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Planar Graphs with the Distance of 6^--Cycles at Least 2 from Each Other Are DP-3-Colorable

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Abstract: DP-coloring as a generalization of list coloring was introduced by Dvořák and Postle recently. In this paper, we prove that every planar graph in which the distance between 6^--cycles is at least 2 is DP-3-colorable, which extends the result of Yin and Yu [Discret. Math. 2019, 342, 2333–2341].

Keywords: planar graphs; DP-coloring; discharging method

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1. Introduction

All graphs are finite and simple in this paper. Let G be a plane graph and V, E, and F be sets of vertices, edges, and faces of G, respectively. Two faces are adjacent if they have a common edge. For a face $f \in F$, we write $f = [v_1, v_2, \cdots, v_k]$ when the vertices on f in a cyclic order are $v_1, v_2, \cdots, v_k$. A k-vertex (k^--vertex, k^+-vertex) is a vertex of degree k (at most k, at least k). A k-face (k^--face, k^+-face) is a face of degree k (at most k, at least k). The notation will be same for cycles. A triangle is a 3-cycle in G. A vertex or an edge of G is triangular when it is on a triangle. We say a chord is triangular in a cycle C if it splits the cycle C into at least one triangle. Let an $(l_1, l_2, \cdots, l_k)$-face be a k-face $f = [v_1 v_2 \cdots v_k]$ with $d(v_i) = l_i$. Let (l_1, l_2)-edge be an edge $e = v_1 v_2$ with $d(v_i) = l_i$. Let $|C|$ be the length (number of edges) of the cycle C. Let $|f|$ be the number of edges incident with f. Let $Ext(C)$ and $Int(C)$ denote the sets of vertices lying outside and inside of C, respectively. A cycle C is called separating if $Ext(C) \neq \emptyset$ and $Int(C) \neq \emptyset$. The distance $d(u, v)$ between two vertices u and v in G is the length (number of edges) of the shortest path between them. The distance $d(C, C')$ between two cycles C and C' in G is the minimum of the distances between vertices $u \in V(C)$ and $v \in V(C')$. A matching of G is a set of independent edges in G.

A proper k-coloring of G is a function $f : V(G) \to \{1, 2, \cdots, k\}$ such that $f(u) \neq f(v)$ for every edge $uv \in E(G)$. Let $\chi(G)$, the chromatic number of G, be the smallest k such that G is k-colorable. A list assignment of G is a mapping L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of colors. An L-coloring of G is a function $f : V \to \bigcup_{v \in V} L(v)$ such that $f(v) \in L(v)$ for every $v \in V$ and $f(u) \neq f(v)$ for every edge $uv \in E(G)$. A graph G is k-choosable if G has a L-coloring for every assignment L with $|L(v)| \geq k$. Let $\chi_l(G)$, the choice number of G, be the smallest k such that G is k-choosable.

It is well known that 3-COLORING is NP-complete for planar graphs. This provides motivation for finding some sufficient conditions for 3-coloring of planar graphs. In 1959, Grötzsch [1] proved that planar graphs with no triangles are 3-colorable. In 1969, Havel [2] asked whether there exists or not a constant d such that if G is a planar graph with the distance of triangles at least d, then G is 3-colorable. Borodin and Glebov [3] proved that every planar graph with no 5-cycles and $d = 2$ is 3-colorable. Dvořák, Kral, and Thomas [4] showed that for every planar graph $d = 10^{100}$ suffices.

List coloring was introduced as a generalization of proper coloring by Vizing [5] and independently by Erdős, Rubin, and Taylor [6]. Thomassen [7] showed that planar graphs with girth at least 5 are 3-choosable. Dvoȓák [8] proved that planar graphs with the distance of 4^--cycles from each other at least 26 are 3-choosable.

There are fewer techniques to approach list problems than ordinary coloring. Identifications of vertices are involved in the reduction configurations for ordinary coloring. However, in list coloring, because different vertices have different lists, it is not possible to use identification of vertices. The concept of DP-coloring as a generalization of list coloring, was introduced by Dvořák and Postle [9].

Definition 1. *Let G be a simple graph, and L be a list assignment of $V(G)$. For each vertex $v \in V(G)$, let $L_v = \{v\} \times L(v)$. For each edge uv in G, let M_{uv} be a partial matching between the sets L_u and L_v and let $\mathcal{M} = \{M_{uv} : uv \in E(G)\}$, called the matching assignment. The matching assignment is called a k-matching assignment if $L(v) = [k]$ for each $v \in V(G)$.*

Definition 2. *A $\mathcal{M}$-coloring of G is a function ϕ that assigns each vertex $v \in V(G)$ a color $\phi(v) \in L(v)$, such that for every $uv \in E(G)$, the vertices $(u, \phi(u))$ and $(v, \phi(v))$ are not adjacent in M_{uv}. We say that G is $\mathcal{M}$-colorable if such a $\mathcal{M}$-coloring exists.*

Definition 3. *The graph G is DP-k-colorable if, for each k-list assignment L and each matching assignment $\mathcal{M}$ over L, it has an $\mathcal{M}$-coloring. The minimum k such that G is DP-k-colorable is the DP-chromatic number of G, denoted by $\chi_{DP}(G)$.*

If every $(u, c_1)(v, c_2) \in E(M_{u,v})$ satisfies $c_1 = c_2$, then $uv \in E(G)$ is straight in a k-matching assignment $\mathcal{M}$. Dvořák and Postle [9] proved that planar graphs with no cycles of length from 4 to 8 are 3-choosable and noted that $\chi_{DP}(G) \le 3$ if G is a planar graph with no 4^--cycles. Liu and Li [10] proved that planar graphs without adjacent cycles of length at most 8 are 3-choosable. Zhao and Miao [11] proved that every planar graph in which the distance between 5^--cycles is at least 2 is DP-3-colorable. Bernshteyn et al. [12–16] gave some results on DP-coloring. DP-3-colorable planar graphs can be found in [17,18] and DP-4-colorable planar graphs can be found in [19–21]. Yin and Yu [22] proved planar graphs with no $\{4, 5, 6\}$-cycles in which the distance between triangles is at least 2 are DP-3-colorable. We present the following result in this paper.

Theorem 1. *Let G be a planar graph in which the distance between 6^--cycles is at least 2. Let C_0 be a 10^--cycle in G. Then, for every DP-3-coloring ϕ_0 of C_0, there exists a DP-3-coloring of G whose restriction to C_0 is ϕ_0.*

Corollary 1. *Every planar graph in which the distance between 6^--cycles is at least 2 is DP-3-colorable.*

Proof. Let G be a planar graph. By Dvořák and Postle [9], if G is 4^--cycle free then G is DP-3-colorable. So, we may assume that G contains a 4^--cycle and the 4^--cycle can be precolored. Then, G has a DP-3-coloring extended from the coloring of the 4^--cycle by Theorem 1 when the distance between 6^--cycles is at least 2 in G. □

2. Proof of Theorem 1

To prove Theorem 1, we use the reductio ad absurdum. Let G be a counterexample with the least number of vertices to Theorem 1. If G is 10^--cycle free then G is DP-3-colorable by Dvořák and Postle [9]. So, we may assume that G contains a 10^--cycle C_0.

The following Lemma 1 to Lemma 8 are about some crucial properties of the minimal counterexample G.

Lemma 1. *If $v \in V(G - C_0)$, then $d(v) \ge 3$.*

Proof. Let $v \in V(G - C_0)$ and $d(v) \le 2$. Because G is a minimal counterexample, we can first extend ϕ_0 of C_0 to $V(G) - \{v\}$. Then we can select a color $\phi(v)$ for v such that

$(v,\phi(v))(u,\phi(u)) \notin E(M_{uv})$ for each neighbor u of v. Therefore, G has been colored, a contradiction. □

Lemma 2. *C_0 is the boundary of the outer face.*

Proof. First, we show that C_0 is not separating. For otherwise, if C_0 is separating, G can be colored by extending the coloring of C_0 to both $Int(C_0)$ and $Ext(C_0)$, a contradiction. Therefore, either $Int(C_0)$ or $Ext(C_0)$ is empty. Then we may assume that $Ext(C_0)$ is empty without loss of generality. So C_0 is the boundary of the outer face. □

Lemma 3. *There exist no separating 10^--cycles.*

Proof. By Lemma 2, C_0 is not a separating 10^--cycle. Let $C \neq C_0$ be a separating 10^--cycle in G. Because G is a minimal counterexample, we can first extend ϕ_0 of C_0 to $G - Int(C)$. Then the coloring of the cycle C can be extended to $Int(C)$. Therefore, G has been colored, a contradiction. □

Lemma 4. *Let C be a cycle in G. If $|C| \leq 7$, then C has no chord. If C, $8 \leq |C| \leq 10$, has a chord e, then either e is triangular, or e splits C into a 7-cycle and a 4-cycle when $|C| = 9$, or e splits C into a 8-cycle and a 4-cycle when $|C| = 10$, or e splits C into a 7-cycle and a 5-cycle when $|C| = 10$.*

Proof. As the distance between 6^- cycles is at least 2 in G, C cannot have a chord if $|C| \leq 7$. If $|C| = 8$, then C can only have a triangular chord. If $|C| = 9$, then either e is triangular or e splits C into a 7-cycle and a 4-cycle. If $|C| = 10$, then either e is triangular or e splits C into an 8-cycle and a 4-cycle, or e splits C into a 7-cycle and a 5-cycle. □

Lemma 5. *C_0 has no chord.*

Proof. If C_0 has a chord e, then e must be one of the cases described in Lemma 4. Because G has no separating 10^--cycles by Lemma 3, G has no other vertices except the vertices on C_0. Then the coloring of C_0 is a coloring of G, a contradiction. □

Lemma 6 ([18])**.** *Let $k \geq 3$ and H be a subgraph of G. If the vertices of H can be ordered as $v_1, v_2, \cdots, v_l$ such that the following hold*

(1) $v_1v_l \in E(G)$, and v_1 has no neighbor in $G - H$,
(2) $d(v_l) \leq k$ and v_l has at least one neighbor outside of H,
(3) for each $2 \leq i \leq l-1$, v_i has at most $k-1$ neighbors in $G[v_1, \cdots, v_{i-1}] \bigcup (G - H)$,
then a DP-k-coloring of $G - H$ can be extended to a DP-k-coloring of G.

A vertex is internal if it is not incident with C_0 and a face is internal if it contains no vertex on C_0.

Lemma 7. *Let f be an internal 7-face in G. If all vertices on f are vertices with degree 3, then f cannot be adjacent to an internal 6^--face f_1 such that all vertices on f_1 are 3-vertices.*

Proof. Let $f = [v_1v_2w_1w_2w_3w_4w_5]$ and $f_1 = [v_1v_2 \cdots v_i](i \in \{3,4,5,6\})$ such that v_1v_2 is the common edge of f and f_1, and all vertices on f and f_1 are vertices with degree 3. Let $H = \{v_1, w_5, w_4, w_3, w_2, w_1, v_2, v_3, \cdots, v_i\}$ $(i \in \{3,4,5,6\})$. Order the vertices in H as $v_1, w_5, w_4, w_3, w_2, w_1, v_2, v_3, \cdots, v_i$ $(i \in \{3,4,5,6\})$. Since f and f_1 are internal faces, no vertex in H is on C_0. Because G is a minimal counterexample, we can first extend ϕ_0 of C_0 to $G - H$. Then by Lemma 6, the coloring of $G - H$ can be extended to a coloring of G, a contradiction. □

Lemma 8. *Let f be an internal 7-face in G. Let f_1 be an internal 6^--face which is adjacent to f. If except one vertex of f, all other vertices on f and f_1 are 3-vertices, then each of following holds:*

(a) *If f contains a (3, 4)-edge and is adjacent to another internal 6^--face f_2 with the common (3, 4)-edge, then f_2 has another vertex with degree at least 4.*

(b) *f cannot be adjacent to another internal 6^--face f_2 such that all vertices on f_2 are 3-vertices.*

Proof. Let $f = [v_1v_2 \cdots v_7]$ and $f_1 = [v_1v_2w_1 \cdots w_i]$ $(i \in \{1,2,3,4\})$ with the common (3, 3)-edge v_1v_2. Since the 6^--cycles in G are at a distance of at least 2 from each other, by symmetry we may assume that the edge v_4v_5 is on f_2 and $f_2 = [v_4v_5u_1 \cdots u_j]$ $(j \in \{1,2,3,4\})$.

(a) Suppose otherwise that all vertices in $\{u_1 \cdots u_j\}$ $(j \in \{1,2,3,4\})$ are vertices with degree 3. If $d(v_4) = 4$, then Let H be the set of vertices listed as: $v_2, v_3, v_4, u_j, \cdots, u_1, v_5, v_6, v_7, v_1, w_i, \cdots, w_1$ $(i \in \{1,2,3,4\})$ and $(j \in \{1,2,3,4\})$. If $d(v_5) = 4$, then Let H be the set of vertices listed as: $v_1, v_7, v_6, v_5, u_1, \cdots, u_j, v_4, v_3, v_2, w_1, \cdots, w_i$ $(i \in \{1,2,3,4\})$ and $(j \in \{1,2,3,4\})$. Since f, f_1 and f_2 are internal faces, no vertex in H is on C_0. Because G is a minimal counterexample, we can first extend ϕ_0 of C_0 to $G - H$. Then by Lemma 6, the coloring of $G - H$ can be extended to a coloring of G, a contradiction.

(b) Suppose otherwise that f_2 is an internal 6^--face and that all vertices on f_2 are 3-vertices. Since f has six vertices with degree 3, by symmetry we assume that $d(v_6) = 3$. Let u be the neighbor of u_1 not on f_2. We can rename the lists of vertices in $\{u_1, v_5, v_4, v_6, v_7\}$ such that each edge in $\{uu_1, u_1v_5, v_4v_5, v_5v_6, v_6v_7\}$ is straight. Consider the graph G' obtained from $G - \{v_6, v_5, v_4, u_1, \cdots, u_j\}$ $(j \in \{1,2,3,4\})$ by identifying v_7 and u. We claim that no new loops, multiple edges or cycles with length 3, 4, 5 or 6 are created. Otherwise, there is a $\{1,2,3,4,5,6\}$-path from v_7 to u in G, which together with v_6, v_5, u_1 forms a cycle C, $5 \leq d(C) \leq 10$. Since f_2 is a 6^--face, C cannot be a 6^--cycle.

- If v_4 is in $Int(C)$ see Figure 1a, then C is a separating $\{7,8,9,10\}$-cycle, it is a contradiction to Lemma 3.
- If v_4 is not in $Int(C)$ see Figure 1b. Since f is a 7-cycle and $d(v_6) = 3$, by Lemma 3 and 4, v_6 must be incident with an edge e in $Int(C)$. The other end vertex of e is either on C or not. If it is on C, then e is a chord of C. By Lemma 4, $8 \leq d(C) \leq 10$ and e is on a 5^--cycle C'. Then the distance between C' and f_2 is at most 1, a contradiction. If it is not on C, then it is in $Int(C)$. So C must be a separating $\{7,8,9,10\}$-cycle, it is a contradiction to Lemma 3. Because none of v_7 and u is on a 6^--cycle, the 6^--cycles in G' are at a distance of at least 2 from each other. Now, we claim that no new chord in C_0 is formed in G'. For otherwise, u is on C_0 and v_7 is adjacent to a vertex v_7' on C_0, then there is a path between v_7' and u on C_0 with length at most five, which forms a $\{6,7,8,9,10\}$-cycle with u_1, v_5, v_6, v_7. Similar to the proof above, it does not occur.
  Since C_0 is still the boundary of the outer face of the embedding of G', the coloring of C_0 can be extended to G' by minimality of G. Now keep the colors of all vertices in G' and color v_7 and u with the color of the identified vertex. Now color v_6, and then color u_1 with the color of v_6. We can do this because the edges in $\{uu_1, u_1v_5, v_4v_5, v_5v_6, v_6v_7\}$ are straight and the color of v_6 is different from the color of v_7 and u. If $|f_2| = 3$, then we color v_4, v_5 in the order. If $|f_2| = 4, 5$ or 6, then we color $u_2, \cdots, u_j, v_4, v_5$ $(j \in \{2,3,4\})$ in the order. Then we obtain a coloring of G, a contradiction.

□

Let f_0 be the outer face of the embedding of G. We are now ready to present a discharging procedure. We set the initial charge of every vertex $v \in V(G)$ to be $\mu(v) = 2d(v) - 6$, of every face $f \neq f_0$ in our fixed plane drawing of G to be $\mu(f) = |f| - 6$, and set $\mu(f_0) = |f_0| + 6$. Then $\sum_{x \in V \bigcup F} \mu(x) = 0$ by Euler's Formula.

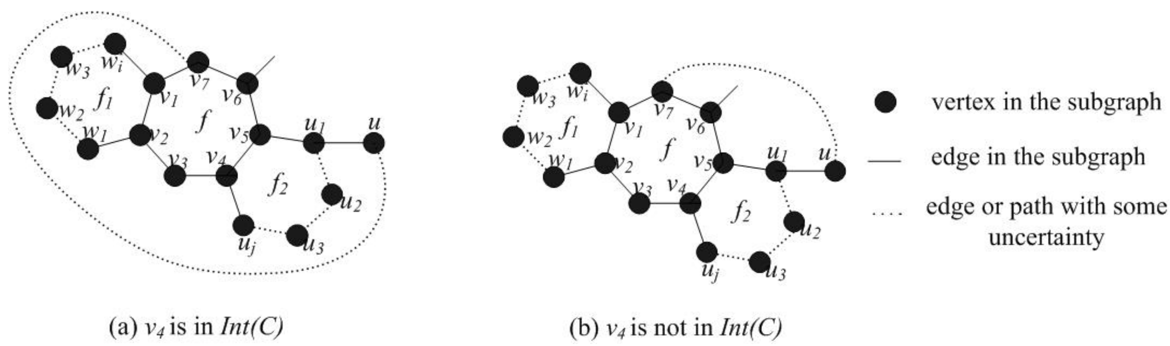

Figure 1. The identification of u and v_7.

let $F_k = \{f : V(f) \bigcap V(C_0) \neq \emptyset$ and f be a k-face$\}$. A 7-face f is special when f is in F_7 and adjacent to two internal 6^--faces. We say a vertex v is rich to a 7^+-face f when v is on f and not on a 5^--face which is adjacent to f.

The discharging rules:

(R1): If v is an internal 4-vertex and on a 5^--face f, then v gives $\frac{3}{2}$ to f.

(R2): If v is an internal 5^+-vertex, then v gives $\frac{3}{2}$ to its incident 5^--face if any and $\frac{1}{2}$ to its incident 7-face if any.

(R3): Each 7^+-face f $(f \neq f_0)$ gives $\frac{1}{2}$ to its adjacent internal $(3,\ 3,\ 4^+)$-face if any, $\frac{1}{8}$ to its adjacent internal $(3,\ 3,\ 3,\ 4^+)$-face if any, 1 to its adjacent internal $(3,\ 3,\ 3)$-face if any, $\frac{1}{2}$ to its adjacent internal $(3,\ 3,\ 3,\ 3)$-face if any and $\frac{1}{5}$ to its adjacent internal $(3,\ 3,\ 3,\ 3,\ 3)$-face if any.

(R4): Each internal 7-face receives $\frac{1}{2}$ from its incident rich 4-vertex.

(R5): After (R3) and (R4), each 7^+-face gives all its remaining charge to f_0.

(R6): The outer face f_0 receives $\mu(v)$ from each $v \in C_0$, gives 1 to each special 7-face if any, 3 to each face in F_3 if any, 2 to each face in F_4 if any and 1 to each face in F_5 if any.

Let $\mu^*(x)$ denote the final charge of $x \in V \bigcup F$. To lead to a contradiction, we will prove that $\mu^*(x) \geq 0$ for all $x \in V \bigcup F \setminus \{f_0\}$ and $\mu^*(f_0) > 0$.

Lemma 9. *For all $v \in V$, $\mu^*(v) \geq 0$.*

Proof. Since f_0 receives $\mu(v)$ from each $v \in C_0$ by (R6) whether $\mu(v)$ is positive or not, $\mu^*(v) = 0$ when v is on C_0. Let v be an internal vertex in G, then by Lemma 1 $d(v) \geq 3$. If $\boldsymbol{d(v) = 3}$, then $\mu^*(v) = 2d(v) - 6 = 0$.

Because the 6^--cycles in G are at a distance of at least 2 from each other, each vertex can be incident with at most one 6^--face. Let $\boldsymbol{d(v) = 4}$. If v is on a 5^--face f, then v gives $\frac{3}{2}$ to f and $\frac{1}{2}$ to its incident 7-face when v is rich to the 7-face by (R1) and (R4). If v is not on a 5^--face, then by (R4) v gives at most $\frac{1}{2}$ to each incident face. Thus, $\mu^*(v) \geq 2d(v) - 6 - \max\{\frac{3}{2} + \frac{1}{2}, \frac{1}{2} \times 4\} = 0$.

Let $\boldsymbol{d(v) \geq 5}$. By (R2), v gives $\frac{3}{2}$ to its incident 5^--face if any and at most $\frac{1}{2}$ to each other incident face. Thus, $\mu^*(v) \geq 2d(v) - 6 - \frac{3}{2} - \frac{1}{2} \times (d(v) - 1) > 0$. □

Lemma 10. *For all $f \in F - \{f_0\}$, $\mu^*(f) \geq 0$.*

Proof. Let $\boldsymbol{|f| = 3}$. If $V(f) \bigcap V(C_0) \neq \emptyset$, then by (R6) f receives 3 from f_0, so $\mu^*(f) = |f| - 6 + 3 = 0$. Now let $V(f) \bigcap V(C_0) = \emptyset$. If $V(f)$ contains at least two 4^+-vertices, then by (R1) and (R2) f receives $\frac{3}{2}$ from each of the 4^+-vertices. So $\mu^*(f) \geq |f| - 6 + \frac{3}{2} \times 2 = 0$. If $V(f)$ contains exactly one 4^+-vertex, then f receives $\frac{3}{2}$ from the 4^+-vertex and receives $\frac{1}{2}$ from each of its adjacent 7^+-faces by (R1), (R2) and (R3). So $\mu^*(f) \geq |f| - 6 + \frac{3}{2} + \frac{1}{2} \times 3 = 0$. If f is an internal (3, 3, 3)-face, then f receives 1 from each of the adjacent 7^+-faces by (R3). So $\mu^*(f) \geq |f| - 6 + 1 \times 3 = 0$.

Let $|f| = 4$. If $V(f) \bigcap V(C_0) \neq \emptyset$, then by (R6) f receives 2 from f_0, so $\mu^*(f) = |f| - 6 + 2 = 0$. Now let $V(f) \bigcap V(C_0) = \emptyset$. If $V(f)$ contains at least two 4^+-vertices, then f receives $\frac{3}{2}$ from each of the 4^+-vertices by (R1) and (R2). So $\mu^*(f) \geq |f| - 6 + \frac{3}{2} \times 2 > 0$. If $V(f)$ contains exactly one 4^+-vertex, then f receives $\frac{3}{2}$ from the 4^+-vertex and receives $\frac{1}{8}$ from each adjacent 7^+-face by (R1), (R2) and (R3). So $\mu^*(f) \geq |f| - 6 + \frac{3}{2} + \frac{1}{8} \times 4 = 0$. If f is an internal (3, 3, 3, 3)-face, then f receives $\frac{1}{2}$ from each adjacent 7^+-face by (R3). So $\mu^*(f) \geq |f| - 6 + \frac{1}{2} \times 4 = 0$.

Let $|f| = 5$. If $V(f) \bigcap V(C_0) \neq \emptyset$, then by (R6) f receives 1 from f_0, so $\mu^*(f) = |f| - 6 + 1 = 0$. Now let $V(f) \bigcap V(C_0) = \emptyset$. If $V(f)$ contains a 4^+-vertex, then f receives $\frac{3}{2}$ from the 4^+-vertex by (R1) and (R2). So $\mu^*(f) \geq |f| - 6 + \frac{3}{2} > 0$. If f is an internal (3, 3, 3, 3, 3)-face, then f receives $\frac{1}{5}$ from each adjacent 7^+-face by (R3). So $\mu^*(f) \geq |f| - 6 + \frac{1}{5} \times 5 = 0$.

Let $|f| = 6$, by our rules f sends out nothing, so $\mu^*(f) = |f| - 6 = 0$.

Let $|f| \geq 7$. By (R3) f needs to give $\frac{1}{2}$ to its adjacent internal (3, 3, 4^+)-faces if any, $\frac{1}{8}$ to its adjacent internal (3, 3, 3, 4^+)-faces if any, 1 to its adjacent internal (3, 3, 3)-faces if any, $\frac{1}{2}$ to its adjacent internal (3, 3, 3, 3)-faces if any and $\frac{1}{5}$ to its adjacent internal (3, 3, 3, 3, 3)-faces if any. Since the distance between 6^--cycles is at least 2, f is adjacent to at most $\lfloor \frac{|f|}{3} \rfloor$ internal 6^--faces. If $|f| \geq 8$, then $\mu^*(f) \geq |f| - 6 - 1 \times \lfloor \frac{|f|}{3} \rfloor \geq 0$.

Let $|f| = 7$. Since the distance between 6^--cycles is at least 2, f is adjacent to at most 2 internal 6^--faces. Let $V(f) \bigcap V(C_0) \neq \emptyset$. If f is adjacent to at most one internal 6^--face, then by R(3) f gives at most 1 to the adjacent 6^--face if any. So $\mu^*(f) \geq |f| - 6 - 1 = 0$. If f is special, then f gives at most 1 to the adjacent 6^--faces and receives 1 from f_0, so $\mu^*(f) = |f| - 6 - 2 + 1 = 0$. Now let $V(f) \bigcap V(C_0) = \emptyset$. If f is adjacent to at most one internal 6^--face, then by R(3) f gives at most 1 to the adjacent 6^--face if any. So $\mu^*(f) \geq |f| - 6 - 1 = 0$. Let f be adjacent to two internal 6^--faces. If none of the 6^--faces is a (3, 3, 3)-face, then by R(3) f gives at most $\frac{1}{2} \times 2$ to the 6^--faces, so $\mu^*(f) \geq |f| - 6 - 1 = 0$. If one of the 6^--faces has at least two 4^+-vertices, then the 6^--face receives nothing from f by (R3), so $\mu^*(f) \geq |f| - 6 - 1 = 0$. So, we assume that f is adjacent to a (3, 3, 3)-face f_1 and another 6^--face f_2 that f_2 has at most one 4^+-vertex. By Lemma 7, f has at least one 4^+-vertex.

- If f_2 shares a (3, 4^+)-edge with f, then f contains another 4^+-vertex v'. For otherwise, f_2 has at least two 4^+-vertices by Lemma 8(1), a contradiction. Since $|f| = 7$ and the distance between 6^--cycles is at least 2, f can be adjacent to at most two 6^--faces. So, if $d(v') = 4$, then v' must be rich and gives $\frac{1}{2}$ to f by (R4). If $d(v') \geq 5$, then v' gives $\frac{1}{2}$ to f by (R2). So $\mu^*(f) \geq |f| - 6 - 1 - \frac{1}{2} + \frac{1}{2} = 0$.
- If f_2 shares a $(3, 3)$-edge with f. If all vertices on f_2 are 3-vertices, then f contains at least two 4^+-vertices. For otherwise, f_2 cannot be a 6^--face that all vertices on f_2 are 3-vertices by Lemma 8(2). Since $|f| = 7$ and the distance between 6^--cycles is at least 2, f can be adjacent to at most two 6^--faces. So, if one of the 4^+-vertices is a 4-vertex, then it must be rich and gives $\frac{1}{2}$ to f by (R4). If one of the 4^+-vertices is a 5^+-vertex, then it gives $\frac{1}{2}$ to f by (R2). So $\mu^*(f) \geq |f| - 6 - 1 \times 2 + \frac{1}{2} \times 2 = 0$. If f_2 contains a 4^+-vertex, then f has a 4^+-vertex v'' because f_1 is a (3, 3, 3)-face by Lemma 7. Since $|f| = 7$ and the distance between 6^--cycles is at least 2, f can be adjacent to at most two 6^--faces. So, if $d(v'') = 4$, then v'' must be rich and gives $\frac{1}{2}$ to f by (R4). If $d(v'') \geq 5$, then v'' gives $\frac{1}{2}$ to f by (R2). So $\mu^*(f) \geq |f| - 6 - 1 \times 2 + \frac{1}{2} \times 2 = 0$.

□

Lemma 11. *$\mu^*(f_0) > 0$.*

Proof. Suppose otherwise that $\mu^*(f_0) \leq 0$. Let $E(G - C_0, C_0)$ denote the set of edges between $G - C_0$ and C_0. Let E' be the set of edges which are in $E(G - C_0, C_0)$ but not on 6^--faces. Let e' be the number of edges in E'. Let x be the charges that f_0 receives by (R5), so $x \geq 0$. Let ℓ_i be the number of faces in F_i ($i \in \{3, 4, 5, 6\}$). Since C_0 has no chord by

Lemma 5, each face in F_3, F_4, F_5 and F_6 has at least two edges in $E(G - C_0, C_0)$. Let ℓ_7 be the number of special 7-faces. By (R5) and (R6),

$$
\begin{aligned}
\mu^*(f_0) &= |f_0| + 6 + \sum_{v \in C_0} (2d(v) - 6) - 3\ell_3 - 2\ell_4 - \ell_5 - \ell_7 + x \\
&= |f_0| + 6 + \sum_{v \in C_0} 2(d(v) - 2) - 2d(C_0) - 3\ell_3 - 2\ell_4 - \ell_5 - \ell_7 + x \\
&= 6 - |f_0| + 2|E(G - C_0, C_0)| - 3\ell_3 - 2\ell_4 - \ell_5 - \ell_7 + x \\
&\geq 6 - |f_0| + 4\ell_3 + 4\ell_4 + 4\ell_5 + 4\ell_6 + 2e' - 3\ell_3 - 2\ell_4 - \ell_5 - \ell_7 + x \\
&= 6 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + 2e' - \ell_7 + x
\end{aligned} \tag{1}
$$

Equality holds when each 6^--face in F_3, F_4, F_5 and F_6 contains two edges in $E(G - C_0, C_0)$.

We consider the cases.

Case 1. If G **has a special 7-face** f, then $\ell_7 > 0$ and $E(G - C_0, C_0) \bigcap E(f) \neq \emptyset$. Because f is adjacent to two internal 6^--faces and the distance between 6^--cycles is at least 2, so each edge in $E(G - C_0, C_0) \bigcap E(f)$ is in E' and f shares exactly one vertex or one edge with C_0. So $\boldsymbol{e' \geq \ell_7}$.

- **Let $\boldsymbol{e' = \ell_7 > 0}$** . If f is a special 7-face, then each edge in $E(G - C_0, C_0) \bigcap E(f)$ is in E' and $|E(G - C_0, C_0) \bigcap E(f)| = 2$. So $\ell_3 = \ell_4 = \ell_5 = \ell_6 = 0$. For otherwise if f' is a 6^--face in F_i $(i \in \{3,4,5,6\})$, then there must be two 8^+-faces adjacent to f' and containing vertices of C_0, then $e' > \ell_7$, a contradiction. Since each special 7-face shares exactly one vertex or one edge with C_0, $\ell_7 \geq |f_0|$. By (1), $\mu^*(f_0) \geq 6 - |f_0| + 2e' - \ell_7 + x = 6 - |f_0| + \ell_7 + x > 0$, a contradiction.
- **Let $\boldsymbol{e' > \ell_7 > 0}$** . By (1), $\mu^*(f_0) \geq 6 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + 2e' - \ell_7 + x \geq 6 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + 2(\ell_7 + 1) - \ell_7 + x = 8 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + \ell_7 + x$. **If $\boldsymbol{|f_0| \leq 8}$**, then $\mu^*(f_0) \geq 8 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + \ell_7 + x > 0$, a contradiction. **If $\boldsymbol{|f_0| = 9}$**, then $\mu^*(f_0) \geq 8 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + \ell_7 + x = -1 + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + \ell_7 + x$. Since $\mu^*(f_0) \leq 0$, $\boldsymbol{\ell_7 \leq 1}$. Recall $\ell_7 > 0$, so $\boldsymbol{\ell_7 = 1}$. Because $e' > \ell_7 = 1$ and $0 \geq \mu^*(f_0) \geq 6 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + 2e' - \ell_7 + x = 6 - 9 + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + 2e' - 1 + x$, so $e' = 2$ and $\ell_3 = \ell_4 = \ell_5 = \ell_6 = x = 0$. It follows that C_0 is adjacent to a 9^+-face f which has at least 7 consecutive 2-vertices. So $x \geq |f| - 6 - \lceil \frac{|f|-9}{3} \rceil > 0$ by (R3), a contradiction. **If $\boldsymbol{|f_0| = 10}$**, then $\mu^*(f_0) \geq 8 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + \ell_7 + x = -2 + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + \ell_7 + x$. Since $\mu^*(f_0) \leq 0$, $\boldsymbol{\ell_7 \leq 2}$. **If $\boldsymbol{\ell_7 = 1}$**, because $e' > \ell_7 = 1$ and $0 \geq \mu^*(f_0) \geq 6 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + 2e' - \ell_7 + x = 6 - 10 + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + 2e' - 1 + x$, then $e' = 2$ and $\ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + x \leq 1$. If $\ell_3 = 1$, then $\ell_4 = \ell_5 = \ell_6 = x = 0$ and C_0 is adjacent to a 7^+-face f which has at least 3 consecutive 2-vertices and is adjacent to the 3-face in F_3. So $x \geq |f| - 6 - \lfloor \frac{|f|-5}{3} \rfloor > 0$ by (R3), a contradiction. If $\ell_3 = 0$, then $\ell_4 = \ell_5 = \ell_6 = 0$, $x \leq 1$ and C_0 is adjacent to a 10^+-face f which has at least 8 consecutive 2-vertices. So $x \geq |f| - 6 - \lceil \frac{|f|-10}{3} \rceil > 2$ by (R3), a contradiction. **If $\boldsymbol{\ell_7 = 2}$**, because $e' > \ell_7$ and $e' > \ell_7 = 2$ and $0 \geq \mu^*(f_0) \geq 6 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + 2e' - \ell_7 + x = 6 - 10 + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + 2e' - 2 + x$, then $e' = 3$ and $\ell_3 = \ell_4 = \ell_5 = \ell_6 = x = 0$. Thus, the two 7-faces in F_7 must share an edge in $E(G - C_0, C_0)$. Then C_0 is adjacent to a 8^+-face f which has at least 7 consecutive 2-vertices. So $x \geq |f| - 6 - \lceil \frac{|f|-8}{3} \rceil > 0$ by (R3), a contradiction.

Case 2. If G **has no special 7-faces**, then $\ell_7 = 0$. Recall that $\boldsymbol{e' \geq 0}$

- **Let $\boldsymbol{e' = 0}$** . By (1), $0 \geq \mu^*(f_0) \geq 6 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + x$, $\ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + x \leq |f_0| - 6$. Since $|f_0| \leq 10$ and the distance between 6^--cycles is at least 2, $\boldsymbol{\ell_3 \leq 3}$.

Let $\boldsymbol{\ell_3 = 3}$. By (1), $\mu^*(f_0) \geq 6 - |f_0| + 3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + x = 9 - |f_0| + 2\ell_4 + 3\ell_5 + 4\ell_6 + x$. **If $\boldsymbol{|f_0| \leq 8}$**, then $\mu^*(f_0) \geq 9 - |f_0| + 2\ell_4 + 3\ell_5 + 4\ell_6 + x > 0$, a contradiction. **If $\boldsymbol{|f_0| = 9}$**, then $\ell_4 = \ell_5 = \ell_6 = x = 0$ and C_0 is adjacent to a 7^+-face f which has at least one 2-vertex and is adjacent to two 3-faces in F_3. So $x \geq |f| - 6 - \lfloor \frac{|f|-5}{3} \rfloor > 0$ by (R3), a contradiction. **If $|f_0| = 10$**, then $\ell_4 = \ell_5 = \ell_6 = 0$ and $x \leq 1$. It follows that C_0 is adjacent to three 7^+-faces and each 7^+-face f contains at least one 2-vertex and is adjacent to two 3-faces in F_3. So $x \geq 3 \times [|f| - 6 - \lfloor \frac{|f|-5}{3} \rfloor] \geq 3$ by (R3), a contradiction.

Let $\ell_3 = 2$. By (1), $\mu^*(f_0) \geq 6 - |f_0| + 2 + 2\ell_4 + 3\ell_5 + 4\ell_6 + x = 8 - |f_0| + 2\ell_4 + 3\ell_5 + 4\ell_6 + x$. **If $|f_0| \leq 7$**, then $\mu^*(f_0) \geq 8 - |f_0| + 2\ell_4 + 3\ell_5 + 4\ell_6 + x > 0$, a contradiction. **If $|f_0| = 8$**, then $\ell_4 = \ell_5 = \ell_6 = x = 0$ and C_0 is adjacent to a 7^+-face f which has at least 2 consecutive 2-vertices and is adjacent to two 3-faces in F_3. So $x \geq |f| - 6 - \lfloor \frac{|f|-6}{3} \rfloor > 0$ by (R3), a contradiction. **If $|f_0| = 9$**, then $\ell_4 = \ell_5 = \ell_6 = 0$ and $x \leq 1$. It follows that C_0 is adjacent to two 7^+-faces that each 7^+-face f contains at least one 2-vertex and is adjacent to two 3-faces in F_3. So $x \geq 2 \times [|f| - 6 - \lfloor \frac{|f|-5}{3} \rfloor] \geq 2$ by (R3), a contradiction. **If $|f_0| = 10$**, then $\ell_5 = \ell_6 = 0$ and $\ell_4 \leq 1$. If $\ell_4 = 1$, then $x = 0$ and C_0 is adjacent to a 7^+-face f which has at least one 2-vertex and is adjacent to two 3-faces in F_3. So $x \geq |f| - 6 - \lfloor \frac{|f|-5}{3} \rfloor > 0$ by (R3), a contradiction. If $\ell_4 = 0$, then $x \leq 2$. It follows that C_0 is adjacent to a 7^+-face f_1 which has at least one 2-vertex and is adjacent to two 3-faces in F_3, and a 8^+-face f_2 which has at least three 2-vertices and is adjacent to two 3-faces in F_3. So $x \geq |f_1| - 6 - \lfloor \frac{|f_1|-5}{3} \rfloor + |f_2| - 6 - \lfloor \frac{|f_1|-7}{3} \rfloor \geq 3$ by (R3), a contradiction.

Let $\ell_3 = 1$. By (1), $\mu^*(f_0) \geq 6 - |f_0| + 1 + 2\ell_4 + 3\ell_5 + 4\ell_6 + x = 7 - |f_0| + 2\ell_4 + 3\ell_5 + 4\ell_6 + x$. **If $|f_0| \leq 6$**, then $\mu^*(f_0) \geq 7 - |f_0| + 2\ell_4 + 3\ell_5 + 4\ell_6 + x > 0$, a contradiction. **If $|f_0| = 7$**, then $\ell_4 = \ell_5 = \ell_6 = x = 0$ and C_0 is adjacent to a 8^+-face f which has at least 5 consecutive 2-vertices and is adjacent to the 3-face in F_3. So $x \geq |f| - 6 - \lfloor \frac{|f|-8}{3} \rfloor > 0$ by (R3), a contradiction. **If $|f_0| = 8$**, then $\ell_4 = \ell_5 = \ell_6 = 0$ and $x \leq 1$. It follows that C_0 is adjacent to a 9^+-face which has at least 6 consecutive 2-vertices and is adjacent to the 3-face in F_3. So $x \geq |f| - 6 - \lfloor \frac{|f|-9}{3} \rfloor \geq 3$ by (R3), a contradiction. **If $|f_0| = 9$**, then $\ell_4 \leq 1$. If $\ell_4 = 1$, then then $\ell_5 = \ell_6 = x = 0$. It follows that C_0 is adjacent to a 7^+-face f which has at least 2 consecutive 2-vertices and is adjacent to the 3-face in F_3 and the 4-face in F_4. So $x \geq |f| - 6 - \lfloor \frac{|f|-6}{3} \rfloor > 0$ by (R3), a contradiction. If $\ell_4 = 0$, then $\ell_5 = \ell_6 = 0$ and $x \leq 2$. It follows that C_0 is adjacent to a 10^+-face which has at least 7 consecutive 2-vertices and is adjacent to the 3-face in F_3. So $x \geq |f| - 6 - \lfloor \frac{|f|-10}{3} \rfloor \geq 4$ by (R3), a contradiction. **If $|f_0| = 10$**, then $\ell_6 = 0$ and $\ell_5 \leq 1$. If $\ell_5 = 1$, then $\ell_4 = x = 0$. It follows that C_0 is adjacent to a 7^+-face f which has at least 2 consecutive 2-vertices and is adjacent to the 3-face in F_3 and the 5-face in F_5. So $x \geq |f| - 6 - \lfloor \frac{|f|-6}{3} \rfloor > 0$ by (R3), a contradiction. If $\ell_5 = 0$, then $\ell_4 \leq 1$. If $\ell_4 = 1$, then $x \leq 1$. It follows that C_0 is adjacent to two 7^+-faces and each 7^+-face f contains at least one 2-vertex and is adjacent to the 3-face in F_3 and the 4-face in F_4. So $x \geq 2 \times [|f| - 6 - \lfloor \frac{|f|-5}{3} \rfloor] \geq 2$ by (R3), a contradiction. If $\ell_4 = 0$, then $x \leq 3$ and C_0 is adjacent to a 11^+-face f which has at least 8 consecutive 2-vertices and is adjacent to the 3-face in F_3. So $x \geq |f| - 6 - \lfloor \frac{|f|-11}{3} \rfloor > 5$ by (R3), a contradiction.

Let $\ell_3 = 0$, then $\ell_4 + \ell_5 + \ell_6 > 0$. For otherwise that $G = C_0$. Since $d(C_0) \leq 10$ and $0 \geq \mu^*(f_0) \geq 6 - |f_0| + 2\ell_4 + 3\ell_5 + 4\ell_6 + x$, $\boldsymbol{\ell_4 \leq 2}$.

Let **$\ell_4 = 2$**. By (1), $\mu^*(f_0) \geq 6 - |f_0| + 4 + 3\ell_5 + 4\ell_6 + x = 10 - |f_0| + 3\ell_5 + 4\ell_6 + x$. **If $|f_0| \leq 9$**, then $\mu^*(f_0) \geq 10 - |f_0| + 3\ell_5 + 4\ell_6 + x > 0$, a contradiction. **If $|f_0| = 10$**, then $\ell_5 = \ell_6 = x = 0$ and C_0 is adjacent to a 7^+-face f which has at least 2 consecutive 2-vertices and is adjacent to the 4-faces in F_4. Thus, $x \geq |f| - 6 - \lfloor \frac{|f|-6}{3} \rfloor > 0$ by (R3), a contradiction.

Let $\ell_4 = 1$. By (1), $\mu^*(f_0) \geq 6 - |f_0| + 2 + 3\ell_5 + 4\ell_6 + x = 8 - |f_0| + 3\ell_5 + 4\ell_6 + x$. **If $|f_0| \leq 7$**, then $\mu^*(f_0) \geq 8 - |f_0| + 3\ell_5 + 4\ell_6 + x > 0$, a contradiction. **If $|f_0| = 8$**, then $\ell_5 = \ell_6 = x = 0$ and C_0 is adjacent to a 8^+-face f which has at least 5 consecutive 2-vertices and is adjacent to the 4-face in F_4. So $x \geq |f| - 6 - \lfloor \frac{|f|-8}{3} \rfloor > 0$ by (R3), a contradiction. **If $|f_0| = 9$**, then $\ell_5 = \ell_6 = 0$, $x \leq 1$ and C_0 is adjacent to a 9^+-face f which has at least 6 consecutive 2-vertices and is adjacent to the 4-face in F_4. So $x \geq |f| - 6 - \lfloor \frac{|f|-9}{3} \rfloor > 1$ by (R3), a contradiction. **If $|f_0| = 10$**, then $\ell_5 = \ell_6 = 0$, $x \leq 2$ and C_0 is adjacent to a

10^+-face f which has at least 7 consecutive 2-vertices and is adjacent to the 4-face in F_4. So $x \geq |f| - 6 - \lfloor \frac{|f|-10}{3} \rfloor > 2$ by (R3), a contradiction.

Let $\ell_4 = 0$, then $\ell_5 + \ell_6 > 0$. For otherwise that $G = C_0$. Since $d(C_0) \leq 10$ and $0 \geq \mu^*(f_0) \geq 6 - |f_0| + 3\ell_5 + 4\ell_6 + x$, $\ell_5 \leq 1$.

Let $\ell_5 = 1$. By (1), $\mu^*(f_0) \geq 6 - |f_0| + 3 + 4\ell_6 + x = 9 - |f_0| + 4\ell_6 + x$. **If $|f_0| \leq 8$**, then $\mu^*(f_0) \geq 9 - |f_0| + 4\ell_6 + x > 0$, a contradiction. **If $|f_0| = 9$**, then $\ell_6 = x = 0$ and C_0 is adjacent to a 8^+-face f which has at least 5 consecutive 2-vertices and is adjacent to the 5-face in F_5. So $x \geq |f| - 6 - \lfloor \frac{|f|-8}{3} \rfloor > 0$ by (R3), a contradiction. **If $|f_0| = 10$**, then $\ell_6 = 0$, $x \leq 1$ and C_0 is adjacent to a 9^+-face f which has at least 6 consecutive 2-vertices and is adjacent to the 5-face in F_5. So $x \geq |f| - 6 - \lfloor \frac{|f|-9}{3} \rfloor > 1$ by (R3), a contradiction.

Let $\ell_5 = 0$, then $\ell_6 > 0$. For otherwise that $G = C_0$. Since $d(C_0) \leq 10$ and $0 \geq \mu^*(f_0) \geq 6 - |f_0| + 4\ell_6 + x$, $\ell_6 = 1$. **If $|f_0| \leq 9$**, then $\mu^*(f_0) \geq 10 - |f_0| + x > 0$, a contradiction. **If $|f_0| = 10$**, then $x = 0$ and C_0 is adjacent to a 8^+-face f which has at least 5 consecutive 2-vertices and is adjacent to the 6-face in F_6. So $x \geq |f| - 6 - \lfloor \frac{|f|-8}{3} \rfloor > 0$ by (R3), a contradiction.

- **Let $e' > 0$** . By (1), $0 \geq \mu^*(f_0) \geq 6 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + 2e' + x$, $\ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + 2e' + x \leq |f_0| - 6$. Since $|f_0| \leq 10$, $e' \leq 2$.

Let $e' = 2$. By (1), $\mu^*(f_0) \geq 6 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + 4 + x = 10 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + x$. **If $|f_0| \leq 9$**, then $\mu^*(f_0) \geq 10 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + x > 0$, a contradiction. **If $|f_0| = 10$**, then $\ell_3 = \ell_4 = \ell_5 = \ell_6 = x = 0$. It follows that C_0 is adjacent to a 7^+-face f which has at least 4 consecutive 2-vertices. So $x \geq |f| - 6 - \lceil \frac{|f|-7}{3} \rceil > 0$ by (R3), a contradiction.

Let $e' = 1$. By (1), $\mu^*(f_0) \geq 6 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + 2 + x = 8 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + x$. **If $|f_0| \leq 7$**, then $\mu^*(f_0) \geq 8 - |f_0| + \ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + x > 0$, a contradiction. **If $|f_0| = 8$**, then $\ell_3 = \ell_4 = \ell_5 = \ell_6 = x = 0$. It follows that C_0 is adjacent to a 9^+-face f which has at least 7 consecutive 2-vertices. So $x \geq |f| - 6 - \lceil \frac{|f|-9}{3} \rceil > 0$ by (R3), a contradiction. **If $|f_0| = 9$**, then $\ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + x \leq 1$. So $\ell_3 \leq 1$. If $\ell_3 = 1$, then $\ell_4 = \ell_5 = \ell_6 = x = 0$ and C_0 is adjacent to a 7^+-face f which has at least 3 consecutive 2-vertices and is adjacent to the 3-face in F_3. So $x \geq |f| - 6 - \lfloor \frac{|f|-5}{3} \rfloor > 0$ by (R3), a contradiction. If $\ell_3 = 0$, then $\ell_4 = \ell_5 = \ell_6 = 0$ and $x \leq 1$. It follows that C_0 is adjacent to a 10^+-face f which has at least 8 consecutive 2-vertices. So $x \geq |f| - 6 - \lceil \frac{|f|-10}{3} \rceil > 1$ by (R3), a contradiction. **If $|f_0| = 10$**, then $\ell_3 + 2\ell_4 + 3\ell_5 + 4\ell_6 + x \leq 2$. So **$\ell_3 \leq 2$**. **If $\ell_3 = 2$**, then $\ell_4 = \ell_5 = \ell_6 = x = 0$ and C_0 is adjacent to a 7^+-face f which has at least one 2-vertex and is adjacent to the 3-faces in F_3. So $x \geq |f| - 6 - \lfloor \frac{|f|-5}{3} \rfloor > 0$ by (R3), a contradiction. **If $\ell_3 = 1$**, then $\ell_4 = \ell_5 = \ell_6 = 0$ and $x \leq 1$. It follows that C_0 is adjacent to two 7^+-faces. If one of the 7^+-faces f is a 8^+-face and contains at least 5 consecutive 2-vertices, then $x \geq |f| - 6 - \lfloor \frac{|f|-7}{3} \rfloor \geq 2$ by (R3), a contradiction. For otherwise that all of 7^+-faces contain at least 3 consecutive 2-vertices and are adjacent to the 3-face in F_3, then $x \geq 2 \times [|f| - 6 - \lfloor \frac{|f|-5}{3} \rfloor] \geq 2$ by (R3), a contradiction. **If $\ell_3 = 0$**, then $2\ell_4 + 3\ell_5 + 4\ell_6 + x \leq 2$. So **$\ell_4 \leq 1$**. **If $\ell_4 = 1$**, then $\ell_5 = \ell_6 = x = 0$ and C_0 is adjacent to a 7^+-face f which has at least 3 consecutive 2-vertices and is adjacent to the 4-face in F_4. So $x \geq |f| - 6 - \lfloor \frac{|f|-5}{3} \rfloor > 0$ by (R3), a contradiction. **If $\ell_4 = 0$**, then $\ell_5 = \ell_6 = 0$, $x \leq 2$ and C_0 is adjacent to a 11^+-face f which has at least 9 consecutive 2-vertices. So $x \geq |f| - 6 - \lceil \frac{|f|-11}{3} \rceil > 2$ by (R3), a contradiction. □

Proof of Theorem 1. From Lemmas 9–11, $\sum_{x \in V \bigcup F} \mu^*(x) > 0$, a contradiction to Euler's Formula. Thus, the counterexample G cannot exist. So, Theorem 1 is true. □

3. Conclusions

The coloring theory of graphs is useful in many fields, such as discrete mathematics, allocation of wireless communication channels, combinatorial optimization, computer theory.

It is well known that 3-COLORING is NP-complete for planar graphs. This provides motivation for finding some sufficient conditions for 3-coloring of planar graphs. DP-coloring is a stronger version of list coloring. Proving a planar graph to be DP-3-colorable is harder than proving a planar graph to be 3-colorable.

It is unknown if there exists a planar graph in which the distance between 6^{-}-cycles at least 1 is not DP-3-colorable.

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