda(F) \subseteq PE(F)$; and
- (ii) if $\lambda^p \in \mathbb{T}_+^{k^p} \setminus \{0\}, E_\lambda(F) \subseteq PE_w(F)$.

Proof. (i) If $\lambda^p \in \text{int}\mathbb{T}_+^{k^p}$ for each $p \in \mathcal{P}$, when $E_\lambda(F) = \emptyset$, it is obviously true; and if $E_\lambda(F) \neq \emptyset$, then for each $\bar{x} \in E_\lambda(F)$, by Definition 2, for any $p \in \mathcal{P}$ and $i \in S^p$ satisfies

$$\bar{x}_i^p > 0 \Rightarrow (F_\lambda^p)_i(\bar{x}) \geq (F_\lambda^p)_l(\bar{x}), \quad \forall l \in S^p.$$

It is below proved that $\bar{x} \in PE(F)$. Proof by contradiction. Assume that $\bar{x} \notin PE(F)$, from Definition 1, then there exists one population $p_0 \in \mathcal{P}$ and one strategy $i_0 \in S^{p_0}$ satisfying $\bar{x}_{i_0}^{p_0} > 0$, but there is another strategy $l_0 \in S^{p_0}$ such that

$$F_{i_0}^{p_0}(\bar{x}) - F_{l_0}^{p_0}(\bar{x}) \in -\mathbb{R}_+^{k^{p_0}} \setminus \{0\}.$$

From $\lambda^{p_0} \in \text{int}\mathbb{T}_+^{k^{p_0}}$, then it holds that

$$(\lambda^{p_0})^T F_{i_0}^{p_0}(\bar{x}) < (\lambda^{p_0})^T F_{l_0}^{p_0}(\bar{x}),$$

namely,

$$(F_\lambda^{p_0})_{i_0}(\bar{x}) < (F_\lambda^{p_0})_{l_0}(\bar{x}),$$

which implies that $\bar{x} \notin E_\lambda(F)$. However, this contradicts the fact $\bar{x} \in E_\lambda(F)$. So $\bar{x} \in PE(F)$, hence $E_\lambda(F) \subseteq PE(F)$ for $\lambda^p \in \mathbb{T}_+^{k^p} \setminus \{0\}$ ($\forall p \in \mathcal{P}$).

(i) Analogous to the above proof, it also holds that $E_\lambda(F) \subseteq PE_w(F)$ for $\lambda^p \in \text{int}\mathbb{T}_+^{k^p}$ ($\forall p \in \mathcal{P}$). \square

Example 1. Consider the bi-objective population game $\tilde{F} = (\tilde{F}_1; \tilde{F}_2)^T$ played by a single population with two strategies. For each state $x \in X = \{x = (x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 = 1\}$,

$$\begin{aligned} \tilde{F}_1(x) &= (\tilde{F}_{11}(x), \tilde{F}_{12}(x)) = (x_1, 2), \\ \tilde{F}_2(x) &= (\tilde{F}_{21}(x), \tilde{F}_{22}(x)) = (1 + x_1, x_2). \end{aligned}$$

Obviously, $PE(F) = PE_w(F) = X$.

For a given weight vector $\lambda = (\lambda_1, \lambda_2) = (1/3, 2/3) \in \text{int}\mathbb{T}_+^2$, the resulting weighted payoffs are below

$$\begin{aligned} (\tilde{F}_\lambda)_1(x) &= \lambda_1 \tilde{F}_{11}(x) + \lambda_2 \tilde{F}_{12}(x) = x_1/3 + 4/3, \\ (\tilde{F}_\lambda)_2(x) &= \lambda_1 \tilde{F}_{21}(x) + \lambda_2 \tilde{F}_{22}(x) = (1 + x_1)/3 + 2x_2/3 = -x_1/3 + 1. \end{aligned}$$

From Definition 2, it deduces that $(1, 0) \in X$ is a unique weighted Nash equilibrium of \tilde{F} for $\lambda = (\lambda_1, \lambda_2) = (1/3, 2/3)$. Therefore, $E_\lambda(\tilde{F}) = \{(1, 0)\} \subset X = PE_w(F) = PE(\tilde{F})$.

Both Theorem 2 and Example 1 jointly represent that the additive weighted method just partly characterizes (weakly) Pareto-Nash equilibria for MPG.

4.2. Characterizations of Pareto-Nash Equilibria for MPPG

In this subsection, for MPPG, more comprehensive characterizations of (weakly) Pareto-Nash equilibria are studied by a vector-valued approach, in which MPPG keep in a multiobjective model. In this situation, Kuhn-Tucker conditions of multiobjective optimization is fundamental.

We note in Section 3.2 that increases in vector-valued potential are generated by all profitable strategy revisions. This shows that (weakly) Pareto-Nash equilibria of MPPG are related with (weakly) Pareto efficient solutions of its vector-valued potential function.

Let $F : X \rightarrow \mathbb{R}^{n \times k}$ be an MPPG with a vector-valued potential function $f : X \rightarrow \mathbb{R}^k$. The resulting multiobjective optimization problem of $f(x)$ is as follows

$$(VP) \quad \begin{aligned} \max \quad & f(x) = (f_1(x), f_2(x), \dots, f_k(x)) \\ \text{s. t.} \quad & g_i(x) = x_i \geq 0, \quad i = 1, 2, \dots, n; \\ & h(x) = 1 - \sum_{i=1}^n x_i = 0. \end{aligned}$$

If $\bar{x} \in X$, $\bar{\omega} \in \mathbb{R}^k$, $\bar{\mu} \in \mathbb{R}_+^n$ and $\bar{v} \in \mathbb{R}$ satisfy the following system

$$\bar{\omega}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}) + \bar{v} \nabla h(\bar{x}) = \mathbf{0}, \tag{2}$$

$$\bar{\mu}^T g(\bar{x}) = 0. \tag{3}$$

Since $\nabla g(x) = (\nabla g_1(x), \nabla g_2(x), \dots, \nabla g_n(x)) \in \mathbb{R}^{n \times n}$ is an identity matrix and thus $\bar{\mu}^T \nabla g(\bar{x}) = \bar{\mu} \in \mathbb{R}^n$, and $\nabla h(x) = (-1, -1, \dots, -1)^T \in \mathbb{R}^n$, and from Remark 1, hence (2) and (3) are rewritten as

$$\bar{\omega}^T E_i(\bar{x}) + \bar{\mu}_i - \bar{v} = 0, \quad i = 1, 2, \dots, n; \tag{4}$$

$$\bar{\mu}_i \bar{x}_i = 0, \quad i = 1, 2, \dots, n. \tag{5}$$

Definition 4. Assume that $F : X \rightarrow \mathbb{R}^{n \times k}$ is an MPPG with a vector-valued potential function $f : X \rightarrow \mathbb{R}^k$. For $\bar{x} \in X$, $\bar{\omega} \in \mathbb{R}^k$, $\bar{\mu} \in \mathbb{R}_+^n$ and $\bar{v} \in \mathbb{R}$ satisfying (2) and (3),

(i) if $\bar{\omega} \in \text{int}\mathbb{R}_+^k$, then \bar{x} is called a strongly Kuhn-Tucker state of f ; and $S\text{-KT}(f)$ denotes the set of all strongly Kuhn-Tucker states;

(ii) if $\bar{\omega} \in \mathbb{R}_+^k \setminus \{\mathbf{0}\}$, then \bar{x} is called a weakly Kuhn-Tucker state of f ; and $W\text{-KT}(f)$ denotes the set of all weakly Kuhn-Tucker states.

For a potential function $f = (f_1, f_2, \dots, f_k)$, a strongly Kuhn-Tucker state is obviously a weakly Kuhn-Tucker state, i.e., $S\text{-KT}(f) \subseteq W\text{-KT}(f)$.

The following theorem characterizes (weakly) Pareto-Nash equilibria of F based on the Kuhn-Tucker first-order conditions of vector-valued optimization problem (VP).

Theorem 3. Assume that $F : X \rightarrow \mathbb{R}^{n \times k}$ is an MPPG with a vector-valued potential function $f : X \rightarrow \mathbb{R}^k$, and let f be first-order continuously differentiable at $\bar{x} \in X$.

(i) If $\bar{x} \in S\text{-KT}(f)$ for some $\bar{\omega} \in \text{int}\mathbb{R}_+^k$, $\bar{\mu} \in \mathbb{R}_+^n$ and $\bar{v} \in \mathbb{R}$, then $\bar{x} \in PE(F)$;

(ii) If $\bar{x} \in W\text{-KT}(f)$ for some $\bar{\omega} \in \mathbb{R}_+^k \setminus \{\mathbf{0}\}$, $\bar{\mu} \in \mathbb{R}_+^n$ and $\bar{v} \in \mathbb{R}$, then $\bar{x} \in PE_w(F)$.

Proof. (i) If $\bar{x} \in S\text{-KT}(f)$ for some $\bar{\omega} \in \text{int}\mathbb{R}_+^k$, $\bar{\mu} \in \mathbb{R}_+^n$ and $\bar{v} \in \mathbb{R}$, we are required to prove $\bar{x} \in PE(F)$. We show it by contradiction. Assume that there exists a certain strategy $i_0 \in \{1, 2, \dots, n\}$ with $\bar{x}_{i_0} > 0$ and another strategy $l_0 \in \{1, 2, \dots, n\}$ such that

$$F_{l_0}(\bar{x}) - F_{i_0}(\bar{x}) \in \mathbb{R}_+^k \setminus \{\mathbf{0}\}. \tag{6}$$

By (5), it holds $\bar{\mu}_{i_0} = 0$. Due to $\bar{\omega} \in \text{int}\mathbb{R}_+^k$, from (4) and (6) it implies that

$$\bar{\mu}_{l_0} = \bar{\omega}^T (F_{i_0}(\bar{x}) - F_{l_0}(\bar{x})) < 0,$$

which is inconsistent with the fact $\bar{\mu} \in \mathbb{R}_+^n$, thus $\bar{x} \in PE(F)$.

(ii) If $\bar{x} \in W\text{-}KT(f)$ for some $\bar{\omega} \in \mathbb{R}_+^k \setminus \{0\}$, $\bar{\mu} \in \mathbb{R}_+^n$ and $\bar{v} \in \mathbb{R}$, it needs to show $\bar{x} \in PE_w(F)$. We also prove this fact by contradiction. Suppose that there is a certain strategy $i_0 \in \{1, 2, \dots, n\}$ with $\bar{x}_{i_0} > 0$ and another strategy $l_0 \in \{1, 2, \dots, n\}$ such that

$$F_{l_0}(\bar{x}) - F_{i_0}(\bar{x}) \in \text{int}\mathbb{R}_+^k. \tag{7}$$

By (5), it holds $\bar{\mu}_{i_0} = 0$. Because of $\bar{\omega} \in \mathbb{R}_+^k \setminus \{0\}$, from (4) and (7) we deduce

$$\bar{\mu}_{l_0} = \bar{\omega}^T (F_{i_0}(\bar{x}) - F_{l_0}(\bar{x})) < 0,$$

which contradicts the fact $\bar{\mu} \in \mathbb{R}_+^n$, thus $\bar{x} \in PE_w(F)$. \square

Example 2. Consider the following MPPG

$$F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix} = \begin{pmatrix} 3(x_1 - \frac{1}{2})^2 & -(6 + 20x_1 + 30x_1^2) \\ 0 & -(6 + 20x_2 + 30x_2^2) \end{pmatrix},$$

the state set is $X = \{x = (x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 = 1\}$ and its vector-valued potential function is

$$f(x) = (f_1(x), f_2(x)) = \left((x_1 - \frac{1}{2})^3, -6x_1 - 6x_2 - 10(x_1^2 + x_1^3 + x_2^2 + x_2^3) \right).$$

According to (2) to (5):

for $\bar{x} = (1, 0) \in X$, there exists $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2) = (200, 1)$, $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2) = (0, 100)$, $\bar{v} = 94$ such that $\bar{x} \in S\text{-}KT(f)$; moreover, there is also $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2) = (4, 0)$, $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2) = (0, 3)$, $\bar{v} = 3$ such that $\bar{x} \in W\text{-}KT(f)$. By Theorem 3, it holds $\bar{x} = (1, 0) \in PE(F) \subseteq PE_w(F)$.

Furthermore, for $\bar{x} = (\bar{x}_1, \bar{x}_2) \in X$ with $\frac{1}{2} \leq \bar{x}_1 < 1$ and $0 < \bar{x}_2 \leq \frac{1}{2}$, there is $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2) = (0, 0)$ and for each $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2) \in \text{int}\mathbb{R}_+^2$ and $\bar{v} \in \mathbb{R}$ such that $\bar{x} \in S\text{-}KT(f) \subseteq W\text{-}KT(f)$, and by Theorem 3, it holds $\bar{x} = (\bar{x}_1, \bar{x}_2) \in PE(F) \subseteq PE_w(F)$.

As a result, from Theorem 3 we obtain

$$PE(F) = PE_w(F) = \{ \bar{x} = (\bar{x}_1, \bar{x}_2) : \frac{1}{2} \leq \bar{x}_1 \leq 1, 0 \leq \bar{x}_2 \leq \frac{1}{2} \} \subset X. \tag{8}$$

However, for any $x = (x_1, x_2) \in X \setminus PE(F)$ (or $X \setminus PE_w(F)$) (where $0 \leq x_1 < \frac{1}{2}, \frac{1}{2} < x_2 \leq 1$), although $\bar{\mu} = (0, 0) \in \mathbb{R}_+^2$, there is neither $\bar{\omega} \in \text{int}\mathbb{R}_+^2$ nor $\bar{\omega} \in \mathbb{R}_+^2 \setminus \{0\}$ nor $\bar{v} \in \mathbb{R}$ such that $x \in S\text{-}KT(f)$ (or $W\text{-}KT(f)$).

Furthermore, Example 2 suggests that $PE(F) = S\text{-}KT(f)$ and $PE_w(F) = W\text{-}KT(f)$ holds for an MPPG with two strategies. Thus, we next derive a necessary condition of (weakly) Pareto-Nash equilibria for an MPPG with two strategies.

Theorem 4. Suppose that $F : X \rightarrow \mathbb{R}^{2 \times k}$ is an MPPG with its vector-valued potential function $f : X \rightarrow \mathbb{R}^k$, and let f be continuously differentiable at $\bar{x} \in X$.

- (i) If $\bar{x} \in PE(F)$, then there exists $\bar{\omega} \in \text{int}\mathbb{R}_+^k$, $\bar{\mu} \in \mathbb{R}_+^2$ and $\bar{v} \in \mathbb{R}$ such that $\bar{x} \in S\text{-}KT(f)$.
- (ii) If $\bar{x} \in PE_w(F)$, then there exists $\bar{\omega} \in \mathbb{R}_+^k \setminus \{0\}$, $\bar{\mu} \in \mathbb{R}_+^2$ and $\bar{v} \in \mathbb{R}$ such that $\bar{x} \in W\text{-}KT(f)$.

Proof. (i) Assume that $\bar{x} \in PE(F)$, we are required to prove that there exists $\bar{\omega} \in \text{int}\mathbb{R}_+^k$, $\bar{\mu} \in \mathbb{R}_+^2$ and $\bar{v} \in \mathbb{R}$ such that $\bar{x} \in S\text{-}KT(f)$. We firstly show that the following linear system (I):

$$\nabla f_t(\bar{x})^T q = F_{.t}(\bar{x})^T q \geq 0, \quad t = 1, 2, \dots, k, \tag{9}$$

$$\nabla f_{t_0}(\bar{x})^T q = F_{.t_0}(\bar{x})^T q > 0, \quad \exists t_0 \in \{1, 2, \dots, k\}, \tag{10}$$

$$\nabla g_j(\bar{x})^T q \geq 0, \quad j \in J(\bar{x}) = \{j \in \{1, 2\} : g_j(\bar{x}) = 0\}, \tag{11}$$

$$\nabla h(\bar{x})^T q = 0. \tag{12}$$

has no solution $q \in \mathbb{R}^2$. Otherwise, if (I) has a solution $q = (q_1, q_2) \in \mathbb{R}^2$, then by (12), it holds that

$$q_2 = -q_1. \tag{13}$$

By (10), we further obtain

$$q_1 \neq 0, q_2 \neq 0. \tag{14}$$

Next, our proof is divided into the following two cases.

Case (a): If $\bar{x} = (\bar{x}_1, \bar{x}_2) = (1, 0) \in PE(F)$, here $g_1(\bar{x}) = \bar{x}_1 = 1, g_2(\bar{x}) = \bar{x}_2 = 0$, so $J(\bar{x}) = \{2\}$ and $\nabla g_2(\bar{x}) = (0, 1)^T$, and by (11), $q_2 \geq 0$; from (13) and (14), then $q_2 > 0$ and $q_1 < 0$; furthermore, (9) and (10) jointly imply $F_{1t}(\bar{x}) \leq F_{2t}(\bar{x})$ for any $t \in \{1, 2, \dots, k\}$ and $F_{1t_0}(\bar{x}) < F_{2t_0}(\bar{x})$ for some $t_0 \in \{1, 2, \dots, k\}$. Consequently,

$$F_2(\bar{x}) - F_1(\bar{x}) \in \mathbb{R}_+^k \setminus \{0\},$$

this however contradicts the above assumption $\bar{x} = (1, 0) \in PE(F)$. Hence the abovementioned linear system (I) possesses no solution in \mathbb{R}^2 . By Tucker’s Theorem of the alternative (Lemma 1), there exists $\bar{\omega} \in \text{int}\mathbb{R}_+^k$, $\bar{\mu}_2 \geq 0$ and $\bar{v} \in \mathbb{R}$ such that

$$\bar{\omega}^T \nabla f(\bar{x}) + \bar{\mu}_2 \nabla g_2(\bar{x}) + \bar{v} \nabla h(\bar{x}) = 0,$$

and since $g_1(\bar{x}) = \bar{x}_1 = 1$, we take $\bar{\mu}_1 = 0$, thereby $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2) = (0, \bar{\mu}_2) \in \mathbb{R}_+^2$ such that both (2) and (3) hold true, therefore $\bar{x} \in S\text{-}KT(f)$.

By the same way, if $\bar{x} = (0, 1) \in PE(F)$, it is also $\bar{x} \in S\text{-}KT(f)$.

Case (b): if $\bar{x} = (\bar{x}_1, \bar{x}_2) \in PE(F) \subset \text{int } X$, here $g_1(\bar{x}) = \bar{x}_1 > 0, g_2(\bar{x}) = \bar{x}_2 > 0$, so $J(\bar{x}) = \emptyset$. We similarly derive (13) and (14) from (10) and (12).

When $q_1 > 0$, from (9) and (10) we deduce

$$F_1(\bar{x}) - F_2(\bar{x}) \in \mathbb{R}_+^k \setminus \{0\},$$

however, it contradicts $\bar{x} = (\bar{x}_1, \bar{x}_2) \in PE(F) \subset \text{int } X$.

When $q_1 < 0$, from (9) and (10), we similarly deduce

$$F_2(\bar{x}) - F_1(\bar{x}) \in \mathbb{R}_+^k \setminus \{0\},$$

which also contradicts $\bar{x} = (\bar{x}_1, \bar{x}_2) \in PE(F) \subset \text{int } X$.

As a result, the forementioned linear system (I) has no solution in \mathbb{R}^2 . By Tucker’s Theorem of the alternative (Lemma 1), there exists $\bar{\omega} \in \text{int}\mathbb{R}_+^k$, $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2) = 0 \in \mathbb{R}_+^2$ and $\bar{v} \in \mathbb{R}$ such that (2) and (3) hold true, therefore $\bar{x} \in S\text{-}KT(f)$.

(ii) Let $\bar{x} \in PE_w(F)$. We need show that there exists $\bar{\omega} \in \mathbb{R}_+^k \setminus \{0\}$, $\bar{\mu} \in \mathbb{R}_+^2$ and $\bar{v} \in \mathbb{R}$ such that $\bar{x} \in W-KT(f)$. Following, we show that the next linear system (II):

$$\nabla f_t(\bar{x})^T q = F_{.t}(\bar{x})^T q > 0, \quad t = 1, 2, \dots, k, \tag{15}$$

$$\nabla g_j(\bar{x})^T q \geq 0, \quad j \in J(\bar{x}) = \{j \in \{1, 2\} : g_j(\bar{x}) = 0\}, \tag{16}$$

$$\nabla h(\bar{x})^T q = 0 \tag{17}$$

has no solution $q \in \mathbb{R}^2$. Otherwise, if (II) has a solution $q = (q_1, q_2) \in \mathbb{R}^2$, then both (13) and (14) stay true from (15) and (17).

Following, we prove it in two cases as follows.

Case (c): when $\bar{x} = (\bar{x}_1, \bar{x}_2) = (1, 0) \in PE_w(F)$, here $g_1(\bar{x}) = \bar{x}_1 = 1, g_2(\bar{x}) = \bar{x}_2 = 0$, hence $J(\bar{x}) = \{2\}$ and $\nabla g_2(\bar{x}) = (0, 1)^T$. From (16), $q_2 \geq 0$; it is further deduced $q_2 > 0$ and $q_1 < 0$ from (13) and (14). By (15), obviously $F_{2t}(\bar{x}) > F_{1t}(\bar{x})$ for each $t = 1, 2, \dots, k$, thus

$$F_2(\bar{x}) - F_1(\bar{x}) \in \text{int}\mathbb{R}_+^k,$$

which contradicts the assumption $\bar{x} = (1, 0) \in PE_w(F)$. Hence the system (II) has no solution in \mathbb{R}^2 . By Motzkin’s Theorem of the alternative (Lemma 2), there exists $\bar{\omega} \in \mathbb{R}_+^k \setminus \{0\}$, $\bar{\mu}_2 \geq 0$ and $\bar{v} \in \mathbb{R}$ such that

$$\bar{\omega}^T \nabla f(\bar{x}) + \bar{\mu}_2 \nabla g_2(\bar{x}) + \bar{v} \nabla h(\bar{x}) = 0,$$

and since $g_1(\bar{x}) = \bar{x}_1 = 1$, we take $\bar{\mu}_1 = 0$, thereby $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2) = (0, \bar{\mu}_2) \in \mathbb{R}_+^2$ satisfies both (2) and (3), therefore $\bar{x} \in W-KT(f)$.

Likewise, if $\bar{x} = (0, 1) \in PE_w(F)$, then $\bar{x} \in W-KT(f)$.

Case (d): if $\bar{x} = (\bar{x}_1, \bar{x}_2) \in PE_w(F) \subset \text{int } X$, here $g_1(\bar{x}) = \bar{x}_1 > 0, g_2(\bar{x}) = \bar{x}_2 > 0$, then $J(\bar{x}) = \emptyset$. Both (13) and (14) hold true from (15) and (17).

When $q_1 > 0$, from (15) we obtain

$$F_1(\bar{x}) - F_2(\bar{x}) \in \text{int}\mathbb{R}_+^k,$$

this contradicts $\bar{x} = (\bar{x}_1, \bar{x}_2) \in PE_w(F) \subset \text{int } X$.

When $q_1 < 0$, from (15) we analogously deduce

$$F_2(\bar{x}) - F_1(\bar{x}) \in \text{int}\mathbb{R}_+^k,$$

which also contradicts $\bar{x} = (\bar{x}_1, \bar{x}_2) \in PE_w(F) \subset \text{int } X$.

Consequently, linear system (II) has no solution in \mathbb{R}^2 . By Motzkin’s Theorem of the alternative (Lemma 2), there exists $\bar{\omega} \in \mathbb{R}_+^k \setminus \{0\}$, $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2) = 0 \in \mathbb{R}_+^2$ and $\bar{v} \in \mathbb{R}$ such that (2) and (3) hold true, therefore $\bar{x} \in W-KT(f)$. \square

Due to Theorem 3 and Theorem 4, the next Corollary 1 makes the characterizations of (weakly) Pareto-Nash equilibria accessible to MPPG with two strategies.

Corollary 1. *Let $F : X \rightarrow \mathbb{R}^{2 \times k}$ be an MPPG, and let its potential function $f : X \rightarrow \mathbb{R}^k$ be continuously differentiable at $\bar{x} \in X$. Then $PE(F) = S-KT(f)$, $PE_w(F) = W-KT(f)$.*

Remark 2. *In particular, if $k = 1$ in Definition 4, a strongly (or weakly) Kuhn-Tucker state of f is simplified as a Kuhn-Tucker state of a scalar potential function. Theorem 3.1.3 of [1] points out that a Kuhn-Tucker state of a scalar potential function coincides with Nash equilibria of population potential games. Thus, Corollary 1 extends Theorem 3.1.3 of [1] from the scalar case to the vector-valued one.*

Therefore, for the characterizations of (weakly) Pareto-Nash equilibria of MPPG, Theorems 3 and 4 by a vector-valued approach are more comprehensive than the additive weighted method (Theorem 2). Theorems 2–4 and Corollary 1 are all novel for MPPG.

5. Conclusions

To meet the urgent need in practical applications, this paper studies MPPG, in which agents devote to synchronously maximizing multiobjective payoff functions with finite strategies via a weaker partial order on the criteria-function set. Thus, characterizations of (weakly) Pareto-Nash equilibria for MPPG are a challenge. It proves one kind of comprehensive characterizations for (weakly) Pareto-Nash equilibria of MPPG, which supply the theoretical gap in MPPG. These results are also practically significant for MPPG.

Firstly, an additive weighted method is adopted to characterize (weakly) Pareto-Nash equilibria for MPG, in which MPG is converted to single-objective population games. Each weighted Nash equilibrium is proven to be a Pareto-Nash equilibrium and a weak one for a given weight vector. This shows an additive weighted method partly characterizes (weakly) Pareto-Nash equilibria for MPG because of missing some important information within multiobjective payoff functions.

Moreover, to comprehensively characterize (weakly) Pareto-Nash equilibria for MPPG, they remain in multiobjective cases learned from the above additive weighted method. Thus, a vector-valued approach plays a crucial role in this situation. Based on Kuhn-Tucker conditions of multiobjective optimization, a strongly (weakly) Kuhn-Tucker state is introduced for vector-valued potential functions and it is formulated as a solution of a linear system. On the one hand, each strongly (weakly) Kuhn-Tucker state is shown to be a (weakly) Pareto-Nash equilibrium for MPPG. On the other hand, by using Tucker's Theorem of the alternative and Motzkin's one of linear systems, the converse of the above statement also holds true for MPPG with two strategies. Concisely, a (weakly) Pareto-Nash equilibrium is equivalent to a strongly (weakly) Kuhn-Tucker state for MPPG with two strategies.

The results make clear that this paper establishes a more comprehensive characterization of (weakly) Pareto-Nash equilibria of MPPG. This paper extends population potential games with a single objective [1] to multiobjective cases. All the results are novel and provide a theoretical basis for further computing (weakly) Pareto-Nash equilibria of MPPG.

Finally, our future work is devoted to computing (weakly) Pareto-Nash equilibria of MPPG with numerical results and its practical applications, where it is vital how to formulate practical problems as MPPG.

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