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On the Growth of Higher Order Complex Linear Differential Equations Solutions with Entire and Meromorphic Coefficients

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Abstract: We revisit the problem of studying the solutions growth order in complex higher order linear differential equations with entire and meromorphic coefficients of $[p,q]$-order, proving how it is related to the growth of the coefficient of the unknown function under adequate assumptions. Our study improves the previous results due to J. Liu - J. Tu - L. Z Shi, L. M. Li - T. B. Cao , and others.

Keywords: entire function; meromorphic function; $[p,q]$-order; linear differential equations

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1. Introduction, Definitions and Notations

Complex linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + ... + A_1(z)f' + A_0(z)f = 0 \tag{1}$$

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + ... + A_1(z)f' + A_0(z)f = F(z), \tag{2}$$

where the coefficients A_0, A_1, ... , A_{k-1}, $(k \geq 2)$, and $F(\not\equiv 0)$ are entire or meromorphic functions, are relevant and they have extensively been studied by many authors (cf. [1–11]). In this line, Juneja-Kapoor-Bajpai studied entire functions of $[p,q]$-order with the aim of accurately discussing the growth of these functions, ([12,13]). Additionally, more recently, Liu-Tu-Shi [14] modified slightly the aforementioned $[p,q]$-order definition investigating properties of the solutions of complex linear differential equations, also see [7].

The study of order of an entire or meromorphic function f studies the symmetries or analogies between the growth of the maximum modulus of f and the growth of exponential and logarithmic functions, since the order of growth of a function relates to the rate of growth of the latter ones, ([7–9,12–14]). In order to handle this comparison, for each real number $r \in [0,\infty)$ belonging to the domain of $f \in \{\exp, \log\}$, we consider $f_1(r) = f(r)$ and $f_0(r) = r$. Additionally, for each of such f and $p \in \mathbb{N}$, we define $f_{p+1}(r) = f(f_p(r))$, this for sufficiently large r when $f = \log$. We will consider $\exp_{-1} r = \log_1 r$ and $\log_{-1} r = \exp_1 r$. Moreover, given a set $E \subset [0,\infty)$, we denote its linear measure by $mE = \int_E dt$, and the logarithmic measure for $E \subset (1,\infty)$, by $m_l E = \int_E \frac{dt}{t}$.

Despite the fact that this paper uses standard notions of Nevanlinna theory, we consider it to be convenient to recall some notation that is related to the number of poles of a meromorphic or entire function that are located within a disk centered at the origin in order to facilitate its reading (cf. [15–17]). Let $n(r,f)$ be the number of poles of a function f (counting multiplicities) in $|z| \leq r$, and where $\overline{n}(r,f)$ is the number of distinct poles of a function f in $|z| \leq r$. Subsequently, we define the integrated counting function $N(r,f)$ by

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f) \log r,$$

and

$$\overline{N}(r,f) = \int_0^r \frac{\overline{n}(t,f) - \overline{n}(0,f)}{t} dt + \overline{n}(0,f) \log r,$$

on the other hand, we define the proximity function $m(r,f)$ by

$$m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\phi}\right) \right| d\phi,$$

where $\log^+ x = \max\{0, \log x\}$. We should think of $m(r,f)$ as a measure of how close f is to infinity on $|z| = r$.

Nevertheless, within that context, we recall that $T(r,f)$ stands for the Nevanlinna characteristic function of the meromorphic function f that is defined on each positive real value r by

$$T(r,f) = m(r,f) + N(r,f).$$

Additionally, $M(r,f)$ stands for the so-called maximum modulus function defined for each non-negative real value r by

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

Now, we recall the following definitions, where p, q are positive integers satisfying $p \geq q \geq 1$.

Definition 1 ([7,14])**.** *Let f be a meromorphic function, the $[p,q]-$order of f is defined by*

$$\sigma_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log_q r}.$$

If f is an entire function, then

$$\sigma_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r,f)}{\log_q r}.$$

Remark 1. *If $p = q = 1$, above definition reduces to standard order. If, just $q = 1$, it reduces to p-th order.*

Definition 2 ([7,14])**.** *The $[p,q]-$ lower order of a meromorphic function f is defined by*

$$\mu_{[p,q]}(f) = \underline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log_q r}.$$

If f is an entire function, then

$$\mu_{[p,q]}(f) = \underline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r,f)}{\log_q r}.$$

Definition 3 ([7,14])**.** *The $[p,q]-$type of a meromorphic function f of $[p,q]$-order σ $\left(0 < \sigma_{[p,q]}(f) = \sigma < \infty\right)$ is defined by*

$$\tau_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_{p-1} T(r,f)}{\left(\log_{q-1} r\right)^{\sigma}}.$$

If f is an entire function, then

$$\tau_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p M(r,f)}{\left(\log_{q-1} r\right)^{\sigma}}.$$

Definition 4 ([7,14])**.** *The $[p,q]-$convergence exponent of the sequence of zeros of a meromorphic function f is defined by*

$$\lambda_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p N\left(r,\frac{1}{f}\right)}{\log_q r}.$$

Definition 5 ([7,14])**.** *The $[p,q]-$convergence exponent of distinct zeros of a meromorphic function f is defined by*

$$\overline{\lambda}_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{N}\left(r,\frac{1}{f}\right)}{\log_q r}.$$

Liu-Tu-Shi [14] consider the Equation (1) with entire functions as coefficients, and then obtain the following results.

Theorem 1 ([14])**.** *Let $A_j, 0 \leq j \leq k-1$, be entire functions satisfying $\max\left\{\sigma_{[p,q]}(A_j),\ j \neq 0\right\} < \sigma_{[p,q]}(A_0) < \infty$, then every nontrivial solution f of (1) satisfies*

$$\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0).$$

Theorem 2 ([14])**.** *Let A_j, $0 \leq j \leq k-1$, be entire functions satisfying $\max\left\{\sigma_{[p,q]}(A_j),\ j \neq 0\right\} \leq \sigma_{[p,q]}(A_0) < \infty$, and*

$$\max\left\{\tau_{[p,q]}(A_j) : \sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_0) > 0\right\} < \tau_{[p,q]}(A_0),$$

then every nontrivial solution f of (1) satisfies

$$\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0).$$

When the coefficients in (2) are meromorphic functions, Li-Cao [7] obtain the following result:

Theorem 3 ([7])**.** *Let $A_j, 0 \leq j \leq k-1$, and $F(\not\equiv 0)$ be meromorphic functions, and let f be a meromorphic solution of (2) satisfying $\max\left\{\sigma_{[p+1,q]}(A_j), \sigma_{[p+1,q]}(F)\right\} < \sigma_{[p+1,q]}(f)$, then we have*

$$\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f).$$

The following section contains the main results that deepen the aforementioned theorems regardinf how fast the solutions of linear differential Equations (1) and (2) may grow.

2. Main Results

In this section, we present our main results.

Theorem 4. *Let* $A_j, 0 \leq j \leq k-1$, *be entire functions satisfying* $\sigma_{[p,q]}(A_0) = \sigma_1$ *and* $\overline{\lim_{r\to\infty}} \sum_{j=1}^{k-1} \frac{m(r,A_j)}{m(r,A_0)} < 1$, *then every nontrivial solution* f *of* (1) *satisfies*

$$\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0) = \sigma_1.$$

Theorem 5. *Let* $A_j, 0 \leq j \leq k-1$, *be entire functions and let* A_0 *be a transcendental function that satisfies*

$$\max\left\{\sigma_{[p,q]}(A_j) \mid j \neq 0\right\} \leq \mu_{[p,q]}(A_0) = \sigma_{[p,q]}(A_0),$$

and $\underline{\lim}_{r\to\infty} \sum_{j=1}^{k-1} \frac{m(r,A_j)}{m(r,A_0)} < 1$ $(r \notin E_1)$, *where* E_1 *is a set of* r *of finite linear measure, then every nontrivial solution of* (1) *satisfies*

$$\sigma_{[p+1,q]}(f) = \mu_{[p,q]}(A_0) = \sigma_{[p,q]}(A_0).$$

For the non-homogeneous case (2), we obtain the following result:

Theorem 6. *Let* $A_j, 0 \leq j \leq k-1$, *and* $F(\not\equiv 0)$ *be meromorphic functions. If* f *is a meromorphic solution of* (2) *satisfying*

$$\overline{\lim_{r\to\infty}} \frac{\sum_{j=0}^{k-1} T(r, A_j) + T(r, F)}{T(r, f)} < 1,$$

then

$$\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f).$$

3. Preliminary Lemmas

In this section, we introduce some lemmas and remark that we will use them in the sequel.

Lemma 1 ([17])**.** *Let* $g : (0, \infty) \to \mathbb{R}$ *and* $h : (0, \infty) \to \mathbb{R}$ *be monotone increasing functions so that*

(1) $g(r) \leq h(r)$ *outside of a set* E_2 *of finite linear measure. Subsequently, for any* $\alpha > 1$, *there exists* $r_0 > 0$, *such that* $g(r) \leq h(\alpha r)$ *for all* $r > r_0$.

(2) $g(r) \leq h(r)$ *outside of a set* E_2 *of finite logarithmic measure. Subsequently, for any* $\alpha > 1$, *there exists* $r_0 > 0$, *such that* $g(r) \leq h(r^{\alpha})$ *for all* $r > r_0$.

Lemma 2 ([17])**.** *Let* f *be a transcendental entire function, and* z *a point with* $|z| = r$, *at which* $|f(z)| = M(r, f)$. *Subsequently, for all* $|z|$ *outside a set* E_3 *of finite logarithmic measure, it holds*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^n (1 + o(1)), \quad (n \in \mathbb{N}, r \notin E_3),$$

where $\nu_f(r)$ *is the central index of* f.

Remark 2. *Because the number of zeros of a polynomial* P *of degree* n *is finite (at most* n*) and, indeed, its central index is* n *for sufficiently large* r, *the above Lemma* 2 *holds for any given entire, transcendental or not, function* f.

Lemma 3 ([14])**.** *Let* f *be an entire function of* $[p, q]-$ *order satisfying* $\sigma_{[p,q]}(f) = \sigma_5$, *then there exists a set* $E_4 \subset (1, \infty)$ *having an infinite logarithmic measure, such that, for all* $r \in E_4$, *it holds*

$$\lim_{r\to\infty} \frac{\log_p T(r, f)}{\log_q r} = \sigma_{[p,q]}(f) = \sigma_5.$$

Lemma 4 ([18]). *Let* $A_j, 0 \le j \le k-1$, *be entire functions in* (1), *with at least one of them transcendental. If* A_s, $s \in \{0, 1, \ldots, k-1\}$, *is the first one (according to the sequence of* $A_0, A_1, \ldots, A_{k-1}$) *satisfying* $\lim\limits_{r\to\infty} \sum\limits_{j=s+1}^{k-1} \frac{m(r,A_j)}{m(r,A_s)} < 1$ $(r \notin E_5)$, *where* $E_5 \subset (1, \infty)$ *is a set with finite linear measure, then* (1) *possesses at most s linearly independent entire solutions satisfying* $\overline{\lim\limits_{r\to\infty}} \frac{\log T(r,f)}{m(r,A_s)} = 0$ $(r \notin E_5)$.

Lemma 5 ([12]). *Let f be an entire function of* $[p,q]-$*order, and let* $\nu_f(r)$ *be the central index of f, then*

$$\overline{\lim_{r\to\infty}} \frac{\log_p \nu_f(r)}{\log_q r} = \sigma_{[p,q]}(f).$$

4. Proof of Main Results

Proof of Theorem 4. From Equation (1), it follows that

$$-A_0(z) = \frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z)\frac{f^{(k-1)}(z)}{f(z)} + \ldots + A_1(z)\frac{f'(z)}{f(z)}. \tag{3}$$

By Remark 2 and (3),

$$m(r, A_0) \le \sum_{j=1}^{k-1} m(r, A_j) + O\{\log rT(r, f)\}, \qquad (r \notin E), \tag{4}$$

where E is a set of finite linear measures.

Assume that

$$\overline{\lim_{r\to\infty}} \sum_{j=1}^{k-1} \frac{m(r, A_j)}{m(r, A_0)} = \alpha < \beta < 1,$$

then for sufficiently large r, we find that

$$\sum_{j=1}^{k-1} m(r, A_j) < \beta m(r, A_0). \tag{5}$$

From (4) and (5), it follows that

$$(1-\beta)m(r, A_0) \le O\{\log rT(r, f)\} \qquad (r \notin E). \tag{6}$$

By Lemma 3, there exists a set $E_4 \subset (1, \infty)$ of r of infinite logarithmic measure, such that, for all z satisfying $|z| = r \in E_4$, we have

$$\lim_{r\to\infty} \frac{\log_p m(r, A_0)}{\log_q r} = \sigma_{[p,q]}(A_0)(= \sigma_1).$$

Subsequently, by the definition of limit, there exists a $\varepsilon > 0$, such that

$$\begin{aligned} \lim_{r\to\infty} \frac{\log_p m(r, A_0)}{\log_q r} &\ge (\sigma_1 - \varepsilon) \\ \Rightarrow \log_p m(r, A_0) &\ge (\sigma_1 - \varepsilon)\log_q r = \log\left(\log_{q-1} r\right)^{(\sigma_1-\varepsilon)} \\ \Rightarrow m(r, A_0) &\ge \exp_{p-1}\left\{\left(\log_{q-1} r\right)^{(\sigma_1-\varepsilon)}\right\}. \end{aligned}$$

By substituting the above inequality in (6), there exists a set $E_4 \subset (1,\infty)$ of r of infinite logarithmic measure, such that, for all z satisfying $|z| = r \in E_4 \backslash E$ and for any $\varepsilon > 0$, we have

$$(1-\beta)\exp_{p-1}\left\{\left(\log_{q-1} r\right)^{\sigma_1-\varepsilon}\right\} \le (1-\beta)m(r,A_0) \le O\{\log rT(r,f)\}, \quad (r \notin E). \tag{7}$$

From (7) and Lemma 1, we deduce

$$\begin{aligned}(1-\beta)\exp_p\left\{(\sigma_1-\varepsilon)\left(\log_q r\right)\right\} &\le O\{\log rT(r,f)\}\\ \Rightarrow (\sigma_1-\varepsilon)\left(\log_q r\right) &\le \log_{p+1} T(r,f) + O(\log r).\end{aligned}$$

Taking limit $r \to \infty$ after dividing both side by $\log_q r$, we obtain that

$$\sigma_{[p+1,q]}(f) \ge \sigma_{[p,q]}(A_0) = \sigma_1. \tag{8}$$

On the other hand, Equation (1) provides

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \le |A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \ldots + |A_1(z)|\left|\frac{f'(z)}{f(z)}\right| + |A_0(z)|. \tag{9}$$

Now, Remark 2 provides a set E_3 of finite logarithmic measure, so that, for all z satisfying $|z| = r \notin E_3$ and $|f(z)| = M(r,f)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1+o(1)), \quad (j = 1,2,\ \ldots\ ,k). \tag{10}$$

Subsequently, (5) and the fact $\sigma_{[p,q]}(A_0) = \sigma_1$ imply that

$$\sigma_{[p,q]}(A_j) < \sigma_{[p,q]}(A_0) = \sigma_1 \quad (j = 0,1,\ \ldots\ ,k-1),$$

$$|A_j(z)| \le \exp_p\left\{\left(\log_{q-1} r\right)^{\sigma_{[p,q]}(A_0)+\varepsilon}\right\}, \quad (j = 0,1,\ \ldots\ ,k-1). \tag{11}$$

Hence, having in mind the definition of $[p,q]-$order,

$$|A_0(z)| \le \exp_p\left\{\left(\log_{q-1} r\right)^{\sigma_{[p,q]}(A_0)+\varepsilon}\right\}. \tag{12}$$

Substituting (10)–(12) into (9), it follows that

$$\begin{aligned}\left(\frac{|\nu_f(r)|}{r}\right)^k |1+o(1)| &\le k\exp_p\left\{\left(\log_{q-1} r\right)^{\sigma_{[p,q]}(A_0)+\varepsilon}\right\}\left(\frac{|\nu_f(r)|}{r}\right)^{k-1} |1+o(1)|\\ \Rightarrow \nu_f(r) &\le kr\exp_p\left\{\left(\log_{q-1} r\right)^{\sigma_{[p,q]}(A_0)+\varepsilon}\right\}\\ &= kr\exp_{p+1}\left\{\left(\sigma_{[p,q]}(A_0)+\varepsilon\right)\left(\log_q r\right)\right\}\end{aligned}$$

$$\Rightarrow \log_{p+1} \nu_f(r) \le \left(\sigma_{[p,q]}(A_0)+\varepsilon\right)\left(\log_q r\right) + \log_{p+1} kr. \tag{13}$$

Because $\varepsilon > 0$ is arbitrary, Lemma 1 and (13) provide

$$\overline{\lim_{r\to\infty}} \frac{\log_{p+1} \nu_f(r)}{\log_q r} \le \sigma_{[p,q]}(A_0). \tag{14}$$

By Lemma 5 and (14), we get

$$\sigma_{[p+1,q]}(f) \leq \sigma_{[p,q]}(A_0) = \sigma_1. \tag{15}$$

From (8) and (15), we conclude that

$$\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0) = \sigma_1.$$

This proves the theorem. □

Proof of Theorem 5. From Lemma 4, it follows that every nontrivial solution f of Equation (1) satisfies $\overline{\lim\limits_{r\to\infty}} \frac{\log T(r,f)}{m(r,A_0)} > 0$ $(r \notin E_5)$; hence, there exists a $\delta > 0$ and a sequence $\{r_n\}_{n=1}^{\infty}$ tending to infinity, so that, for sufficiently large $r_n \notin E_5$ and for every nontrivial solution f of Equation (1), we have

$$\log T(r_n, f) > \delta m(r_n, A_0). \tag{16}$$

Lemma 3 provides a set $E_4 \subset (1,\infty)$ of infinite logarithmic measure, such that, for all $r \in E_4 \backslash E_5$ and for any $\varepsilon > 0$, we have

$$\delta \exp_{p-1}\left\{ \left(\log_{q-1} r_n\right)^{\sigma_{[p,q]}(A_0)-\varepsilon} \right\} \leq \delta m(r_n, A_0), \tag{17}$$

i.e., by (16) and (17),

$$\delta \exp_{p-1}\left\{ \left(\log_{q-1} r_n\right)^{\sigma_{[p,q]}(A_0)-\varepsilon} \right\} \leq \delta m(r_n, A_0) < \log T(r_n, f). \tag{18}$$

Lemma 1 and Equation (18) imply that

$$\sigma_{[p+1,q]}(f) \geq \sigma_{[p,q]}(A_0).$$

As $\mu_{[p,q]}(A_0) = \sigma_{[p,q]}(A_0)$, it follows that

$$\sigma_{[p+1,q]}(f) \geq \sigma_{[p,q]}(A_0) = \mu_{[p,q]}(A_0). \tag{19}$$

On the other hand, from Equation (1),

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq |A_{k-1}(z)| \left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \ldots + |A_1(z)| \left|\frac{f'(z)}{f(z)}\right| + |A_0(z)|. \tag{20}$$

Because $\max\left\{\sigma_{[p,q]}(A_j) \,|\, j \neq 0\right\} \leq \mu_{[p,q]}(A_0) = \sigma_{[p,q]}(A_0)$, for sufficiently large r and for any given $\varepsilon > 0$, we have

$$|A_j(z)| \leq \exp_p\left\{ \left(\log_{q-1} r\right)^{\sigma_{[p,q]}(A_0)+\varepsilon} \right\}, \quad j = 1, 2, \ldots, k-1. \tag{21}$$

Again, having in mind the definitions of $[p,q]-$order, we have

$$|A_0(z)| \leq \exp_p\left\{ \left(\log_{q-1} r\right)^{\sigma_{[p,q]}(A_0)+\varepsilon} \right\}. \tag{22}$$

Now taking Lemma 2 into account, we may assure that there exists some set E_3 of finite logarithmic measure, so that whenever $|z| = r \notin E_3$ and $|f(z)| = M(r,f)$, it holds that

$$\left(\frac{f^{(j)}(z)}{f(z)}\right) = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)), \qquad (j = 1, 2, \ldots, k). \tag{23}$$

Substituting (21)–(23) into (20), we obtain

$$\begin{aligned}\left(\frac{|\nu_f(r)|}{r}\right)^k |1+o(1)| &\leq k\exp_p\left\{\left(\log_{q-1} r\right)^{\sigma_{[p,q]}(A_0)+\varepsilon}\right\}\left(\frac{|\nu_f(r)|}{r}\right)^{k-1} |1+o(1)| \\ \Rightarrow \nu_f(r) &\leq kr\exp_p\left\{\left(\log_{q-1} r\right)^{\sigma_{[p,q]}(A_0)+\varepsilon}\right\} \\ &= kr\exp_{p+1}\left\{\left(\sigma_{[p,q]}(A_0)+\varepsilon\right)\left(\log_q r\right)\right\}\end{aligned}$$

$$\Rightarrow \log_{p+1}\nu_f(r) \leq \left(\sigma_{[p,q]}(A_0)+\varepsilon\right)\left(\log_q r\right) + \log_{p+1} kr. \tag{24}$$

Because $\varepsilon > 0$ is arbitrary, from Lemma 1 and (24), we deduce

$$\overline{\lim_{r\to\infty}}\frac{\log_{p+1}\nu_f(r)}{\log_q r} \leq \sigma_{[p,q]}(A_0). \tag{25}$$

Lemma 5 and (25) imply that

$$\sigma_{[p+1,q]}(f) \leq \sigma_{[p,q]}(A_0).$$

Because $\mu_{[p,q]}(A_0) = \sigma_{[p,q]}(A_0)$, we have

$$\sigma_{[p+1,q]}(f) \leq \sigma_{[p,q]}(A_0) = \mu_{[p,q]}(A_0). \tag{26}$$

Consequently, by (19) and (26),

$$\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0) = \mu_{[p,q]}(A_0).$$

This proves the theorem. □

Proof of Theorem 6. Let us rewrite Equation (2) as

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z)\frac{f^{(k-1)}(z)}{f(z)} + \ldots + A_1(z)\frac{f'(z)}{f(z)} + A_0(z)\right). \tag{27}$$

If f has got a zero at z_0 of order β ($\beta > k$), and if $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ are all of them analytic at z_0, then F has obtained a zero at z_0 of order $\beta - k$. Therefore

$$N\left(r,\frac{1}{f}\right) \leq k\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{F}\right) + \sum_{j=0}^{k-1} N(r,A_j). \tag{28}$$

The classical lemma on logarithmic derivative and (27) bring out that the inequality

$$m\left(r,\frac{1}{f}\right) \leq m\left(r,\frac{1}{F}\right) + \sum_{j=0}^{k-1} m(r,A_j) + O(\log rT(r,f)), \tag{29}$$

holds for $r \notin E$, E being a set of finite linear measure.

Analogously from (28) and (29), it follows that the inequality

$$\begin{aligned}T(r,f) &= T\left(r,\frac{1}{f}\right) + O(1) = N\left(r,\frac{1}{f}\right) + m\left(r,\frac{1}{f}\right) + O(1) \\ &\leq k\overline{N}\left(r,\frac{1}{f}\right) + T(r,F) + \sum_{j=0}^{k-1} T(r,A_j) + O(\log rT(r,f)).\end{aligned} \tag{30}$$

holds for $r \notin E$, where E is a set of r of finite linear measure.

Suppose that

$$\overline{\lim_{r\to\infty}} \frac{\sum\limits_{j=0}^{k-1} T(r, A_j) + T(r, F)}{T(r, f)} = \delta < c < 1.$$

Subsequently, for sufficiently large r and for any given $\varepsilon, 0 < \varepsilon < c - \delta$, it holds

$$\sum_{j=0}^{k-1} T(r, A_j) + T(r, F) \le (\delta + \varepsilon) T(r, f) < cT(r, f). \tag{31}$$

Substituting (31) into (30), we obtain that

$$\begin{aligned} T(r, f) &\le k\overline{N}\left(r, \frac{1}{f}\right) + cT(r, f) + O(\log rT(r, f)) \\ \Rightarrow (1 - c)T(r, f) &\le k\overline{N}\left(r, \frac{1}{f}\right) + O(\log rT(r, f)) \end{aligned}$$

$$\Rightarrow T(r, f) \le \frac{2k}{1 - c}\overline{N}\left(r, \frac{1}{f}\right) + O(\log rT(r, f)), \quad (r \notin E). \tag{32}$$

First, take logarithm and divide by $\log_q r$ in both side of (32) and then take limit $r \to \infty$, we can obtain that

$$\overline{\lambda}_{[p+1,q]}(f) \ge \sigma_{[p+1,q]}(f).$$

Definitions make immediate the reverse inequalities

$$\overline{\lambda}_{[p+1,q]}(f) \le \lambda_{[p+1,q]}(f) \le \sigma_{[p+1,q]}(f).$$

Therefore,

$$\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f).$$

This proves the theorem. □

5. Discussion

Keeping the results already established in mind, one may explore, for analogous theorems in which the coefficients of differential equations are bi-complex valued, entire and meromorphic functions of $[p, q]$-order, with p and q being any two integers with $p \ge q \ge 1$. Further, the case in which the coefficients of differential equations generated by analytic functions of $[p, q]$-order in the unit disc may be considered by future researchers in this area. Moreover, the investigation of the problems under the flavor of $[p, q]$ index pair of both complex and bi-complex valued entire and meromorphic functions is still a virgin domain for the new researchers.

6. Open Problem

The methodologies that were adopted in this paper can be treated algebraically under the flavor of bicomplex numbers, and these may be regarded as an Open Problem to the future workers of this branch.

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