mathematics

MDPI

Article

Fixed Point Theory Using ψ Contractive Mapping in C^*-Algebra Valued B-Metric Space

Rahmah Mustafa [1,*], Saleh Omran [2] and Quang Ngoc Nguyen [3,*]

[1] Department of Mathematics, Faculty of Science, AL-Baha University, Alaqiq 65779-77388, Saudi Arabia
[2] Department of Mathematics, South Valley University, Safaga Road, Km 6 Qena, Qena 83523, Egypt; salehomran@yahoo.com
[3] Department of Communications and Computer Engineering, School of Fundamental Science and Engineering, Waseda University, Shinjuku-ku, Tokyo 169-0051, Japan
* Correspondence: rmalhasani@bu.edu.sa (R.M.); quang.nguyen@aoni.waseda.jp (Q.N.N.)

Abstract: In this paper, fixed point theorems using ψ contractive mapping in C^*-algebra valued b-metric space are introduced. By stating multiple scenarios that illustrate the application domains, we demonstrate several applications from the obtained results. In particular, we begin with the definition of the positive function and then recall some properties of the function that lay the fundamental basis for the research. We then study some fixed point theorems in the C^*-algebra valued b-metric space using a positive function.

Keywords: C*-algebra valued; b-metric spaces; fixed point theory; common Banach fixed point; contractive mapping

check for updates

Citation: Mustafa, R.; Omran, S.; Nguyen, Q.N. Fixed Point Theory Using ψ Contractive Mapping in C^*-Algebra Valued B-Metric Space. *Mathematics* **2021**, *9*, 92. https://doi.org/10.3390/math9010092

Received: 20 December 2020
Accepted: 2 January 2021
Published: 4 January 2021

1. Introduction

Ma et al. [1] introduced the concept of C^*-algebra-valued metric space and studied some fixed point theorems for the self-mapping with certain contractive conditions. In addition, the notion of C^*-algebra-valued metric space is generalized to that of C^*-algebra-valued b-metric space, where b is an element of C^*-algebra greater than 1 and the triangle inequality is modified into $\mathrm{d}(\mathrm{x},\mathrm{y}) \leq \mathrm{b}\ (\ \mathrm{d}(\mathrm{x},\mathrm{z}) + \mathrm{d}(\mathrm{z},\mathrm{y}))$. Then, various fixed point theorems are obtained for self-map with contractive condition [2]. Besides, though Alsulami et al. [3] investigated that fixed point results in C^*-algebra-valued metric space can be obtained by using the classical Banach fixed point theorems, C^*-algebra valued b–metric space is still an interesting and challenging topic with promising applications, which has not received sufficient consideration in fixed point theory [4,5]. The metric space has been found in many applications in computer science, major data and neural networks [6,7].

Along this line, in this paper, we recall the concept of C^*-algebra-valued b-metric space with new and practical insights and prove that Banach fixed point is a suitable mapping under various contractive conditions. This is critical since C^*-algebra is a crucial subject in functional analysis and operator theory, which plays a key role in noncommutative geometry and theoretical physics, e.g., quantum mechanics and string theory [8–10].

Throughout this paper, we denote A as an unital C^*-algebra, and $A_h = \{a \in A : a = a^*\}$. Specifically, an element $a \in A$ is a positive element, if $a = a^* and\ \sigma(a) \subseteq R^+$, where $\sigma(a)$ is the spectrum of a. There is a natural partial order on A_h given by $a \leq b\ if\ f\ \theta \leq (b-a)$, where θ means the zero element in A. Then, let $A^+\ and\ A'$ denote the set $\{a \in A\ : \theta \leq a\}$ and the set $\{a \in A\ : ab = ba,\ for\ all\ b \in A\}$, respectively and $|a| = (a^*a)^{\frac{1}{2}}$.

Definition 1. *Let X be a nonempty set and $b \in A$ such that $b \geq\ I$. Supposethat the mapping $d : X \times X \longrightarrow A$ is held, the following constraints exist [2].*
(1) $\theta \leq d(x,y)$ and $d(x,y) = \theta$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$;

(3) $d(x,y) \leq b(d(x,z)+d(z,y))$ *for all* $x,y,z \in X$.

Then, d is called C^**-algebra valued b-metric on X and* (X,A,d) *is called* C^**-algebra-valued b-metric space.*

Definition 2. *Let* (X,A,d) *be* C^**-algebra b- valued metric space. Suppose that* $\{x_n\}$ *is a sequence in X and* $x \in X$*. If for any* $\varepsilon > 0$*, there exists N such that for all* $n > N$, $\|d(x_n,x)\| \leq \varepsilon$ *then* $\{x_n\}$ *is said to be convergent with respect to A, and* $\{x_n\}$ *converges to x, i.e., we take* $\lim\limits_{n\to\infty} x_n = x$*. If for any* $\varepsilon > 0$*, there exists N such that for all* $n,m > N$, $\|d(x_n,x_m)\| \leq \varepsilon$, *then* $\{x_n\}$ *is called a Cauchy sequence in X* [2].

(X,A,d) *is called a complete* C^**-algebra-valued b-metric space if every Cauchy sequence is convergent in X.*

Example 1. *Let* $\mathbb{C}$ *be a* C^**-algebra. Then,* $\mathbb{C}$ *with space* l_p $(0<p<1)$*, where* $l_p = \{\{x_n\} \subset \mathbb{C} : \sum\limits_{n=1}^{\infty} |x_n|^p < \infty\}$*, together with the function* $d : l_p \times l_p \longrightarrow \mathbb{C}$ *and* $d(x,y) = (\sum\limits_{n=1}^{+\infty} |x_n - y_n|^p)^{\frac{1}{p}}$ *where* $x = x_n$, $y = y_n \in l_p$*, is* C^**-algebra-valued b-metric space. By applying a fundamental calculation, we can deduce that* $d(x,z) \leq 2^{\frac{1}{p}}(d(x,y)+d(y,z))$.

2. Methods and Key Results

Let A be a C^*-algebra and suppose that φ is a linear functional on A. Define $\varphi^*(a) = \varphi(a^*)$ for all $a \in A$. Then, φ^* is also a linear function A. And the function φ is called self-adjoint if $\varphi^* = \varphi$.

Every linear function on A can be represented in the form $\varphi = \varphi_1 + i\varphi_2$, where φ_1, φ_2 are self-adjoint. Specifically, $\varphi_1 = \frac{1}{2}(\varphi + \varphi^*)$, $\varphi_2 = \frac{1}{2i}(\varphi - \varphi^*)$.

A linear function φ on A is called positive if $\varphi(a^*a) \geq 0$ *for all* $a \in A$. We denote the positivity of φ by $\varphi \geq 0$. For two self-adjoint linear function φ_1, φ_2, we have $(\varphi_2 - \varphi_1) \geq 0$ when $\varphi_2 \geq \varphi_1$.

Definition 3. *If* $\varphi : A \to B$ *is a linear mapping in* C^**-algebra, it is said to be positive if* $\varphi(A^+) \subseteq \varphi(B^+)$*. In this case,* $\varphi(A_h) \subseteq \varphi(B_h)$*, and the restriction map:* $\varphi : A_h \to B_h$ *is increasing* [10].

If $B = \mathbb{C}$ *then the positive linear map is called positive linear functional, and it satisfies the following Propositions 1 and 2.*

Proposition 1. *Let A be a* C^**-algebra with 1, then a positive functional is bounded and* $\varphi(1) = \|\varphi\|$ [10].

Proposition 2. *Let A be a* C^**-algebra with 1 and let* φ *be a bounded linear functional on A such that* $\varphi(a) = \|\varphi\|\|a\|$*. There exists positive element* $a \in A$ *such that* φ *is a positive linear functional* [10].

We replace the real valued function $\psi_{\mathbb{R}}$ in [11] by C^*-valued function ψ, a positive linear function, to obtain a new ψ contraction condition.

Definition 4. *Let* $\Psi_{\mathbb{R}}$ *be the set of continuous functions* $\psi_{\mathbb{R}} : [0,\infty[\to [0,\infty[$ *satisfying the following conditions* [11]*:*

(i) $\psi_{\mathbb{R}} = 0$ *if and only if* $t = 0$*; and*
(ii) *For all sequences* $\{t_n\}$ *of elements in* $[0,\infty)$*, if* $\{\psi(t_n)\}$ *is a decreasing sequence, then the sequence* $\{t_n\}$ *is bounded, i.e.,* $\sup\limits_n\{t_n\} < \infty$.

We observe that if a function: $[0,\infty[\to [0,\infty[$ *is a continuous function satisfying either of the following conditions, then it must belong to* $\Psi_{\mathbb{R}}$:

(iii) $\psi_{\mathbb{R}}$ *is nondecreasing in* $[0,\infty[$;
(iv) $\psi_{\mathbb{R}} \geq Mt^a$ *for all* $t > 0$ *where* $M, a > 0$.

Definition 5. *Let the function* $\psi : A^+ \to A^+$ *be positive if having the following constraints:*

(i) ψ *is continuous and nondecreasing;*
(ii) $\psi(a) = 0$ *if and only if* $a = 0$;
(iii) $\lim_{n\to\infty} \psi^n(a) = 0$.

Definition 6. *Suppose that* A *and* B *are* C^**-algebra. A mapping* $\psi : A \to B$ *is said to be* C^**-homomorphism if:*

(i) $\psi(ax + by) = a\psi(x) + b\psi(y)$ *for all* $a, b \in \mathbb{C}$ *and* $x, y \in A$;
(ii) $\psi(xy) = \psi(x)\psi(y)$ *for all* $x, y \in A$;
(iii) $\psi(x^*) = \psi(x)^*$ *for all* $x \in A$;
(iv) ψ *maps the unit in* A *to the unit in* B *[12].*

Definition 7. *Let* A *and* B *be* C^**-algebra spaces and let* $\psi : A \to B$ *be a homomorphism then* ψ *is called an* $*$*-isomorphism if it is one-to-one* $*$*-homomorphism [13].*

We state that C^**-algebra* A *is* $*$*-isomorphic to a* C^**-algebra* B *if there exists* $*$*-isomorphism of* A *onto* B.

Property 1. *Let* A *and* B *be* C^**-algebra spaces and* $\psi : A \to B$ *is a* C^**-homomorphism* $for\ all\ x \in A$, *we have* $\sigma(\psi(x)) \subset \sigma(x)$ *and* $\|\psi(x)\| \leq \|\psi\|$ *[13].*

Corollary 1. *Every* C^**-homomorphism is bounded [12].*

Corollary 2. *Suppose that* ψ *is* C^**-isomorphism from* A *to* B*, then* $\sigma(\psi(x)) = \sigma(x)$ *and* $\|\psi(x)\| = \|x\|$ $for\ all\ x \in A$ *[12].*

Lemma 1. *Every* $*$*-homomorphism is positive [12].*

In this section, we investigate several use-cases using fixed point theorem in C^*-algebra-valued b-metric space with a positive function as a proof-of-concept, then discuss the results and apply the obtained results to some potential applications in fixed point theory.

Theorem 1. *Let* (X, A, d) *be a complete* C^**-algebra valued b-metric space. Let* $T : X \to X$ *satisfy the following condition:*

$$d(Tx, Ty) \leq a^* d(x, y)a - \psi(d(x, y))$$

where ψ *is* $*$*-homomorphism and* $\lim_{a\to\infty} \psi(a) = \infty$ *and* $\|b\|\|a\|^2 < 1$*. Then,* T *has a fixed point.*

Proof.

Let $x_{n+1} = Tx_n$. For each $n \geq 1$, then:

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq a^* d(x_n, x_{n-1})a - \psi(d(x_n, x_{n-1})) \\ &\leq \cdots \\ &\leq (a^*)^n d(x_1, x_0)a^n - \psi^n(d(x_1, x_0)). \end{aligned}$$

Then for $n > m$:

$$\begin{gathered} d(x_n, x_m) \leq b\, d(x_n, x_{n-1}) + b^2 d(x_{n-1}, x_{n-2}) + \cdots + b^{n-m} d(x_{m+1}, x_m) \\ \leq b(a^*)^{n-1} d(x_1, x_0)a^{n-1} + \cdots + b^{n-m}(a^*)^m d(x_1, x_0)a^m - b\psi(d(x_1, x_0)) - b^2\psi^2(d(x_1, x_0)) \\ - \cdots - b^{n-m}\, \psi^m(d(x_1, x_0)). \end{gathered}$$

Therefore:

$$
\begin{gathered}
\|d(x_n, x_m)\| \\
\leq \|b\|\|a\|^{2(n-1)}d(x_1, x_0) + \cdots + \|b^{n-m}\|\|a\|^{2m}d(x_1, x_0)a^m - \|b\|\|\psi(d(x_1, x_0))\| - \cdots \\
-\|b^{n-m}\|\|\psi^m(d(x_1, x_0))\| \\
\leq \|b\|\left[\|a\|^{2(n-1)}d(x_1, x_0) - \|\psi(d(x_1, x_0))\|\right] \\
+\|b^{n-m}\|\left[\|a\|^{2m}d(x_1, x_0) - \|\psi^m(d(x_1, x_0))\|\right] \xrightarrow{n,m \to +\infty} 0
\end{gathered}
$$

Then $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $\lim_{n \to +\infty} x_n = x$. Moreover, due to the continuity of T, $\lim_{n \to +\infty} x_n = x = limTx_{n-1} = Tx$. Hence, $Tx = x$.

Let y be another fixed point where:
$d(x, y) = d(Tx_n, Ty_n) \leq (a^*)^n d(x, y)a^n - \psi^n(d(x, y))$.
We have
$\|d(x, y)\| \leq \|a\|^{2n}\|d(x, y)\| - \|\psi^n(d(x, y))\| \xrightarrow{n \to +\infty} 0$, thus $x = y$. □

Lemma 2. *Let (X, A, d) be a C^*-algebra valued b-metric space such that $d(x, y) \in A^+$, $for\ all\ x, y \in X$ where $x \neq y$. Let $\phi : A^+ \to A^+$ be a function with the following properties:*

(i) $\phi(a) = 0\ iff\ a = 0$;
(ii) $\phi(a) < a,\ for\ a \in A^+$;
(iii) *Either $\phi(a) \leq d(x, y)$ or $d(x, y) \leq \phi(a)$, where $a \in A^+$ and $x, y \in X$.*

The C^*-algebra version of the lemma 2 is given in [14].

Theorem 2. *Let (X, A, d) be a complete C^*-algebra valued b-metric space. Let $T : X \to X$ be a contractive and mapping function:*

$$\psi(d(Tx, Ty)) \leq \phi(d(x, y)),$$

where ψ is $*$-homomorphism and $\phi : A^+ \to A^+$ is a continuous function with the constraint $\psi(a) < \phi(a)$. Then, T has a fixed point.

Proof. Let $x_0 \in X$, we define:

$$
\begin{gathered}
x_1 = Tx_0,\ x_2 = Tx_1, \cdots,\ x_n = Tx_{n-1}. \\
\psi(d(x_{n+1}, x_n)) = \psi(d(Tx_n, Tx_{n-1})) \leq \phi(d(x_n, x_{n-1}))
\end{gathered}
$$

using the constraint of Theorem 2, we have:

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}).$$

Then:

$$\|d(x_{n+1}, x_n)\| \leq \|d(x_n, x_{n-1})\|$$

Hence, the sequence $d(x_{n+1}, x_n)$ is norm decreasing and consequently, there exists $r \geq 0$ such that $d(x_{n+1}, x_n) \to r$ as $n \to +\infty$. Let $n \to +\infty$, from Property 1, we have $\psi(r) \leq \phi(r)$. From the condition of the theorem, we then deduce $r = 0$, and $d(x_{n+1}, x_n) \to 0$. This implies $\|d(x_{n+1}, x_n)\| \to 0$.

Let $n > m$ then:

$$d(x_n, x_m) \leq bd(x_n, x_{n-1}) + b^2 d(x_{n-1}, x_{n-2}) + \cdots + b^{n-m}d(x_{m-1}, x_m).$$

Using the condition of the theorem, we have:

$$\begin{gathered}\psi(d(x_n,x_m)) \le \psi(bd(x_n,x_{n-1})) + \psi\big(b^2 d(x_{n-1},x_{n-2})\big) + \cdots + \psi(b^{n-m} d(x_{m-1},x_m) \\ \le \phi(bd(x_n,x_{n-1})) + \phi\big(b^2 d(x_{n-1},x_{n-2})\big) + \cdots + \phi(b^{n-m} d(x_{m-1},x_m)) \\ = \psi(b)\psi(d(x_n,x_{n-1})) + \psi(b^2)\psi(d(x_{n-1},x_{n-2})) + \cdots + \psi(b^{n-m})\psi d(x_{m-1},x_m) \\ \le \phi(b)\phi(d(x_n,x_{n-1})) + \phi(b^2)\phi(d(x_{n-1},x_{n-2})) + \cdots + \phi(b^{n-m})\phi(d(x_{m-1},x_m)). \\ \|\psi(d(x_n,x_m))\| \le \|\phi\|\|b\|\|d(x_n,x_{n-1})\| + \|\phi\|\|b^2\|\|d(x_{n-1},x_{n-2})\| + \cdots \\ + \|\phi\|\|b^{n-m}\|\|d(x_{m-1},x_m))\| \ \to 0 \ as\ n,m \to +\infty\end{gathered}$$

Then $\{x_n\}$ is a Cauchy sequence. Since X is complete, then there exists $x \in X$ such that $\lim\limits_{n\to+\infty} x_n = x$. Moreover, due to the continuity of T, $\lim\limits_{n\to+\infty} x_n = x = \lim Tx_{n-1} = Tx$. Thus, $Tx = x$.

Let y be another fixed point. We have:

$$d(x,y) = d(Tx_n, Ty_n) \text{ then } \psi(d(x,y)) = \psi(d(Tx_n,Ty_n)) \le \phi(d(x_n,y_n)).$$

So, $\|\psi(d(x,y))\| = \|\psi(d(Tx_n,Ty_n))\| \le \|\phi(d(x_n,y_n))\| \to 0 \ \ as\ n \to +\infty$.
Hence $d(x,y) = 0$, and $x = y$. □

Theorem 3. *Suppose that there exists a metric space $d \in X$ such that (X, A, d) is a complete C^*-algebra valued b-metric space. Let $T : X \to X$ be a contractive mapping function and* $\psi(d(Tx,Ty)) \le \psi(M(x,y)) - \phi(d(x,y))$, and

$$M(x,y) = a_1 d(x,y) + a_2[d(Tx,y) + d(Ty,x)] + a_3[d(Tx,x) + d(Ty,y)]$$

where $b \in A'_+$, $a_1, a_2, a_3 \ge 0$, $a_1 + 2a_2 b + 2a_3 \le 1$, ψ and ϕ are $*$-homomorphisms and with the constraint $\psi(a) < \phi(a)$. Then, T has a fixed point.

Proof. Let $x_0 \in X$, and define
$x_1 = Tx_0, x_2 = Tx_1, \cdots, x_n = Tx_{n-1}$.
We have:

$$\begin{gathered}\psi(d(x_{n+1},x_n)) = \psi(d(Tx_n,Tx_{n-1})) \\ \le \psi(M(x_n,x_{n-1})) - \phi(d(x_n,x_{n-1})) \\ = \psi(a_1 d(x_n,x_{n-1}) + a_2[d(x_{n+1},x_{n-1}) + d(x_n,x_n)] + a_3[d(x_{n+1},x_n) + d(x_n,x_{n-1})]) - \\ \phi(d(x_n,x_{n-1})) \\ = \psi(a_1)\psi(d(x_n,x_{n-1}) + \psi(a_2)\psi[d(x_{n+1},x_{n-1}) + d(x_n,x_n)] + \ \psi(\ a_3)[d(x_{n+1},x_n) + d(x_n,x_{n-1})]) - \phi(d(x_n,x_{n-1})).\end{gathered}$$

Therefore

$$\begin{gathered}\|\psi(d(x_{n+1},x_n))\| \le \|\psi a_1\|\|\psi d(x_n,x_{n-1})\| + \|\psi a_2\|\|\psi[d(x_{n+1},x_{n-1}) + d(x_n,x_n)]\| + \\ \|\psi a_3\|\|\psi[d(x_{n+1},x_n) + d(x_n,x_{n-1})]\| - \|\phi(d(x_n,x_{n-1}))\| \to 0 \ as\ n \to +\infty.\end{gathered}$$

Given that ϕ *and* ψ are strongly monotone functions, we have:

$$\begin{gathered}d(x_{n+1},x_n) \le a_1 d(x_n,x_{n-1}) + a_2 b[d(x_{n+1},x_n) + d(x_n,x_{n-1})] + a_3[d(x_{n+1},x_n) + d(x_n,x_{n-1})] \\ (1 - a_2 b - a_3) d(x_{n+1},x_n) \le (a_1 + a_2 b + a_3) d(x_n,x_{n-1}) \\ d(x_{n+1},x_n) \le \tfrac{a_1+a_2b+a_3}{1-a_2b-a_3}\ d(x_n,x_{n-1}) \\ d(x_{n+1},x_n) \le h\, d(x_n,x_{n-1}) \text{ where } h = \tfrac{a_1+a_2b+a_3}{1-a_2b-a_3} \le 1.\end{gathered}$$

Then we have:

$$\|d(x_n,x_{n-1})\|\|d(x_{n+1},x_n)\| \le \|h\|\ \|d(x_n,x_{n-1})\| \to 0 \ as\ n,m \to +\infty.$$

Let $n > m$:

$$d(x_n,x_m) \le bd(x_n,x_{n-1}) + b^2 d(x_{n-1},x_{n-2}) + \cdots + b^{n-m} d(x_{m-1},x_m).$$

Applying the constraint of Theorem 3 then:

$$\psi(d(x_n,x_m)) \le \psi(bd(x_n,x_{n-1})) + \psi\big(b^2 d(x_{n-1},x_{n-2})\big) + \cdots + \psi(b^{n-m} d(x_{m-1},x_m)$$
$$= \psi(b)\psi(d(x_n,x_{n-1})) + \psi\big(b^2\big)\psi(d(x_{n-1},x_{n-2})) + \cdots + \psi(b^{n-m})\psi(d(x_{m-1},x_m))$$
$$\psi(d(x_n,x_m)) \le \psi(bMd(x_n,x_{n-1})) - \phi(bd(x_n,x_{n-1}))$$
$$+\psi\big(b^2 Md(x_{n-1},x_{n-2})\big) - \phi\big(b^2 d(x_{n-1},x_{n-2})\big) + \cdots + \psi(b^{n-m} Md(x_{m-1},x_m) - \phi(b^{n-m} d(x_{m-1},x_m))$$
$$= \psi(a_1 bd(x_n,x_{n-1}) + a_2 b\ [d(x_{n+1},x_{n-1}) + d(x_n,x_n)] +\ a_3 b[d(x_{n+1},x_n) + d(x_n,x_{n-1})]) - \phi(bd(x_n,x_{n-1})) + \cdots +$$
$$\psi(a_1 b^{n-m} d(x_{m-1},x_m) + a_2 b^{n-m}\ [d(x_{m-2},x_m) + d(x_{m-1},x_{m-1})] +\ a_3 b[d(x_{m-2},x_{m-1}) + d(x_{m-1},x_m)]) - \phi(bd(x_{m-1},x_m)).$$

Therefore:

$$\psi(d(x_n,x_m)) \le \psi(a_1)\psi(b)\psi(d(x_n,x_{n-1}) + \psi(a_2)\psi(b)\psi\ [d(x_{n+1},x_{n-1}) + d(x_n,x_n)] +$$
$$\psi(a_3)\psi(b)\psi[d(x_{n+1},x_n) + d(x_n,x_{n-1})]) - \phi(bd(x_n,x_{n-1})) + \cdots +$$
$$\psi(a_1)\psi(b^{n-m})\psi(d(x_{m-1},x_m) + \psi(a_2)\psi(b^{n-m})\psi\ [d(x_{m-2},x_m) + d(x_{m-1},x_{m-1})] +$$
$$\psi(a_3)\psi(b)\psi[d(x_{m-2},x_{m-1}) + d(x_{m-1},x_m)]) - \phi(bd(x_{m-1},x_m)).$$

Since the property of $\phi\ and\ \psi$ is strongly monotone, we have:

$$d(x_n,x_m) \le a_1 bd(x_n,x_{n-1}) + a_2 b\ [d(x_{n+1},x_{n-1}) + d(x_n,x_n)]$$
$$+a_3 b[d(x_{n+1},x_n) + d(x_n,x_{n-1})] + \cdots + a_1 b^{n-m} d(x_{m-1},x_m)$$
$$+a_2 b^{n-m}\ [d(x_{m-2},x_m) + d(x_{m-1},x_{m-1})] + a_3 b[d(x_{m-2},x_{m-1}) + d(x_{m-1},x_m)].$$

We then have:

$$\|d(x_n,x_m)\| \le \|a_1\|\|b\|\|d(x_n,x_{n-1})\| + \|a_2\|\|b\|\|\ [d(x_{n+1},x_{n-1}) + d(x_n,x_n)]\|$$
$$+\|a_3\|\|b\|\|[d(x_{n+1},x_n) + d(x_n,x_{n-1})]\| + \cdots + \|a_1\|\|b^{n-m}\|\|d(x_{m-1},x_m)\|$$
$$+\|a_2\|\|b^{n-m}\|\ \|[d(x_{m-2},x_m) + d(x_{m-1},x_{m-1})]\|$$
$$+\|a_3\|\|b\|\|[d(x_{m-2},x_{m-1}) + d(x_{m-1},x_m)]\| \to 0\ as\ n,m \to +\infty.$$

Then, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $\lim_{n\to+\infty} x_n = x$. Moreover, due to the continuity of T, $\lim_{n\to+\infty} x_n = x = \lim Tx_{n-1} = Tx$. So, $Tx = x$.

Let y be another fixed point. We have:
$d(x,y) = d(Tx_n, Ty_n)$ then

$$\psi(d(x,y)) = \psi(d(Tx_n,Ty_n)) \le \psi(M(x_n,y_n)) - \phi(d(x_n,y_n))$$
$$\|\psi(\ d(Tx_n,Ty_n)\| \le \|\psi a_1\|\|\psi d(x_n,y_n)\| + \|\psi a_2\|\|\psi[d(Tx_n,y_n) + d(Ty_n,x_n)]\|$$
$$+\|\psi a_3\|\|\psi[d(Tx_n,y_n) + d(Ty_n,x_n)]\| - \|\phi(d(x_n,y_n))\|.$$

Using the property of ϕ, we get:

$$\|\psi(\ d(Tx_n,Ty_n)\| \le \|\psi a_1\|\|\psi d(x_n,y_n)\| + \|\psi a_2\|\|\psi[d(Tx_n,y_n) + d(Ty_n,x_n)]\|$$
$$+\|\psi a_3\|\|\psi[d(Tx_n,x_n) + d(Ty_n,y_n)]\|.$$

where ψ is strongly monotone, then:

$$\|\ d(x,y)\| \le \|a_1\|\|d(x_n,y_n)\| + \|a_2\|\|[d(x,y_n) + d(y,x_n)]\| \to 0 \text{ as } n \to +\infty.$$

Thus, $d(x,y) = 0$, and $x = y$. □

Next, we briefly discuss the results and applications of the proposed findings by presenting a relevant example from the perspectives of both application and research.

Example 2. *Let* $X = [0,\ 2]$ *and* $A = \mathbb{C}$ *with a norm* $\|z\| = |z|$ *be a* $C^* - algebra$*. We define* $\mathbb{C}^+ = \{z = (x,\ y) \in \mathbb{C} : x = Re(z) \ge 0,\ y = Im(z) \ge 0\}$.

The partial order $\le$ *with respect to the* $C^* - algebra\ \mathbb{C}$ *is the partial order in* $\mathbb{C}$,
$z_1 \le z_2\ if\ Re(z_1) \le Re(z_2)$*, and* $Im(z_1) \le Im(z_2)$ *for any two elements* $z_1,\ z_2$ *in* $\mathbb{C}$.

Let $d : X \times X \to \mathbb{C}$
Suppose that:

$$d(x, y) = 2(|x - y|, |x - y|) \ for\ x, y \in X$$

Then, (X, d) *is a* $C^* - algebra$ *valued b-metric space where* $b = 1$ *with the required properties of theorem 3.*

Let $\psi, \phi : \mathbb{C}^+ \to \mathbb{C}^+$ *such that they can be defined as follows:*
For $t = (x, y) \in \mathbb{C}^+$,

$$\psi(t) = \begin{cases} (x,y),\ if\ x \le 2 \text{ and } y \le 2, \\ (x^2,y),\ if\ x > 2 \text{ and } y \le 2, \\ (x,y^2),\ if\ x \le 2 \text{ and } y > 2, \\ (x^2,y^2),\ if\ x > 2 \text{ and } y > 2. \end{cases}$$

and for $s = (s_1, s_2) \in \mathbb{C}^+$ *with* $v = min\{s_1, s_2\}$,

$$\phi(s) = \begin{cases} \left(\frac{v^2}{4}, \frac{v^2}{4}\right),\ if\ v \le 2 \\ \left(\frac{1}{4}, \frac{1}{4}\right),\ if\ v > 2 \end{cases};$$

Then, ψ *and* ϕ *have the properties mentioned in Definitions 3 and 5.*
Let $T : X \to X$ *be defined as follows:*

$$T(x) = \begin{cases} 0,\ if\ 0 \le x \le 1 \\ \frac{1}{8},\ if\ 1 < x \le 2 \end{cases};$$

Then, T has the required properties mentioned in Theorem 3.
Let $a_1 = \frac{1}{4}$, $a_2 = \frac{1}{16}$ *and* $a_3 = \frac{1}{16}$.
It can be verified that:

$$\psi(d(Tx, Ty)) \le \psi(M(x, y)) - \phi(d(x, y)),\ for\ all\ x, y \in X\ with\ y \preccurlyeq x.$$

Hence, the conditions of Theorem 3 are satisfied. We then show that 0 is a fixed point of T.

The above example with a minor modification was given in [14] as a special case of the cone metric space introduced in [15]. More information regarding cone metric space can be found in [16].

3. Conclusions

C^*-algebra theory is a critical subject in functional analysis and operator theory that plays a central role in fixed point theory and applications. In this context, several researchers have obtained fixed point results for mappings under multiple contractive conditions in the framework of different types of metric spaces.

In this paper, we present a new insight of C^*-algebra-valued b-metric-space in the perspective of the fixed point theory using contractive mapping. By using contractive mapping in the b-metric space, we discussed the existence and the uniqueness of the fixed point with mappings satisfying a contractive condition. As a result, we obtained ψ as an interesting and important result for the general case of C^*-algebra -valued metric spaces.

Author Contributions: Conceptualization, R.M. and S.O.; methodology, R.M. and S.O; validation, R.M., S.O. and Q.N.N.; formal analysis, R.M.; investigation, R.M.; resources, R.M.; data curation, R.M.; writing—original draft preparation, R.M.; writing—review and editing, Q.N.N. and R.M.; visualization, R.M.; supervision, S.O.; project administration, Q.N.N.; funding acquisition, Q.N.N. All authors have read and agreed to the published version of the manuscript.

Funding: This study was supported by the School of Fundamental Science and Engineering, Faculty of Science and Engineering, Waseda University, Tokyo, Japan.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data sharing is not applicable to this article.

Acknowledgments: The authors thank the editors and reviewers for their insightful comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ma, Z.; Jiang, L.; Sun, H. C*-algebra-valued metric spaces and related fixed point theorems. *Fixed Point Theory Appl.* **2014**, *206*. [CrossRef]
2. Ma, Z.; Jiang, L. C*-Algebra-valued b-metric spaces and related fixed point theorems. *Fixed Point Theory Appl.* **2015**, *1*, 1–12. [CrossRef]
3. Alsulami, H.H.; Agarwal, R.P.; Karapınar, E.; Khojasteh, F. A short note on C*-valued contraction mappings. *J. Inequalities Appl.* **2016**, *50*. [CrossRef]
4. Kamran, T.; Postolache, M.; Ghiura, A.; Batul, S.; Ali, R. The Banach contraction principle in C*-algebra-valued b-metric spaces with application. *Fixed Point Theory Appl.* **2016**, *10*. [CrossRef]
5. Özer, Ö.; Omran, S. On the generalized C*-valued metric spaces related with Banach fixed point theory. *Int. J. Adv. Appl. Sci.* **2017**, *4*, 35–37. [CrossRef]
6. Todorčević, V. *Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics*; Springer Nature Switzerland AG 2019; Springer: Cham, Switzerland, 2019; Print ISBN 978-3-030-22590-2, Online ISBN 978-3-030-22591-9.
7. Jovanović, R.Ž.; Sretenović, A.A.; Živković, B.D. Ensemble of various neural networks for prediction of heating energy consumption. *Energy Build.* **2015**, *94*, 189–199. [CrossRef]
8. Connes, A. *Noncommutative Geometry*; Academic Press: Cambridge, MA, USA, 1994; ISBN 978-0-12-185860-5.
9. Schroer, B. Lectures on algebraic quantum field theory and operator algebra. *Acta Math. Sin.* **2001**, *23*, 2205–2212.
10. Murphy, G.J. *C*-Algebras and Operator Theory*; Academic Press: London, UK, 1990.
11. Akkouchi, M. On a fixed point theorem of DW Boyd and JS Wong. *Acta Math. Vietnam* **2002**, *27*, 231–237.
12. Zhu, K. *An Introduction to Operator Algebras*; CRC Press: Boca Raton, FL, USA, 1961.
13. Sakai, S. *C*-Algebras and W*-Algebras*; Springer: Berlin, Heidelberg, 1998; Print ISBN 978-3-540-63633-5, Online ISBN 978-3-642-61993-9.
14. Choudhury, B.S.; Metiya, N. Fixed point and common fixed point results in ordered cone metric spaces. *An. Univ. Ovidius Constanta-Ser. Mat.* **2012**, *20*, 55–72. [CrossRef]
15. Huang, L.G.; Zhang, X. Cone metric spaces and fixed point theorems of contractive mappings. *J. Math. Anal. Appl.* **2007**, *332*, 1468–1476. [CrossRef]
16. Aleksic, S.; Kadelburg, Z.; Mitrovic, Z.D.; Radenovic, S. A new survey: Cone metric spaces. *J. Int. Math. Virtual Inst.* **2019**, *9*, 93–121.