

Article

Four-Fold Formal Concept Analysis Based on Complete Idempotent Semifields

Francisco José Valverde-Albacete [†] and Carmen Peláez-Moreno ^{*,†}

Department of Signal Theory and Communications, Universidad Carlos III de Madrid, 28911 Leganés, Spain; fva@tsc.uc3m.es

* Correspondence: carmen@tsc.uc3m.es; Tel.: +34-91-624-8771

† These authors contributed equally to this work.

Abstract: Formal Concept Analysis (FCA) is a well-known supervised boolean data-mining technique rooted in Lattice and Order Theory, that has several extensions to, e.g., fuzzy and idempotent semirings. At the heart of FCA lies a Galois connection between two powersets. In this paper we extend the FCA formalism to include all four Galois connections between four different semivectors spaces over idempotent semifields, at the same time. The result is $\bar{\mathcal{K}}$ -four-fold Formal Concept Analysis ($\bar{\mathcal{K}}$ -4FCA) where $\bar{\mathcal{K}}$ is the idempotent semifield biasing the analysis. Since complete idempotent semifields come in dually-ordered pairs—e.g., the complete max-plus and min-plus semirings—the basic construction shows dual-order-, row-column- and Galois-connection-induced dualities that appear simultaneously a number of times to provide the full spectrum of variability. Our results lead to a fundamental theorem of $\bar{\mathcal{K}}$ -four-fold Formal Concept Analysis that properly defines quadrilattices as 4-tuples of (order-dually) isomorphic lattices of vectors and discuss its relevance vis-à-vis previous formal conceptual analyses and some affordances of their results.

Keywords: formal concept analysis; extended formal concept analysis; complete idempotent semifields; schedule algebra; max-plus algebra; tropical algebra



Citation: Valverde-Albacete, F.J.; Peláez-Moreno, C. Four-Fold Formal Concept Analysis Based on Complete Idempotent Semifields. *Mathematics* **2021**, *9*, 173. <https://doi.org/10.3390/math9020173>

Academic Editor: Vassilis G. Kaburlasos
Received: 30 November 2020
Accepted: 12 January 2021
Published: 15 January 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

This paper tries to show how Linear Algebra over idempotent semifields in general—and Galois connections in particular—contributes to the program of Lattice Computing (LC) [1] and its attempt to provide “an evolving collection of tools and methodologies that process lattice-ordered data”, as a means of establishing an information processing paradigm belonging to the wider field of Computational Intelligence [2], with an explicit aim at modelling Cyber-Physical Systems [3].

Recall that an idempotent semifield is an idempotent semiring—therefore with a natural order associated to it—with a multiplicative group structure where the unit element is not the maximum ([4], Section 2 and references therein). This is an algebra that resembles a standard field but whose additive structure is idempotent and so it has no additive inverse, and for that same reason exhibits strong order-related properties [5].

In fact, a complete idempotent semifield is already a complete lattice. A companion paper [6] and the present one are examples of how to “compute *in* lattices”, in the sense that they are analogs of operations and computations in standard algebras, e.g., fields, but mingle in those the requirements and the affordances of using a complete lattice as a basic algebra.

At the same time—as this paper will show—Linear Algebra constructs over fields produce analogues over idempotent semifields that give rise to (complete) lattices, so we almost always end up “computing *with* lattices”.

This idea is strengthened by the fact that the ternary semiring in Example 1 that subsumes the boolean semiring is embedded in every complete idempotent semifield,

enabling a direct generalization of boolean constructions from Discrete Algebra—sets, graphs, formal power series, etc.

Example 1. *The smallest complete idempotent semifield is the boolean semiring $\mathbb{2}$. The next smallest is $\mathfrak{B} = \langle \{\perp, e, \top\}, \dot{\oplus}, \dot{\otimes}, \cdot^{-1}, \perp, e \rangle$ with carrying set $\{\perp, e, \top\}$, order $\perp < e < \top$ and operations:*

$$\begin{array}{c|ccc} \dot{\oplus} & \perp & e & \top \\ \hline \perp & \perp & e & \top \\ e & e & e & \top \\ \top & \top & \top & \top \end{array} \quad
 \begin{array}{c|ccc} \dot{\otimes} & \perp & e & \top \\ \hline \perp & \perp & \perp & \perp \\ e & \perp & e & \top \\ \top & \perp & \top & \top \end{array} \quad
 \begin{array}{c|c} x & x^{-1} \\ \hline \perp & \top \\ e & e \\ \top & \perp \end{array} \tag{1}$$

where \perp is the zero element and e the unit. The semifield of Booleans $\mathbb{2}$ can be embedded in this semiring restricting the domain of the operations to $\{\perp, \top\}$. Its dual $\mathfrak{B}^{-1} = \langle \{\perp, e, \top\}, \dot{\oplus}, \dot{\otimes}, \cdot^{-1}, \top, e \rangle$ is the semifield over the same carrying set, with the same inversion, dual order $\perp <^d e <^d \top$, and operations:

$$\begin{array}{c|ccc} \dot{\oplus} & \perp & e & \top \\ \hline \perp & \perp & \perp & \perp \\ e & \perp & e & e \\ \top & \perp & e & \top \end{array} \quad
 \begin{array}{c|ccc} \dot{\otimes} & \perp & e & \top \\ \hline \perp & \perp & \perp & \top \\ e & \perp & e & \top \\ \top & \top & \top & \top \end{array} \tag{2}$$

The dotted notation was introduced by Moreau [7]. In general, it can be conceived as a mnemonic to remember where the multiplication “point towards” $\perp \dot{\otimes} \top = \perp$ and $\perp \dot{\oplus} \top = \top$. \square

Notice that the usual positive semifields—the positive reals, the tropical semifield and the max-plus algebra—are incomplete—barring the Boolean semifield $\mathbb{2}$ —lacking an inverse for the bottom element in the order. To complete an idempotent semifield \mathcal{K} we endow it with this maximum or top, \top , and declare this the inverse of the bottom $\top = \perp^{-1}$. Actually, the top completion provides not one but a pair of dually-ordered semifields $(\overline{\mathcal{K}}, \underline{\mathcal{K}})$ with the inversion in the multiplicative group acting as the duality inducing operator. This entails that expressions will have two (differently completed) multiplications, additions, etc., causing a notational problem that has not yet been agreed upon in the community (see the Appendix A in [6]). We adopt in this paper the dotted notation used in Examples 1 and 2, with the convention that any undotted notation refers to incomplete semifields.

Example 2 (The minimax algebra). *One of the best known examples of dual pairs of completed semifields is $(\overline{\mathbb{R}}_{\max,+}, \overline{\mathbb{R}}_{\min,+})$, the minimax algebra [8]. This is composed of:*

- *The direct semifield $\overline{\mathbb{R}}_{\max,+}$, the completed max-plus algebra [9]—also morphological algebra [10], lattice algebra [11], or polar semifield—*

$$\overline{\mathbb{R}}_{\max,+} = \langle \mathbb{R} \cup \{-\infty, \infty\}, \max, +, \cdot^{-1}, -\infty, 0, \infty \rangle$$

- *Its dual semifield is $\overline{\mathbb{R}}_{\min,+}$, the (completed) min-plus algebra, or tropical semiring.*

$$\overline{\mathbb{R}}_{\min,+} = \langle \mathbb{R} \cup \{-\infty, \infty\}, \min, \dot{+}, \cdot^{-1}, \infty, 0, -\infty \rangle$$

In minimax algebra, the notation solves problems like $-\infty + \infty = -\infty$ and $-\infty \dot{+} \infty = \infty$. \square

This, as well, has historical reasons that have led some researchers to postulate Linear Algebra over the max-plus semifield as the “algebra of combinatorics” [12].

1.1. Galois Connections between Idempotent Semimodules and Formal Concept Analysis

Idempotent semimodules—also known as idempotent spaces [13]—are the products of idempotent semifields analogously to vector spaces over the fields \mathbb{R} and \mathbb{C} [6] (§ 2.1). When \mathcal{K} is an idempotent semifield, Cohen and collaborators [14] (Theorem 42, adapted) proved that a matrix $R \in \mathcal{K}^{g \times m}$ defines a Galois connection between the complete idempotent semimodules \mathcal{K}^g and \mathcal{K}^m mediated by the scalar product $\langle \cdot \mid R \mid \cdot \rangle : \mathcal{K}^g \times \mathcal{K}^m \rightarrow \mathcal{K}$, with $(x, y) \mapsto \langle x \mid R \mid y \rangle = x^{\otimes} \otimes R \otimes y^{-1}$, where the matrix–vector operations arise naturally from those of the semifield, $(\cdot)^{-1}$ is the entry-wise inversion and $x^{\otimes} = (x^{-1})^{\top}$. The dual adjuncts or polars of the connection are:

$$x_{R,\varphi}^{\uparrow} = \vee \{y \in \mathcal{K}^m \mid \langle x \mid R \mid y \rangle \leq \varphi\} \quad y_{R,\varphi}^{\downarrow} = \vee \{x \in \mathcal{K}^g \mid x^{\otimes} \otimes R \otimes y^{-1} \leq \varphi\} \quad (3)$$

where φ is any invertible, e.g., non-null element of the semifield, and the joins are taken with respect to the natural order in the semifield. Note that this construction evidences the existence of many lattices relevant to a linear form over idempotent spaces: the ambient spaces, the images, etc.

In prior works [15–17], we studied the applicability of such results to provide a generalization of Formal Concept Analysis (FCA) for matrices with real entries. FCA is an attempt at re-structuring lattice theory with the explicit aim of “... promoting better communication between lattice theorists and potential users of lattice theory” [18]. Its canonical exposition is [19], but more succinctly its essence can be captured in a single, extensive theorem:

Theorem 1 (Fundamental theorem of FCA). *Let G be a set of formal objects, M a set of formal attributes and (G, M, I) be a formal context with $I \in \mathcal{2}^{g \times m}$. Then:*

1. **The context analysis phase.**

(a) *The polar operators $\cdot^{\uparrow} : \mathcal{2}^G \rightarrow \mathcal{2}^M$ and $\cdot^{\downarrow} : \mathcal{2}^M \rightarrow \mathcal{2}^G$.*

$$A^{\uparrow} = \{m \in M \mid \forall g \in A, gIm\} \quad B^{\downarrow} = \{g \in G \mid \forall m \in B, gIm\}$$

form a Galois connection $(\cdot^{\uparrow}, \cdot^{\downarrow}) : \mathcal{2}^G \rightleftarrows \mathcal{2}^M$ whose formal concepts are the pairs (A, B) of closed elements such that $A^{\uparrow} = B \Leftrightarrow A = B^{\downarrow}$ whence the set of formal concepts is

$$\mathfrak{B}(G, M, I) = \{(A, B) \in \mathcal{2}^G \times \mathcal{2}^M \mid A^{\uparrow} = B \Leftrightarrow A = B^{\downarrow}\}.$$

(b) *Formal concepts are partially ordered with the hierarchical order*

$$(A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 \Leftrightarrow B_1 \supseteq B_2.$$

and the set of formal concepts with this order $\langle \mathfrak{B}(G, M, I), \leq \rangle$ is a complete lattice $\underline{\mathfrak{B}}(G, M, I)$ called the concept lattice of (G, M, I) .

(c) *In $\underline{\mathfrak{B}}(G, M, I)$ infima and suprema are given by:*

$$\bigwedge_{t \in T} (A_t, B_t) = \left(\bigcap_{t \in T} A_t, \left(\bigcap_{t \in T} A_t \right)^{\uparrow} \right) \quad \bigvee_{t \in T} (A_t, B_t) = \left(\left(\bigcap_{t \in T} B_t \right)^{\downarrow}, \bigcap_{t \in T} B_t \right) \quad (4)$$

(d) *The basic functions $\bar{\gamma} : G \rightarrow V$ and $\bar{\mu} : M \rightarrow V$*

$$g \mapsto \bar{\gamma}(g) = (\{g\}^{\uparrow\downarrow}, \{g\}^{\uparrow}) \quad m \mapsto \bar{\mu}(m) = (\{m\}^{\downarrow}, \{m\}^{\downarrow\uparrow})$$

are mappings such that $\bar{\gamma}(G)$ is supremum-dense in $\underline{\mathfrak{B}}(G, M, I)$, $\bar{\mu}(M)$ is infimum-dense in $\underline{\mathfrak{B}}(G, M, I)$.

2. **The context synthesis phase.**

- (a) A complete lattice $\mathbb{L} = \langle L, \leq \rangle$ is isomorphic to (read “can be built as”) the concept lattice $\mathfrak{B}(G, M, I)$ if and only if there are mappings $\bar{\gamma} : G \rightarrow L$ and $\bar{\mu} : M \rightarrow L$ such that
 - $\bar{\gamma}(G)$ is supremum-dense in \mathbb{L} , $\bar{\mu}(M)$ is infimum-dense in \mathbb{L} , and
 - gIm is equivalent to $\bar{\gamma}(g) \leq \bar{\mu}(m)$ for all $g \in G$ and all $m \in M$.
- (b) In particular, consider the doubling context of \mathbb{L} , $K(\mathbb{L}) = (L, L, \leq)$, and the standard context of \mathbb{L} , $S(\mathbb{L}) = (J(\mathbb{L}), M(\mathbb{L}), \leq)$ where $J(\mathbb{L})$ and $M(\mathbb{L})$ are the sets of join- and meet-irreducibles, respectively, of \mathbb{L} , then

$$\mathfrak{B}(K(\mathbb{L})) \cong \mathbb{L} \cong \mathfrak{B}(S(\mathbb{L})).$$

Proof. See [19] [Ch. 1], or the slow buildup of results in [20] [Ch. 7]. \square

Note that FCA can both be understood as a data analysis technique for Boolean tables—the analysis phase—and as a synthesis technique for a Boolean matrix that “respects” the order of some lattice—the synthesis phase. Almost 40 years of developments has uncovered a treasure trove of results that span applied mathematics [19], computer and data science [21,22] (See also the proceedings of the conference series ICFCA, International Conference on Formal Concept Analysis, and CLA, Concept Lattices and their Applications).

For the LC programme, the crucial synthesis half of the theorem asserts that concept lattices have universal representation capabilities for complete lattices. Otherwise said, any instantiation of the LC paradigm for discrete or boolean data is subject to being operated and interpreted using the results of FCA. In particular, every encoding of information into a lattice can equivalently be represented by a formal context, and vice-versa.

Importantly, in [23] [§ 3.8] it was proven that construction (3) obtains the Galois Connection at the heart of FCA when the complete idempotent semifield is \mathfrak{B} , $\varphi = e$, and the entries in the matrix are restricted to $\{\perp, \top\}$. This was built on results from [17] that defined precisely an extension to FCA using idempotent-semifield valued formal contexts, to be explained next.

1.2. \mathcal{K} -Formal Concept Analysis as Linear Algebra over Idempotent Semifields

When $\bar{\mathcal{K}}$ is a complete idempotent semifield, an analogue to FCA for matrices with entries in a semifield $\bar{\mathcal{K}}$ -Formal Concept Analysis ($\bar{\mathcal{K}}$ -FCA) was proposed and developed in [15,16]. This line of work culminated in the realization and proof that although the motivating notions for the generalization were order-theoretic, its operation could be entirely algebraic if done with completed idempotent semifields [24,25]. These developments were extensively based in previous work in the (completed and incomplete) tropical and schedule algebras [8,9,13,14,26–29]. It was instantiated with the minimax algebra and applied to, for example, the analysis of genomic [30], and positive data [31], or even Exploratory Data Analysis [32]. Note that a similar approach to extending FCA had previously been developed for matrices with entries in a fuzzy-semiring, today grouped under the name FCA in a fuzzy setting [33–35], but fuzzy semirings are not idempotent semifields, except for $\mathfrak{2}$ itself.

The next proposition, coming from [14] for finite values of the threshold parameter φ , was extended to deal with the non-finite cases in [25], and to include the notion of bikernel:

Proposition 1 (Galois connections between complete idempotent semimodules). *Let $\bar{\mathcal{K}}$ be a complete idempotent semifield and $R \in \mathcal{K}^{g \times m}$ a matrix with values in its carrier set. Consider the vector spaces $\mathcal{X} = \bar{\mathcal{K}}^g$ and $\mathcal{Y} = \bar{\mathcal{K}}^m$, and an element $\varphi \in \bar{\mathcal{K}}$. Then the bracket $\langle x \mid R \mid y \rangle_{\bar{\mathcal{K}}}^{\varphi} =$*

$x^{\otimes} \dot{\otimes} R \dot{\otimes} y^{-1}$ induces a Galois connection $(\cdot^{\uparrow}_{R,\varphi}, \cdot^{\downarrow}_{R,\varphi}) : \mathcal{X} \rightleftarrows \mathcal{Y}$ between the spaces through the poles

$$x^{\uparrow}_{R,\varphi} = R^{\top} \dot{\otimes} x^{-1} \dot{\otimes} \varphi^{-1} \qquad y^{\downarrow}_{R,\varphi} = R \dot{\otimes} y^{-1} \dot{\otimes} \varphi^{-1} \tag{5}$$

whose composition generate closure operators:

$$\pi_{R,\varphi}(x) = R \dot{\otimes} (R^{\otimes} \dot{\otimes} x \dot{\otimes} \varphi) \dot{\otimes} \varphi^{-1} \qquad \pi_{R^{\top},\varphi}(y) = R^{\top} \dot{\otimes} (R^{-1} \dot{\otimes} y \dot{\otimes} \varphi) \dot{\otimes} \varphi^{-1} \tag{6}$$

which define two bijective sets, \mathfrak{B}_G^{φ} and \mathfrak{B}_M^{φ} over which the polars are dual order bijections and the closures are identities, respectively.

$$\mathfrak{B}_G^{\varphi} = (Y)^{\downarrow}_{R,\varphi} = \pi_{R,\varphi}(X) \qquad \mathfrak{B}_M^{\varphi} = (X)^{\uparrow}_R = \pi_{R^{\top},\varphi}(Y) \tag{7}$$

The following observations can readily be made:

- This proposition only provides the Galois connection for $\overline{\mathcal{K}}$ -FCA equivalent to the analysis part of Theorem 1. The synthesis part is only partially supported by the above-mentioned procedure of embedding FCA into $\overline{\mathcal{K}}$ -FCA by means of restricting the entries of the matrix to only the extreme values. We would like a more general result than this.
- In [24] it was proven that $\overline{\mathcal{K}}$ -FCA is best-understood in terms of the linear algebra of the semivector spaces over $\overline{\mathcal{K}}$ —or semimodules—that are the isomorphic sets of (7). The proof of the proposition uses both algebra over $\overline{\mathcal{K}}$ and $\underline{\mathcal{K}}$, so it is actually a mixture of algebras that does the trick of reducing FCA to linear algebra.
- For concrete algebras, we typically consider that the below-dotted reference is for semifields whose order aligns with the usual order in (\mathbb{R}, \leq) , if available, and this is the one we call $\overline{\mathcal{K}}$. Changing this semifield to its order-dual $\underline{\mathcal{K}}$ is what we call changing the bias of the analysis, and this procedure should obtain a $\underline{\mathcal{K}}$ -FCA.

The next subsection presents a further consideration.

1.3. The 4-Fold Connection

It is very well known that any matrix $R \in \mathbb{2}^{g \times m}$ defines actually four types of connections [36–38] (see Appendix A). Note that the other three connections are not considered to be standard FCA, and indeed are used for many other applications where adjunctions are the focus, such as rough set analysis [37] and knowledge graphs [39,40], or dual Galois connections, such as the analysis of independence in terms of antichains [41].

With this parallel in mind, in research that even pre-dates the inception of FCA, Cuninghame–Green investigated the four semimodules associated to a matrix with entries in the minimax algebra and found a number of results that parallel the standard ones [8]. This was used to extend FCA for matrices in a complete idempotent semifield, in these four directions under the name Extended \mathcal{K} -Formal Concept Analysis [17,24,42]. Since the applications of the other three connections that are not strictly a Galois connection seem to be increasing in number, we believe it is time to reify their construction, and we call this result the $\overline{\mathcal{K}}$ -four-fold Formal Concept Analysis ($\overline{\mathcal{K}}$ -4FCA) (Section 3 and [23]). This work is an extension and completion of that.

Furthermore, if we consider that the matrix R takes value in the carrier set $\overline{\mathcal{K}}$ of a complete semimodule $\overline{\mathcal{K}}$, since inversion is an involution over this set, we may also consider that the manipulation of the matrix is done in the dual complete semifield $\underline{\mathcal{K}}$, whence the same constructions that obtained $\overline{\mathcal{K}}$ -4FCA allows us to obtain $\underline{\mathcal{K}}$ -four-fold Formal Concept Analysis ($\underline{\mathcal{K}}$ -4FCA), that is to say, four more connections that are dual to the previous in a very specific meaning of the word. This paper also explores this possibility and what it entails.

Note that this paper has a sibling paper that discusses the Fundamental Theorem of Linear Algebra over idempotent semifields and its relation to an idempotent Singular Value Decomposition and matrix reconstruction [6], but in this paper we restrict ourselves to the issues that are pertinent to all flavours of $\overline{\mathcal{K}}$ -FCA and their duals $\underline{\mathcal{K}}$ -FCA.

1.4. Reading Guide

This paper is an updated and completed version of [23] including material from [43]. We investigate in it:

- (1) the four-fold Galois connections related to an idempotent semifield-valued matrix;
- (2) what are the implications of a change of bias, that is, using $\underline{\mathcal{K}}$ for the analysis;
- (3) the relation to LC, in general, and to Formal Concept Analysis, in particular, and
- (4) the relationship of these techniques with other types of FCA.

For that purpose, we refer the reader to [6] (Section 2.1) for an introduction to linear algebra over idempotent semifields and their vector spaces. We take a brief look at their homomorphisms (Section 2.1) and then we present our results in Section 3. We start by presenting four-fold connections in detail (Section 3.1) going over their adjuncts as specialized types of homomorphisms (Sections 3.1.2–3.1.4), bikernels (Section 3.1.1), dualities (Section 3.1.5), leading into our presentation of $\overline{\mathcal{K}}$ -four-fold Formal Concept Analysis in Section 3.2. This evolves from the framing of the construction (Section 3.2.1) through the definition of quadrilattices and quadriconcepts (Section 3.2.2) and a Fundamental Theorem for $\overline{\mathcal{K}}$ -four-fold Formal Concept Analysis closely following that of FCA (Section 3.2.3). Given the relevance of duality in our framework introduced by the use pairs of dual completed idempotent semifields, we are led to discuss such issues as dualities (Section 3.2.4) reduction and reconstruction of contexts (Section 3.2.5) in an effort to understand the basics and interpretation of $\overline{\mathcal{K}}$ -4FCA. A brief discussion (Section 4) in the context of LC closes the paper.

2. Linear Algebra over Complete Idempotent and Positive Semifields

An introduction to the linear algebra over complete idempotent semifields is already available in [6] (Section 2.1). Recall that a semimodule over a semiring \mathcal{K} is the generalization of a module over a ring [6] (Section 2.1). In this section, we only clarify the universal algebra foundations of the subsemimodules induced by a homomorphism between complete semimodules.

2.1. Homomorphisms of Complete \mathcal{K} -semimodules

Let \mathcal{K} be a complete idempotent semifield. A map $F : \mathcal{X} \rightarrow \mathcal{Y}$ from a \mathcal{K} -semimodule \mathcal{X} to a \mathcal{K} -semimodule \mathcal{Y} is linear or a homomorphism of \mathcal{K} -semimodules, $F \in Hom(\mathcal{X}, \mathcal{Y})$, if it is additive and homogeneous

$$\begin{aligned} F(x_1 \oplus x_2) &= F(x_1) \oplus F(x_2) & x_1, x_2 \in \mathcal{X} \\ F(\lambda \odot x) &= \lambda \odot F(x) & \lambda \in \mathcal{K}, x \in \mathcal{X} \end{aligned}$$

so that $F(\epsilon_{\mathcal{X}}) = \epsilon_{\mathcal{Y}}$. The following notions may be defined with a set-theoretic flavour and the details supplied with universal-algebraic consideration:

Definition 1 (bikernel of a mapping [27]). *Let $F : X \rightarrow Y$ be a mapping. Then:*

- *The bikernel of F , $KER(F)$ is the equivalence:*

$$KER(F) = \{(x_1, x_2) \in X^2 \mid F(x_1) = F(x_2)\} \tag{8}$$

with blocks $[x]_{KER(F)} = F^{-1}(F(x))$ so that $KER(F) = \{F^{-1}(F(x)) \mid x \in X\}$.

- Its orthogonal (set) is

$$\text{KER}(F)^\top = \{y \in Y \mid y = F(x_1) = F(x_2), (x_1, x_2) \in \text{KER}(F)\} = \text{IM}(F) \tag{9}$$

Note that:

- In this paper we use kernel to refer to monotone, contractive idempotent endomorphisms of ordered sets in Appendix A. This justifies the new name of “bikernel”.
- The equivalence classes have no uniform cardinality. Some may be unitary while others have infinite elements.
- The notion of “orthogonality” does not add anything special to the concept of “image” in this set-theoretic context, but it will for algebras.

We can relate these notions with the adequate instance of the First Isomorphism Theorem from Universal Algebra [44] (Chapter II).

Theorem 2. Let \mathcal{K} be a complete idempotent semifield and let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a homomorphism of \mathcal{K} -semimodules. Then:

- $\text{IM}(F)$ is a \mathcal{K} -subsemimodule of \mathcal{Y} ,
- $\text{KER}(F)$ is a congruence on \mathcal{X} , that is, an equivalence on \mathcal{X} that is also a \mathcal{K} -semimodule of \mathcal{X}^2 , and
- $\text{IM}(F)$ is isomorphic to $\mathcal{X} / \text{KER}(F)$.

If $\{u_i\}_{i \in I}$ is a basis for (free) \mathcal{X} , there is a unique homomorphism $F : \mathcal{X} \rightarrow \mathcal{Y}$ for a family $\{y_j\}_{j \in J}$ of elements in a semimodule \mathcal{Y} such that $F(u_i) = y_j$ [45]. In such case, the homomorphism $F : \mathcal{K}^g \rightarrow \mathcal{K}^m$ can be written as $F(x) = A \otimes x$, where $A \in \mathcal{K}^{m \times g}$ is a matrix. This is the basic representation of homomorphisms for the rest of this paper.

3. $\overline{\mathcal{K}}$ -Four-Fold Formal Concept Analysis

In this section we first introduce the order theory-motivated concept of the four-fold connection, and then we cast $\overline{\mathcal{K}}$ -four-fold Formal Concept Analysis ($\overline{\mathcal{K}}$ -4FCA) into it. Some results from [15–17,23–25,42] are here summarized with new results woven into the exposition along with some proofs of results that were left implicit.

3.1. The Four-Fold Galois Connection over Complete Idempotent \mathcal{K} -Semimodules

Galois connections are generally ubiquitous in order theory and crucial in FCA (see Appendix A). In idempotent semifields, they generalize the concept of linear transformations between spaces and their inverses to a certain extent, since given a matrix $R \in \mathcal{K}^{g \times m}$ there are four possible types of Galois connections between the spaces associated to the matrix due to scalar products in the semifield $\overline{\mathcal{K}}$. The basic construction was introduced in Proposition 1, as a Galois connection, without a proof. We re-state it now along with the other three connections to show a number of issues that emerge time and again when dealing with Galois connections in idempotent semifields.

Theorem 3 ($\overline{\mathcal{K}}$ -four-fold Galois connection). Let $\overline{\mathcal{K}}$ be a complete idempotent semifield, and consider the vector spaces $\mathcal{X} = \overline{\mathcal{K}}^g$ and $\mathcal{Y} = \overline{\mathcal{K}}^m$, a matrix $R \in \overline{\mathcal{K}}^{g \times m}$ and a scalar $\varphi \in \overline{\mathcal{K}}$. Then:

1. The bracket $\langle x \mid R \mid y \rangle_{\overline{\mathcal{K}}}^{\text{OI}} = x^{\otimes} \dot{\otimes} R \dot{\otimes} y^{-1}$ induces a Galois connection $(\cdot \uparrow_{R, \varphi}, \cdot \downarrow_{R, \varphi}) : \mathcal{X} \rightleftarrows \mathcal{Y}$ between the spaces through the polar maps: Equation (5), whose composition generate closure operators: Equation (6), which define two bijective sets, the system of φ -extents $\mathfrak{B}_{\mathcal{X}}^{\varphi}$ and the system of φ -intents $\mathfrak{B}_{\mathcal{Y}}^{\varphi}$ over which the polars are dual order bijections and the closures are identities, respectively. Equation (7).

2. The bracket $\langle x \mid R \mid y \rangle_{\mathcal{K}}^{\text{IO}} = x^{\top} \dot{\otimes} R \dot{\otimes} y$ induces a co-Galois connection $(\cdot^{\uparrow}_{R^{-1},\varphi}, \cdot^{\downarrow}_{R^{-1},\varphi}) : \mathcal{X} \rightleftarrows \mathcal{Y}$ between the scaled spaces through the co-polar maps:

$$x^{\uparrow}_{R^{-1},\varphi} = R^{\otimes} \dot{\otimes} x^{-1} \dot{\otimes} \varphi \qquad y^{\downarrow}_{R^{-1},\varphi} = R^{-1} \dot{\otimes} y^{-1} \dot{\otimes} \varphi \qquad (10)$$

whose composition generate interior operators or kernels:

$$\kappa_{R^{-1},\varphi}(x) = R^{-1} \dot{\otimes} (R^{\top} \dot{\otimes} x \dot{\otimes} \varphi^{-1}) \dot{\otimes} \varphi \qquad \kappa_{R^{\otimes},\varphi}(y) = R^{\otimes} \dot{\otimes} (R \dot{\otimes} y \dot{\otimes} \varphi^{-1}) \dot{\otimes} \varphi \qquad (11)$$

which define two bijective sets the system of φ -neighbourhoods of objects or φ -co-extents \mathfrak{N}_X^{φ} and φ -neighbourhoods of attributes or φ -co-intents $\mathfrak{N}_{GM}^{\varphi}$ over which the co-polar maps are dual-order bijections and the kernels the identities, respectively.

$$\mathfrak{N}_X^{\varphi} = (Y)^{\downarrow}_{R^{-1},\varphi} = \kappa_{R^{-1},\varphi}(X) \qquad \mathfrak{N}_M^{\varphi} = (X)^{\uparrow}_{R^{-1},\varphi} = \kappa_{R^{\otimes},\varphi}(Y) \qquad (12)$$

3. The bracket $\langle x \mid R \mid y \rangle_{\mathcal{K}}^{\text{OO}} = x^{\otimes} \dot{\otimes} R \dot{\otimes} y$ induces a left adjunction $((\cdot)^{\exists}_{R,\varphi}, (\cdot)^{\forall}_{R,\varphi}) : \mathcal{X} \rightleftarrows \mathcal{Y}$ between the scaled spaces through the left adjunct pair of maps:

$$x^{\exists}_{R,\varphi} = R^{\otimes} \dot{\otimes} x \dot{\otimes} \varphi \qquad y^{\forall}_{R,\varphi} = R \dot{\otimes} y \dot{\otimes} \varphi^{-1} \qquad (13)$$

whose compositions are the closure of extents and kernel of co-intents:

$$\pi_{R,\varphi}(x) = (x^{\exists}_{R,\varphi})^{\forall}_{R,\varphi} \qquad \kappa_{R^{\otimes},\varphi}(y) = (y^{\forall}_{R,\varphi})^{\exists}_{R,\varphi} \qquad (14)$$

which define an order bijection between the system of φ -extents \mathfrak{B}_X^{φ} and φ -co-intents \mathfrak{N}_Y^{φ} over which the closure of extents and kernel of co-intents are identities.

$$\mathfrak{B}_X^{\varphi} = (Y)^{\forall}_{R,\varphi} = \pi_{R,\varphi}(X) \qquad \mathfrak{N}_Y^{\varphi} = (X)^{\exists}_{R,\varphi} = \kappa_{R^{\otimes},\varphi}(Y) \qquad (15)$$

4. The bracket $\langle x \mid R \mid y \rangle_{\mathcal{K}}^{\text{II}} = x^{\top} \dot{\otimes} R \dot{\otimes} y^{-1}$ induces an adjunction on the right $((\cdot)^{\forall}_{R^{\top},\varphi}, (\cdot)^{\exists}_{R^{\top},\varphi}) : \mathcal{X} \rightleftarrows \mathcal{Y}$ between the scaled spaces through the pair of adjunct maps:

$$x^{\forall}_{R^{\top},\varphi} = R^{\top} \dot{\otimes} x \dot{\otimes} \varphi^{-1} \qquad y^{\exists}_{R^{\top},\varphi} = R^{-1} \dot{\otimes} y \dot{\otimes} \varphi \qquad (16)$$

whose compositions are the kernel of co-extents and the closure of extents:

$$\kappa_{R^{-1},\varphi}(x) = (x^{\forall}_{R^{\top},\varphi})^{\exists}_{R^{\top},\varphi} \qquad \pi_{R^{\top},\varphi}(y) = (y^{\exists}_{R^{\top},\varphi})^{\forall}_{R^{\top},\varphi} \qquad (17)$$

which define another bijection between the system of φ -co-extents \mathfrak{N}_X^{φ} and φ -intents \mathfrak{B}_Y^{φ} over which the kernel of co-extents and the closure of intents are identities,

$$\mathfrak{N}_X^{\varphi} = (Y)^{\exists}_{R^{\top},\varphi} = \kappa_{R^{-1},\varphi}(X) \qquad \mathfrak{B}_Y^{\varphi} = (X)^{\forall}_{R^{\top},\varphi} = \pi_{R^{\top},\varphi}(Y) \qquad (18)$$

Proof. Statement 1 is included in Proposition 1, so we first prove that. By definition, $x^{\uparrow}_{R,\varphi} = \vee \{y \in \mathcal{Y} \mid \langle x \mid R \mid y \rangle_{\mathcal{K}}^{\text{OI}} \geq \varphi\}$. We solve for y in the inequality:

$$\langle x \mid R \mid y \rangle_{\mathcal{K}}^{\text{OI}} = x^{\otimes} \dot{\otimes} R \dot{\otimes} y^{-1} \geq \varphi \iff y^{-1} \geq \left(x^{\otimes} \dot{\otimes} R \right) \dot{\setminus} \varphi = R^{\otimes} \dot{\otimes} x \dot{\otimes} \varphi$$

whence, inverting, $y \leq R^{\top} \dot{\otimes} x^{-1} \dot{\otimes} \varphi^{-1}$. By Lemma 2 in [6] (§ 2.1.5), the right-hand side term is known to be the greatest solution to the equation $R^{-1} \dot{\otimes} y = x^{-1} \dot{\otimes} \varphi^{-1}$, whence it

is also the looked-for join. Row-column dually on $y_{R,\varphi}^\downarrow = \vee\{x \in \mathcal{X} \mid \langle x \mid R \mid y \rangle_{\overline{\mathcal{K}}}^{\text{OI}} \geq \varphi\}$, the rest of (5) follows.

Clearly, $x_R^\uparrow = R^\top \dot{\otimes} x^{-1} \dot{\otimes} \varphi^{-1} \geq_{\mathcal{Y}} y \iff x \leq_{\mathcal{X}} R \dot{\otimes} y^{-1} \dot{\otimes} \varphi^{-1} = y_R^\downarrow$ where the double implication was obtained by residuating the original inequality and inverting afterwards. This proves that the dual adjoints form a Galois connection: by the results in Appendix A, (6) and (7) follow.

The proof of statement 2 is order-dual to that of statement 1. While those of 3 and 4, also mutually order-dual, follow by just inverting one of the orders, either that for \mathcal{X} or that for \mathcal{Y} . \square

Note that the manner in which this proof proceeds will apply to many other proofs: one of the basic cases is proven, then another one follows by order-duality, then the other two, also mutually order-dual, follow by inverting one of the spaces in the first two cases. We will refer to this proof schema as four-fold GC-duality.

This four-fold connection can be seen as the diagram in Figure 1.

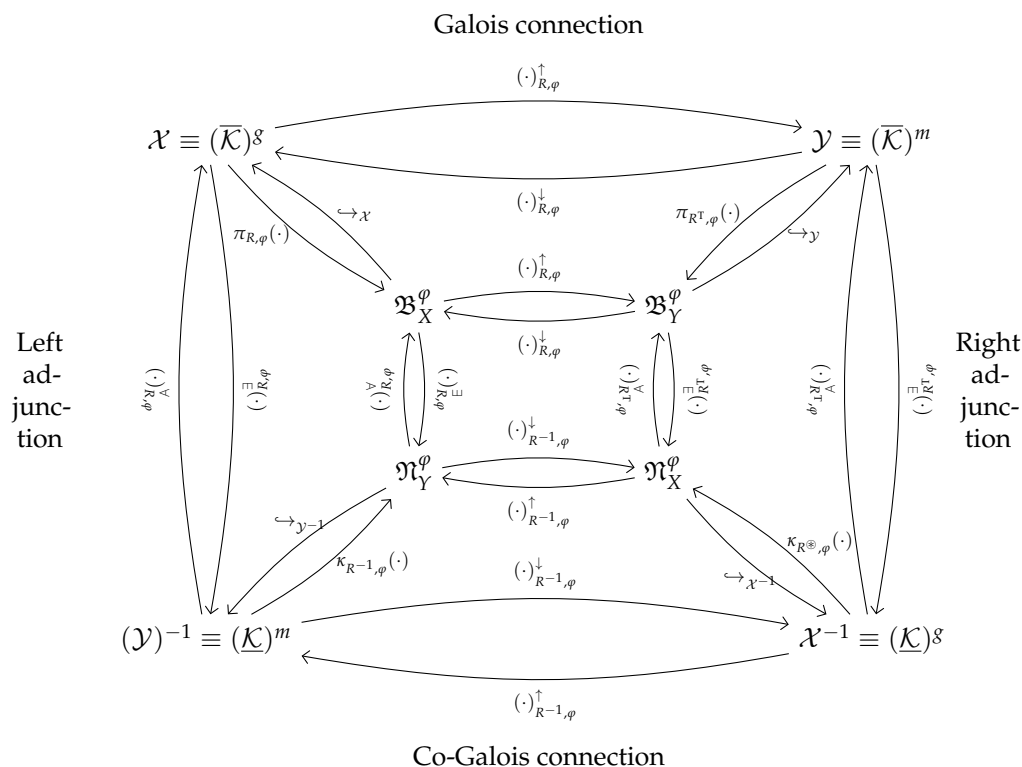


Figure 1. A domain diagram of the four-fold connection between the spaces implicit in a $\overline{\mathcal{K}}$ -valued matrix $R \in \overline{\mathcal{K}}^{g \times m}$ (adapted from [43]). Each of the labels stands in the quadrant where that kind of generalized Galois connection is realized in the diagram. Some maps, like the polar $(\cdot)_{R,\varphi}^\uparrow : \mathcal{X} \rightarrow \mathfrak{B}_Y^\varphi$ and its dual polar are missing for clarity.

This picture is a grand summary but provides very little detail about the structure inside the ranges of the adjoints, dual adjoints, closure and kernel functions, and the bikernels which we investigate in the following sections.

3.1.1. The Bikernels of the Four-fold Connection

In FCA, the quotient sets of the polars are important because they determine a fundamental limitation of the formal context: sets of objects (or attributes) between which it cannot distinguish. They are used mainly for reducing the context and the concept lattice ([19], notes to Chapter 1). Note that:

- In the semifield setting, the bikernels of \mathcal{K} -homomorphisms are already congruences of \mathcal{K} -semimodules. Therefore, the route provided by Definition 2 of [6] (§ 2.1.4) is clearly the way to define such congruences for adjunctions: from (13) we can see that one adjunct is a $\overline{\mathcal{K}}$ -homomorphism while the other is a $\underline{\mathcal{K}}$ -homomorphism. These adjuncts already appear in [46] (Remark 3) in explicit matrix form, without the scaling. The row-column dual approach leads to the right adjunction of (16). This entails that in the context of the 4-fold GC we will have, in general, both types of congruence, so from now on we distinguish them as $\overline{\mathcal{K}}$ -congruences and $\underline{\mathcal{K}}$ -congruences.
- However, for other types of Galois connections, the situation is not so clear. This approach would have to be researched anew for the Galois and co-Galois connection from (5) and (10), since these are join- and meet-inverting, respectively, hence neither $\overline{\mathcal{K}}$ - nor $\underline{\mathcal{K}}$ -homomorphisms. In particular, although bikernels of these other types of adjuncts are equivalence relations, their relation to their ambient semimodule seems more convoluted.

In our approach, we prefer to concentrate on bikernels for closure and kernel operators and this treatment has to be extended using Definition 1 rather than that of [6] (§ 2.1.6).

Definition 2. Let $R \in \overline{\mathcal{K}}^{s \times m}$. For the four-fold connection in Theorem 3 the bikernels induced by the closure operators are the equivalence relations:

$$\begin{aligned} \text{KER}(\pi_{R,\varphi}(\cdot)) &= \{(x_1, x_2) \in \mathcal{X}^2 \mid \pi_{R,\varphi}(x_1) = \pi_{R,\varphi}(x_2)\} \\ \text{KER}(\pi_{R^T,\varphi}(\cdot)) &= \{(y_1, y_2) \in \mathcal{Y}^2 \mid \pi_{R^T,\varphi}(y_1) = \pi_{R^T,\varphi}(y_2)\} \end{aligned} \tag{19}$$

and those of the kernel operators are:

$$\begin{aligned} \text{KER}(\kappa_{R^{-1},\varphi}(\cdot)) &= \{(x_1, x_2) \in \mathcal{X}^2 \mid \kappa_{R^{-1},\varphi}(x_1) = \kappa_{R^{-1},\varphi}(x_2)\} \\ \text{KER}(\kappa_{R^{\otimes},\varphi}(\cdot)) &= \{(y_1, y_2) \in \mathcal{Y}^2 \mid \kappa_{R^{\otimes},\varphi}(y_1) = \kappa_{R^{\otimes},\varphi}(y_2)\} \end{aligned} \tag{20}$$

Since dealing with the normalized spaces and the pivot φ is cumbersome, from now on we exclude them from the notation as mentioned beforehand.

Although the bikernels are the quotient sets of the closure and kernel operators in Theorem 3, the next lemma shows that they coincide with the bikernels induced by the adjuncts.

Lemma 1. In the conditions of Theorem 3, the bikernels of the adjuncts coincide with those of Definition 2.

$$\begin{aligned} \text{KER}(\pi_R(\cdot)) &= \text{KER}\left(\left(\cdot\right)_R^\uparrow\right) = \text{KER}\left(\left(\cdot\right)_R^\exists\right) \\ \text{KER}(\pi_{R^T}(\cdot)) &= \text{KER}\left(\left(\cdot\right)_R^\downarrow\right) = \text{KER}\left(\left(\cdot\right)_{R^T}^\exists\right) \\ \text{KER}(\kappa_{R^{\otimes}}(\cdot)) &= \text{KER}\left(\left(\cdot\right)_{R^{-1}}^\downarrow\right) = \text{KER}\left(\left(\cdot\right)_R^\forall\right) \\ \text{KER}(\kappa_{R^{-1}}(\cdot)) &= \text{KER}\left(\left(\cdot\right)_{R^{-1}}^\uparrow\right) = \text{KER}\left(\left(\cdot\right)_{R^T}^\forall\right). \end{aligned} \tag{21}$$

Proof. For the first equation of (21) we have to prove $(x_1, x_2) \in \text{KER}(\pi_R(\cdot)) \Leftrightarrow (x_1, x_2) \in \text{KER}\left(\left(\cdot\right)_R^\uparrow\right)$, with bikernels of the polars

$$\text{KER}\left(\left(\cdot\right)_R^\uparrow\right) = \{(x_1, x_2) \in \mathcal{X} \mid x_1^\uparrow = x_2^\uparrow\} \quad \text{KER}\left(\left(\cdot\right)_R^\downarrow\right) = \{(y_1, y_2) \in \mathcal{Y} \mid y_1^\downarrow = y_2^\downarrow\}.$$

For $(x_1, x_2) \in \text{KER}(\pi_R(\cdot))$, call $(x_1^\uparrow)_R^\downarrow = (x_2^\uparrow)_R^\downarrow = a$. Then, applying again the polars $((x_1^\uparrow)_R^\downarrow)_R^\uparrow = ((x_2^\uparrow)_R^\downarrow)_R^\uparrow = a^\uparrow = b$. Considering that $(\cdot)_R^\uparrow$ and $(\cdot)_R^\downarrow$ are mutually inverses

on \mathfrak{B}_X^φ and \mathfrak{B}_Y^φ , we have $x_{1R}^\uparrow = x_{2R}^\uparrow = b$, which proves the forward inclusion. Likewise, for $(x_1, x_2) \in \text{KER}\left((\cdot)^\uparrow_R\right)$ call $x_{1R}^\uparrow = x_{2R}^\uparrow = b$, whence $(x_{1R}^\uparrow)^\downarrow = (x_{2R}^\uparrow)^\downarrow = b^\downarrow = a$, the backwards inclusion.

We next prove $\text{KER}(\kappa_{R^\otimes}(\cdot)) = \text{KER}\left((\cdot)^\downarrow_{R^{-1}}\right)$. If $(y_1, y_2) \in \text{KER}\left((\cdot)^\downarrow_R\right)$ then $(y_1)^\downarrow_R = (y_2)^\downarrow_R = a$ whence $((y_1)^\downarrow_R)^\exists = ((y_2)^\downarrow_R)^\exists = a^\exists = d$ whence $(y_1, y_2) \in \text{KER}(\kappa_{R^\otimes}(\cdot))$. On the other hand, if $(y_1, y_2) \in \text{KER}(\kappa_{R^\otimes}(\cdot))$ then $((y_1)^\downarrow_R)^\exists = ((y_2)^\downarrow_R)^\exists = d$ whence $(y_1)^\downarrow_{RR} = (y_2)^\downarrow_{RR} = d^\downarrow = a$, so $(y_1)^\downarrow_R = (y_2)^\downarrow_R = a$ —by the properties of the left adjunction and the fact that $a = d^\downarrow \iff a^\exists = d$ —whence $(y_1, y_2) \in \text{KER}\left((\cdot)^\downarrow_R\right)$. Four-fold GC-dually. \square

Actually this allows us to characterize the bikernels and their isomorphic images by means of the first theorem on isomorphisms on \mathcal{K} -semimodules, Theorem 2.

Corollary 1. *In the conditions of Theorem 3,*

1. $\text{KER}(\pi_R(\cdot))$ is a $\overline{\mathcal{K}}$ -congruence on \mathcal{X} isomorphic to $\text{IM}\left((\cdot)^\exists_R\right) \cong \overline{\mathfrak{N}}_Y^\varphi$, a $\overline{\mathcal{K}}$ -subsemimodule of \mathcal{Y} .
2. $\text{KER}(\kappa_{R^\otimes}(\cdot))$ is a $\underline{\mathcal{K}}$ -congruence on \mathcal{Y} isomorphic to $\text{IM}\left((\cdot)^\downarrow_R\right) \cong \underline{\mathfrak{B}}_X^\varphi$, a $\underline{\mathcal{K}}$ -subsemimodule of \mathcal{X} .
3. $\text{KER}(\pi_{R^\top}(\cdot))$ is a $\overline{\mathcal{K}}$ -congruence on \mathcal{Y} isomorphic to $\text{IM}\left((\cdot)^\exists_{R^\top}\right) \cong \overline{\mathfrak{N}}_X^\varphi$, a $\overline{\mathcal{K}}$ -subsemimodule of \mathcal{X} .
4. $\text{KER}(\kappa_{R^{-1}}(\cdot))$ is a $\underline{\mathcal{K}}$ -congruence on \mathcal{X} isomorphic to $\text{IM}\left((\cdot)^\downarrow_{R^\top}\right) \cong \underline{\mathfrak{B}}_Y^\varphi$, a $\underline{\mathcal{K}}$ -subsemimodule of \mathcal{Y} .

Proof. For 1, consider the left adjunction $((\cdot)^\exists_R, (\cdot)^\downarrow_R : \overline{\mathcal{X}} \rightleftharpoons \overline{\mathcal{Y}}$. Since $(\cdot)^\exists_R$ is a $\overline{\mathcal{K}}$ -homomorphism, by isomorphism Theorem 2, we have that $\text{KER}\left((\cdot)^\uparrow_R\right) = \text{KER}(\pi_R(\cdot))$ is a $\overline{\mathcal{K}}$ -congruence isomorphic to $\text{IM}\left((\cdot)^\exists_R\right) \cong \overline{\mathfrak{N}}_Y^\varphi$. The second statement follows from the $\underline{\mathcal{K}}$ -homomorphism $(\cdot)^\downarrow_R$. The other two statements from considering the right adjunction. \square

Note that the isomorphism is set up by means of the definitions of the congruence classes:

$$\begin{aligned} \pi_R^{-1}(a) &= \{x \in X \mid \pi_R(x) = a\} & \pi_{R^\top}^{-1}(b) &= \{y \in Y \mid \pi_{R^\top}(y) = b\} \\ \kappa_{R^\otimes}^{-1}(d) &= \{y \in Y \mid \kappa_{R^\otimes}(y) = d\} & \kappa_{R^{-1}}^{-1}(c) &= \{x \in X \mid \kappa_{R^{-1}}(x) = c\} \end{aligned} \tag{22}$$

which suggests, when possible, to name the classes of equivalence after their canonical-element, that is, $[x]_{\text{KER}(\pi_R(\cdot))} = \pi_R^{-1}(a)$ where $a = \pi_R(x)$, and analogously for intents, co-extents and co-intents. The order structure of the set e.g., \mathfrak{B}_X^φ with respect to that of $\overline{\mathcal{K}}$ is important, hence we introduce the notation, e.g. $\underline{\mathfrak{B}}_X^\varphi$ (resp. $\overline{\mathfrak{B}}_X^\varphi$) to indicate that the set is aligned in its joins with $\overline{\mathcal{K}}$ (resp. $\underline{\mathcal{K}}$).

Corollary 2. *In the conditions of Theorem 3,*

$$\begin{aligned} \text{KER}(\pi_R(\cdot)) &= \{\pi_R^{-1}(a) \mid a \in \underline{\mathfrak{B}}_X^\varphi\} & \text{KER}(\pi_{R^\top}(\cdot)) &= \{\pi_{R^\top}^{-1}(b) \mid b \in \underline{\mathfrak{B}}_Y^\varphi\} \\ \text{KER}(\kappa_{R^\otimes}(\cdot)) &= \{\kappa_{R^\otimes}^{-1}(d) \mid d \in \overline{\mathfrak{N}}_Y^\varphi\} & \text{KER}(\kappa_{R^{-1}}(\cdot)) &= \{\kappa_{R^{-1}}^{-1}(c) \mid c \in \overline{\mathfrak{N}}_X^\varphi\} \end{aligned} \tag{23}$$

Proof. Each class $D \in \text{KER}(\kappa_{R^\otimes}(\cdot))$ maps onto a co-intent $d \in \overline{\mathfrak{N}}_Y^\varphi$. In the opposite direction, for each open element d that class is $D \equiv \kappa_{R^\otimes}^{-1}(d)$, which is not void, since d itself belongs to it. Four-fold-GC dually. \square

Note that we also have $\pi_R^{-1}(a) = ((\cdot)_R^\forall)^{-1}(d) = ((\cdot)_R^\downarrow)^{-1}(b)$ and analogously for the other closure and the interior operators, but the notation with the adjuncts is certainly more cumbersome.

The congruence classes inherit the structure of the semimodule in question:

Corollary 3. *In the conditions of Theorem 3,*

1. *The congruence classes of $\text{KER}(\pi_R(\cdot))$ are join-subsemilattices of \mathcal{X} .*
2. *The congruence classes of $\text{KER}(\pi_{R^\top}(\cdot))$ are join-subsemilattices of \mathcal{Y} .*
3. *The congruence classes of $\text{KER}(\kappa_{R^{-1}}(\cdot))$ are meet-subsemilattices of \mathcal{X} .*
4. *The congruence classes of $\text{KER}(\kappa_{R^\otimes}(\cdot))$ are meet-subsemilattices of \mathcal{Y} .*

Proof. We prove the statement for $\text{KER}(\pi_R(\cdot))$. Let $x_2, x_1 \in \pi_R^{-1}(a)$ with $a_R^\uparrow = b \iff b_R^\downarrow = a$. Recall that \oplus is the join in \mathcal{X} and $(\cdot)_R^\uparrow$ transforms joins into meets, whence: $(x_1 \vee x_2)_R^\uparrow = (x_1 \oplus x_2)_R^\uparrow = x_{1R}^\uparrow \dot{\oplus} x_{2R}^\uparrow = b \dot{\oplus} b = b$, whence $\pi_R((x_1 \oplus x_2)) = b_R^\downarrow = a$ so that $x_1 \oplus x_2 \in \pi_R^{-1}(a)$. Four-fold-GC dually. \square

Since the closures and kernels are endofunctions, the following lemma is not surprising.

Lemma 2. *In the conditions of Theorem 3,*

1. *Every class in $\text{KER}(\pi_R(\cdot))$ intersects the system of extents \mathfrak{B}_X^φ at a single point.*
2. *Every class in $\text{KER}(\pi_{R^\top}(\cdot))$ intersects the system of intents \mathfrak{B}_Y^φ at a single point.*
3. *Every class in $\text{KER}(\kappa_{R^{-1}}(\cdot))$ intersects the system of co-extents $\overline{\mathfrak{N}}_X^\varphi$ at a single point.*
4. *Every class in $\text{KER}(\kappa_{R^\otimes}(\cdot))$ intersects the system of co-intents $\overline{\mathfrak{N}}_Y^\varphi$ at a single point.*

Proof. For claim 1, see [25] [propo 3.23]. We next prove claim 4, to review the procedure. Let $\kappa_{R^\otimes}(y) = d$, whence $d \in [y]_{\text{KER}(\kappa_{R^\otimes}(\cdot))}$. Call $d_R^\forall = a$, where a is the extent bijectively related to co-intent d by the left adjunction. Since $a_R^\exists = d$, this also means that $d \in \overline{\mathfrak{N}}_Y^\varphi$. However, if we suppose that there is a different $d' \in [y]_{\text{KER}(\kappa_{R^\otimes}(\cdot))} \cap \overline{\mathfrak{N}}_Y^\varphi$, then we will have $d'_R^\forall = a$ and therefore $(d'_R^\forall)^\exists = a_R^\exists = d$, so d' is not open whence $d' \notin \overline{\mathfrak{N}}_Y^\varphi$, a contradiction. Four-fold GC-dually. \square

Perhaps more importantly, the intersections have canonical status.

Corollary 4. *In the conditions of Theorem 3,*

1. *Every extent $a \in \mathfrak{B}_X$ is the maximum of its congruence class $a = \check{a} = \vee \pi_R^{-1}(a)$.*
2. *Every intent $b \in \mathfrak{B}_Y$ is the maximum of its congruence class $b = \check{b} = \vee \pi_{R^\top}^{-1}(b)$.*
3. *Every co-extent $c \in \overline{\mathfrak{N}}_X$ is the minimum of its congruence class $c = \hat{c} = \wedge \kappa_{R^{-1}}^{-1}(c)$.*
4. *Every co-intent $d \in \overline{\mathfrak{N}}_Y$ is the minimum of its congruence class $d = \hat{d} = \wedge \kappa_{R^\otimes}^{-1}(d)$.*

Proof. As proven in the previous lemma, $a \in \pi_R^{-1}(a)$. Since a is the closure of the elements in the class, $\forall x \in \pi_R^{-1}(a), a \geq x$ whence $a = \check{a} = \vee \pi_R^{-1}(a)$ as defined in (30) of [6] (§ 2.1.6). Four-fold GC-dually. \square

3.1.2. Extremal Exploration in the Four-Fold Connection

The next Proposition extends a number of results from [25] [Sec. 3.7]. It deals with specific values of the φ parameter.

Proposition 2. *In the conditions of Theorem 3,*

1. If $\varphi = \perp$, then $\underline{\mathfrak{B}}_X^\perp \cong \overline{\mathfrak{N}}_X^\perp \cong \underline{\mathfrak{B}}_Y^\perp \cong \overline{\mathfrak{N}}_Y^\perp \cong \mathbf{1}$ and

$$\begin{aligned} \text{KER}(\pi_{R,\perp}(\cdot)) &= \{\mathcal{X}\} & \text{KER}(\pi_{R^\top,\perp}(\cdot)) &= \{\mathcal{Y}\} \\ \text{KER}(\kappa_{R^\otimes,\perp}(\cdot)) &= \{\mathcal{Y}^{-1}\} & \text{KER}(\kappa_{R^{-1},\perp}(\cdot)) &= \{\mathcal{X}^{-1}\} \end{aligned} \tag{24}$$

with

$$\forall x \in \mathcal{X}, \pi_{R,\perp}(x) = \top^s, \kappa_{R^{-1},\perp}(x) = \perp^s \quad \forall y \in \mathcal{Y}, \pi_{R^\top,\perp}(y) = \top^m, \kappa_{R^\otimes,\perp}(y) = \perp^m \tag{25}$$

2. If $\varphi \neq \perp$, then:

$$\begin{aligned} \{\top^s, R \dot{\otimes} \perp^m\} &\subseteq \underline{\mathfrak{B}}_X^\varphi & \{R^\top \dot{\otimes} \perp^s, \top^m\} &\subseteq \underline{\mathfrak{B}}_Y^\varphi \\ \{R^\otimes \dot{\otimes} \top^s, \perp^m\} &\subseteq \overline{\mathfrak{N}}_X^\varphi & \{\perp^s, R^{-1} \dot{\otimes} \top^m\} &\subseteq \overline{\mathfrak{N}}_Y^\varphi \end{aligned} \tag{26}$$

with

$$\begin{aligned} \forall x \in \mathcal{N}(R^\otimes), \pi_{R,\varphi}(x) &= R \dot{\otimes} \perp^m & \forall y \in \mathcal{N}(R^{-1}), \pi_{R^\top,\varphi}(y) &= R^\top \dot{\otimes} \perp^s \\ \forall y \in \mathcal{N}(R), \kappa_{R^\otimes,\varphi}(y) &= R^\otimes \dot{\otimes} \top^s & \forall x \in \mathcal{N}(R^\top), \kappa_{R^{-1},\varphi}(x) &= R^{-1} \dot{\otimes} \top^m \end{aligned} \tag{27}$$

3. Furthermore, if $\varphi = \top$, then $\underline{\mathfrak{B}}_X^\top \cong \underline{\mathfrak{B}}_Y^\top \cong \overline{\mathfrak{N}}_X^\top \cong \overline{\mathfrak{N}}_Y^\top \cong \mathbf{2}$ with:

$$\begin{aligned} \text{KER}(\pi_{R,\varphi}(\cdot)) &= \{\mathcal{N}(R^\otimes), \mathcal{X} \setminus \mathcal{N}(R^\otimes)\} & \text{KER}(\pi_{R^\top,\varphi}(\cdot)) &= \{\mathcal{N}(R^{-1}), \mathcal{Y} \setminus \mathcal{N}(R^{-1})\} \\ \text{KER}(\pi_{R,\varphi}(\cdot)) &= \{\mathcal{N}(R^\otimes), \mathcal{X} \setminus \mathcal{N}(R^\otimes)\} & \text{KER}(\pi_{R^\top,\varphi}(\cdot)) &= \{\mathcal{N}(R^{-1}), \mathcal{Y} \setminus \mathcal{N}(R^{-1})\} \end{aligned} \tag{28}$$

Proof. For case 1 first recall the polars of (5). Then, regardless of what vectors $x \in \mathcal{X}$ or $y \in \mathcal{Y}$ we introduce, Equation (25) follows by putting $\varphi = \perp$ in them. Since all of the space \mathcal{X} maps to a single \perp -intent $x_{R,\perp}^\uparrow = \top^m$, then the bikernel has precisely the one class defined by that element, and GC-dually for \perp -extents, which proves (24). To obtain the statements regarding $\overline{\mathfrak{N}}_X^\perp$ and $\overline{\mathfrak{N}}_Y^\perp$, $\text{KER}(\kappa_{R^{-1},\perp}(\cdot))$ and $\text{KER}(\kappa_{R^\otimes,\perp}(\cdot))$ we operate, four-fold dually, with the co-Galois connection of (11).

The case where $\varphi \neq \perp$ highlights the importance of the null space, for if $x \in \mathcal{X}$ is such that $R^\top \dot{\otimes} x^{-1} = \top^m$, equivalently $R^\otimes \dot{\otimes} x = \perp^m \iff x \in \mathcal{N}(R^\otimes)$, then we have that $x_{R,\varphi}^\uparrow = R^\top \dot{\otimes} x^{-1} \dot{\otimes} \varphi^{-1} = \top^m$, and GC-dually if $y \in \mathcal{N}(R^{-1})$ then $y_{R,\varphi}^\downarrow = \top^s$. This proves also the fact that the nullspace $\mathcal{N}(R^\otimes)$ is one of the classes of the bikernel of $\pi_{R,\varphi}(\cdot)$, and GC-dually for the nullspace $\mathcal{N}(R^{-1})$, that is (27). Now, consider that if $x = \top^s$, then $x_{R,\varphi}^\uparrow = R^\top \dot{\otimes} \perp^s$, for $\varphi \neq \perp$. Furthermore, $(\top^s_{R,\varphi}^\uparrow)_{R,\varphi}^\downarrow = R \dot{\otimes} (R^\otimes \dot{\otimes} \top^s) = \top^s$, where the last step follows because the composition of the polars is a closure. Again, the rest of the statements follow four-fold dually from (11).

Finally, for case 3 note that the inner product defining intents and extents reads $x^\otimes \dot{\otimes} R \dot{\otimes} y^{-1} = \top$, which is a matrix equation in \mathcal{K} . Since this is an entire, zero-sum-free semifield, see [6] (§ 2.1.1), the only solutions are: for non-null $y \in \mathcal{Y}$ $x^\otimes \dot{\otimes} R = \top \iff R^\top \dot{\otimes} x^{-1} = \top$ —which entails that $x \in \mathcal{N}(R^\otimes)$ —and for non-null $x \in \mathcal{X}$ the solutions are $R \dot{\otimes} y^{-1} = \top \iff y \in \mathcal{N}(R^{-1})$. Since this is the specific sub-case of 2 where $\varphi = \top$, we know that precisely for $x \in \mathcal{N}(R^\otimes)$, $x_{R,\top}^\uparrow = \top^m$, and GC-dually for $y \in \mathcal{N}(R^{-1})$, $y_{R,\top}^\downarrow = \top^s$. Since $\top^m \notin \mathcal{N}(R^{-1})$ and $\top^s \notin \mathcal{N}(R^\otimes)$, we conclude $\underline{\mathfrak{B}}_Y^\top$ has exactly two elements

$\mathfrak{B}_Y^\top = \{R^\top \dot{\otimes} \perp^s, \top^m\}$ and GC-dually $\mathfrak{B}_X^\top = \{R \dot{\otimes} \perp^m, \top^s\}$. The structure of the bikernels is clear: since we have two intents and because $(\pi_{R, \top}(\cdot))^{-1}(\top^m) = \mathcal{N}(R^\otimes)$, we must have $(\pi_{R, \top}(\cdot))^{-1}(R^\top \dot{\otimes} \perp^s) = \mathcal{X} \setminus \mathcal{N}(R^\otimes)$, and GC-dually for the bikernel of \mathcal{Y} . The rest of the statement follows by using the co-Galois connection of (11). \square

In proving Proposition 2, we obtained some easy corollaries:

Corollary 5. *In the conditions of Theorem 3,*

1. *If $\varphi = \perp$ then*

$$\mathfrak{B}_X^\perp = \{\top^s\} \quad \mathfrak{B}_Y^\perp = \{\top^m\} \quad \overline{\mathfrak{N}}_Y^\perp = \{\perp^m\} \quad \overline{\mathfrak{N}}_X^\perp = \{\perp^s\}$$

2. *If $\varphi = \top$ then*

$$\mathfrak{B}_X^\top = \{R \dot{\otimes} \perp^m, \top^s\} \quad \mathfrak{B}_Y^\top = \{R^\top \dot{\otimes} \perp^s, \top^m\} \quad \overline{\mathfrak{N}}_Y^\top = \{R^\otimes \dot{\otimes} \top^s, \perp^m\} \quad \overline{\mathfrak{N}}_X^\top = \{R^\otimes \dot{\otimes} \top^m, \perp^s\}$$

3. *If $R = \top^{s \times m}$ is totally saturated, then $\mathfrak{B}_X^\varphi = \{\top^s\}$, $\mathfrak{B}_Y^\varphi = \{\top^m\}$, $\overline{\mathfrak{N}}_X^\varphi = \{\perp^s\}$, $\overline{\mathfrak{N}}_Y^\varphi = \{\perp^m\}$.*

4. *When $\varphi \neq \perp$, if $R \in K^{s \times m}$ has no saturated rows,*

$$\{\perp^s, \top^s\} \subseteq \mathfrak{B}_X^\varphi \quad \{\perp^s, \top^s\} \subseteq \overline{\mathfrak{N}}_X^\varphi$$

and if $R \in K^{s \times m}$ has no saturated columns,

$$\{\perp^m, \top^m\} \subseteq \mathfrak{B}_Y^\varphi \quad \{\perp^m, \top^m\} \subseteq \overline{\mathfrak{N}}_Y^\varphi$$

5. *When $\varphi \neq \perp$ the nullspaces are blocks of the bikernels:*

$$\begin{aligned} \mathcal{N}(R^\otimes) &= \pi_R^{-1}(R \dot{\otimes} \perp^m) \in \text{KER}(\pi_R(\cdot)) & \mathcal{N}(R^{-1}) &= \pi_{R^\top}^{-1}(R^\top \dot{\otimes} \perp^s) \in \text{KER}(\pi_{R^\top}(\cdot)) \\ \mathcal{N}(R) &= \kappa_{R^{-1}}^{-1}(R^\otimes \dot{\otimes} \top^s) \in \text{KER}(\kappa_{R^\otimes}(\cdot)) & \mathcal{N}(R^\top) &= \kappa_{R^\otimes}^{-1}(R^{-1} \dot{\otimes} \top^m) \in \text{KER}(\kappa_{R^{-1}}(\cdot)) \end{aligned}$$

Proof. Statements 1 and 2 were obtained in the previous demonstration. For 3, if $R = \top^{s \times m}$, then $\forall \varphi, \forall x, x_{R, \varphi}^\perp = R \dot{\otimes} x^{-1} = \top^m$. Four-fold GC-dually for the rest of the sets of extents, co-extents and co-intents. Statement 4 follows from (26) and the fact that if R has no full rows, then $R \dot{\otimes} \perp^m = \perp^s$, and four-fold GC-dually for the rest of results. Finally, statement 5 is a consequence of the previous lemmas and (27) in Proposition 2. It shows that the nullspaces are the classes related to top and bottom elements in the set of fixpoints. \square

Note that for extremal $\varphi \in \{\perp, \top\}$, the description of the closure and interior systems and the bikernels is completed by Proposition 2 and Corollary 5.

3.1.3. Invertible φ -Exploration in the Four-Fold Connection

We next introduce the most typical case of the four-fold connection with invertible φ .

Theorem 4 ($\overline{\mathcal{K}}$ -four-fold connection, invertible φ). *Let $\overline{\mathcal{K}}$ be a complete idempotent semifield, and consider the vector spaces $\mathcal{X} = \overline{\mathcal{K}}^s$ and $\mathcal{Y} = \overline{\mathcal{K}}^m$, a matrix $R \in \overline{\mathcal{K}}^{s \times m}$, an invertible element $\varphi = \gamma \otimes \mu \in \overline{\mathcal{K}}$ and the scaled spaces*

$$\begin{aligned} \tilde{\mathcal{X}}^\gamma &= \overline{\mathcal{K}}^s \dot{\otimes} \gamma & \tilde{\mathcal{Y}}^\mu &= \overline{\mathcal{K}}^m \dot{\otimes} \mu & (29) \\ (\tilde{\mathcal{Y}}^\mu)^{-1} &= \underline{\mathcal{K}}^m \dot{\otimes} \mu^{-1} = \widetilde{(\underline{\mathcal{K}}^m)}^{(\mu^{-1})} & (\tilde{\mathcal{X}}^\gamma)^{-1} &= \underline{\mathcal{K}}^s \dot{\otimes} \gamma^{-1} = \widetilde{(\underline{\mathcal{K}}^s)}^{(\gamma^{-1})} \end{aligned}$$

Then

1. The bracket $\langle x \mid R \mid y \rangle_{\mathcal{K}}^{\text{OI}} = x^{\otimes} \dot{\otimes} R \dot{\otimes} y^{-1}$ induces a Galois connection $(\cdot^{\uparrow}_R, \cdot^{\downarrow}_R) : \tilde{\mathcal{X}}^{\gamma} \dashv \tilde{\mathcal{Y}}^{\mu}$ between the scaled spaces through the polar maps:

$$x^{\uparrow}_R = R^{\text{T}} \dot{\otimes} x^{-1} \qquad y^{\downarrow}_R = R \dot{\otimes} y^{-1} \qquad (30)$$

whose composition generate closure operators:

$$\pi_R(x) = (x^{\uparrow}_R)^{\downarrow}_R = R \dot{\otimes} (R^{\otimes} \dot{\otimes} x) \qquad \pi_{R^{\text{T}}}(y) = (y^{\downarrow}_R)^{\uparrow}_R = R^{\text{T}} \dot{\otimes} (R^{-1} \dot{\otimes} y) \qquad (31)$$

which define two bijective sets, the system of φ -extents \mathfrak{B}_X^{γ} and the system of φ -intents \mathfrak{B}_Y^{μ} over which the polars are dual order bijections and the closures are identities, respectively.

$$\mathfrak{B}_X^{\gamma} = (\tilde{\mathcal{Y}}^{\mu})^{\downarrow}_R = \pi_R(\tilde{\mathcal{X}}^{\gamma}) \qquad \mathfrak{B}_Y^{\mu} = (\tilde{\mathcal{X}}^{\gamma})^{\uparrow}_R = \pi_{R^{\text{T}}}(\tilde{\mathcal{Y}}^{\mu}) \qquad (32)$$

2. The bracket $\langle x \mid R \mid y \rangle_{\mathcal{K}}^{\text{IO}} = x^{\text{T}} \dot{\otimes} R \dot{\otimes} y$ induces a co-Galois connection $(\cdot^{\uparrow}_{R^{-1}}, \cdot^{\downarrow}_{R^{-1}}) : (\tilde{\mathcal{X}}^{\gamma})^{-1} \dashv (\tilde{\mathcal{Y}}^{\mu})^{-1}$ between the scaled spaces through the co-polar maps:

$$x^{\uparrow}_{R^{-1}} = R^{\otimes} \dot{\otimes} x^{-1} \qquad y^{\downarrow}_{R^{-1}} = R^{-1} \dot{\otimes} y^{-1} \qquad (33)$$

whose composition generate interior operators or kernels:

$$\kappa_{R^{-1}}(x) = (x^{\uparrow}_{R^{-1}})^{\downarrow}_{R^{-1}} = R^{-1} \dot{\otimes} (R^{\text{T}} \dot{\otimes} x) \qquad \kappa_{R^{\otimes}}(y) = (y^{\downarrow}_{R^{-1}})^{\uparrow}_{R^{-1}} = R^{\otimes} \dot{\otimes} (R \dot{\otimes} y) \qquad (34)$$

which define two bijective sets; the system of φ -neighbourhoods of objects or φ -co-extents $\overline{\mathfrak{N}}_X^{\gamma}$ and φ -neighbourhoods of attributes or φ -co-intents $\overline{\mathfrak{N}}_Y^{\mu}$ over which the co-polar maps are dual-order bijections and the kernels the identities, respectively.

$$\overline{\mathfrak{N}}_X^{\gamma} = ((\tilde{\mathcal{Y}}^{\mu})^{-1})^{\downarrow}_{R^{-1}} = \kappa_{R^{-1}}((\tilde{\mathcal{X}}^{\gamma})^{-1}) \qquad \overline{\mathfrak{N}}_Y^{\mu} = ((\tilde{\mathcal{X}}^{\gamma})^{-1})^{\uparrow}_{R^{-1}} = \kappa_{R^{\otimes}}((\tilde{\mathcal{Y}}^{\mu})^{-1}) \qquad (35)$$

3. The bracket $\langle x \mid R \mid y \rangle_{\mathcal{K}}^{\text{OO}} = x^{\otimes} \dot{\otimes} R \dot{\otimes} y$ induces a left adjunction $(\cdot^{\exists}_R, \cdot^{\forall}_R) : \tilde{\mathcal{X}}^{\gamma} \dashv (\tilde{\mathcal{Y}}^{\mu})^{-1}$ between the scaled spaces through the left adjunct pair of maps:

$$x^{\exists}_R = R^{\otimes} \dot{\otimes} x \qquad y^{\forall}_R = R \dot{\otimes} y \qquad (36)$$

whose compositions are the closure of extents and kernel of co-intents:

$$\pi_R(x) = (x^{\exists}_R)^{\forall}_R \qquad \kappa_{R^{\otimes}}(y) = (y^{\forall}_R)^{\exists}_R \qquad (37)$$

which define an order bijection between the system of φ -extents \mathfrak{B}_X^{γ} and φ -co-intents $\overline{\mathfrak{N}}_Y^{\mu}$ over which the closure of extents and kernel of co-intents are identities.

$$\mathfrak{B}_X^{\gamma} = ((\tilde{\mathcal{Y}}^{\mu})^{-1})^{\forall}_R = \pi_R(\tilde{\mathcal{X}}^{\gamma}) \qquad \overline{\mathfrak{N}}_Y^{\mu} = (\tilde{\mathcal{X}}^{\gamma})^{\exists}_R = \kappa_{R^{\otimes}}((\tilde{\mathcal{Y}}^{\mu})^{-1}) \qquad (38)$$

4. The bracket $\langle x \mid R \mid y \rangle_{\mathcal{K}}^{\text{II}} = x^{\text{T}} \dot{\otimes} R \dot{\otimes} y^{-1}$ induces an adjunction on the right $(\cdot^{\forall}_{R^{\text{T}}}, \cdot^{\exists}_{R^{\text{T}}}) : (\tilde{\mathcal{X}}^{\gamma})^{-1} \dashv \tilde{\mathcal{Y}}^{\mu}$ between the scaled spaces through the pair of adjunct maps:

$$x^{\forall}_{R^{\text{T}}} = R^{\text{T}} \dot{\otimes} x \qquad y^{\exists}_{R^{\text{T}}} = R^{-1} \dot{\otimes} y \qquad (39)$$

whose compositions are the kernel of co-extents and the closure of extents:

$$\kappa_{R^{-1}}(x) = (x_{R^T}^{\forall})_{R^T}^{\exists} \qquad \pi_{R^T}(y) = (y_{R^T}^{\exists})_{R^T}^{\forall} \qquad (40)$$

which define another bijection between the system of φ -co-extents $\overline{\mathfrak{N}}_X^\gamma$ and φ -intents $\underline{\mathfrak{B}}_Y^\mu$ over which the kernel of co-extents and the closure of intents are identities,

$$\overline{\mathfrak{N}}_X^\gamma = (\tilde{Y}^\mu)_{R^T}^{\exists} = \kappa_{R^{-1}}((\tilde{\mathcal{X}}^\gamma)^{-1}) \qquad \underline{\mathfrak{B}}_Y^\mu = ((\tilde{\mathcal{X}}^\gamma)^{-1})_{R^T}^{\forall} = \pi_{R^T}(\tilde{Y}^\mu) \qquad (41)$$

Proof. Note that since γ and μ are invertible, then the dotting (for these scalars) can be ignored. So given a finite $\varphi = \gamma \otimes \mu$, since $\gamma \otimes \mu = \gamma \otimes \mu = \gamma \dot{\otimes} \mu$, the inequality $\langle x \mid R \mid y \rangle_{\overline{\mathcal{K}}}^{\text{OI}} \geq \varphi$ may be rewritten using residuation as:

$$\begin{aligned} x^{\otimes} \dot{\otimes} R \dot{\otimes} y^{-1} \geq \gamma \dot{\otimes} \mu &\Leftrightarrow \gamma \setminus (x^{\otimes} \dot{\otimes} R \dot{\otimes} y^{-1}) / \mu \geq e \Leftrightarrow (\gamma^{-1} \otimes x^{\otimes}) \dot{\otimes} R \dot{\otimes} (y^{-1} \otimes \mu) \geq e \\ &\Leftrightarrow (x \otimes \gamma)^{\otimes} \dot{\otimes} R \dot{\otimes} (y \otimes \mu)^{-1} \geq e \Leftrightarrow (\tilde{x}^\gamma)^{\otimes} \dot{\otimes} R \dot{\otimes} (\tilde{y}^\mu)^{-1} \geq e \end{aligned}$$

the products by a scalar $x \otimes \gamma = \tilde{x}^\gamma$ and $y \otimes \mu = \tilde{y}^\mu$ have the interpretation of (finite) scalings in the original spaces. Similarly, for the co-Galois connection-inducing bracket $\langle x \mid R \mid y \rangle_{\overline{\mathcal{K}}}^{\text{IO}} \geq \varphi$ we rewrite

$$\begin{aligned} x^T \dot{\otimes} R \dot{\otimes} y \geq \gamma \dot{\otimes} \mu &\Leftrightarrow \gamma \setminus (x^T \dot{\otimes} R \dot{\otimes} y) / \mu \geq e \Leftrightarrow \gamma^{-1} \dot{\otimes} x^T \dot{\otimes} R \dot{\otimes} y \dot{\otimes} \mu^{-1} \geq e \\ &\Leftrightarrow (x \dot{\otimes} \gamma^{-1})^T \dot{\otimes} R \dot{\otimes} (y \dot{\otimes} \mu^{-1}) \geq e \end{aligned}$$

this shows that the scaling in this case is $(\tilde{Y}^\mu)^{-1} = \underline{\mathcal{K}}^m \dot{\otimes} \mu^{-1}$ and $(\tilde{\mathcal{X}}^\gamma)^{-1} = \underline{\mathcal{K}}^s \dot{\otimes} \gamma^{-1}$. Rewriting the bracket inequalities for left and right adjunctions lead to the same scalings.

Therefore, we are in the situation of Theorem 3, over the scaled spaces $\tilde{\mathcal{X}}^\gamma$ and \tilde{Y}^μ with invertible $\varphi = e$. Recall that e is the unit element, whence all the results follow from there. \square

Note that the precise notation with generic φ and the scaled spaces is cumbersome. We will use it only sparingly, mainly when scaled and non-scaled magnitudes appear in the same equation.

This four-fold connection can be seen as the diagram in Figure 2.

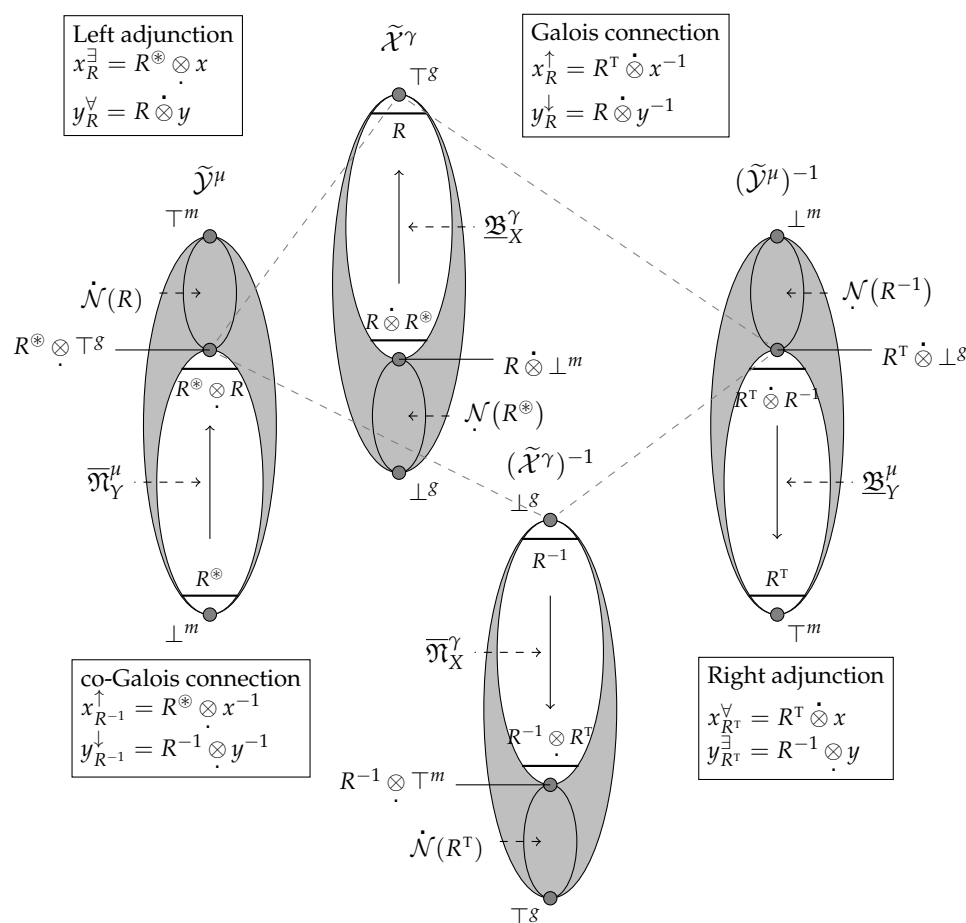


Figure 2. Domain diagram of the four-fold connection between the spaces implicit in a $\bar{\mathcal{K}}$ -valued matrix R (extended from [43]). Some maps, like the polar $\cdot \uparrow_R : \tilde{\mathcal{X}}^\gamma \rightarrow \mathfrak{B}_Y^\mu$ and its dual polar are missing for clarity. The dotted lines highlight the components of the top quadrilateral; similarly, the bottom, meet- and join-dense subsets can be easily spotted (see Section 3.2.2).

This picture is a grand summary but provides very little detail about the structure inside the ranges of the adjoints, dual adjoints, closures and kernel functions, which we investigate next. The core of the following proposition comes from [23], but it was not proven there. In it we have emphasized the semifields generating the semimodules.

Proposition 3. *In the conditions of Theorem 4,*

1. *The systems of φ -extents, -intents, -co-extents and -co-intents carry a $\bar{\mathcal{K}}$ - or $\underline{\mathcal{K}}$ -semimodule structure generated by the rows and columns of the matrix:*

$$\begin{aligned} \mathfrak{B}_X^\gamma &= \langle R \rangle_{\underline{\mathcal{K}}} & \mathfrak{B}_Y^\mu &= \langle R^\top \rangle_{\underline{\mathcal{K}}} \\ \bar{\mathfrak{N}}_Y^\mu &= \langle R^\circ \rangle_{\bar{\mathcal{K}}} & \bar{\mathfrak{N}}_X^\gamma &= \langle R^{-1} \rangle_{\bar{\mathcal{K}}} \end{aligned} \tag{42}$$

In particular, they are not sub-semimodules of the ambient spaces but their order duals, $\mathfrak{B}_X^\gamma \subseteq (\tilde{\mathcal{X}}^\gamma)^{-1}$, $\mathfrak{B}_Y^\mu \subseteq (\tilde{\mathcal{Y}}^\mu)^{-1}$, $\bar{\mathfrak{N}}_Y^\mu \subseteq \tilde{\mathcal{Y}}^\mu$, and $\bar{\mathfrak{N}}_X^\gamma \subseteq \tilde{\mathcal{X}}^\gamma$.

2. *These structures are isomorphic or dually isomorphic:*

$$\bar{\mathfrak{B}}_X^\gamma \cong (\bar{\mathfrak{B}}_Y^\mu)^d \cong \bar{\mathfrak{N}}_Y^\mu \cong (\bar{\mathfrak{N}}_X^\gamma)^d \tag{43}$$

3. In fact, they carry a complete lattice structure with operations:

$$\begin{aligned}
 a_i \in \underline{\mathfrak{B}}_X^\gamma, & \quad \wedge_{i \in I} a_i = \sum_{i \in I}^\bullet a_i & \quad \vee_{i \in I} a_i = \pi_R(\sum_{i \in I}^\bullet a_i) \\
 b_i \in \underline{\mathfrak{B}}_Y^\mu, & \quad \wedge_{i \in I} b_i = \sum_{i \in I}^\bullet b_i & \quad \vee_{i \in I} b_i = \pi_{R^T}(\sum_{i \in I}^\bullet b_i) \\
 d_i \in \overline{\mathfrak{N}}_Y^\mu, & \quad \wedge_{i \in I} d_i = \kappa_{R^\otimes}(\sum_{i \in I}^\bullet d_i) & \quad \vee_{i \in I} d_i = \sum_{i \in I}^\bullet d_i \\
 c_i \in \overline{\mathfrak{N}}_X^\gamma, & \quad \wedge_{i \in I} c_i = \kappa_{R^{-1}}(\sum_{i \in I}^\bullet c_i) & \quad \vee_{i \in I} c_i = \sum_{i \in I}^\bullet c_i
 \end{aligned} \tag{44}$$

Proof. For 1, the result involving the closures were proven in [25] [§ 3.2] on the standard Galois connection. For illustrative purposes we are going to prove it for the interior systems using the co-Galois connection: let $\kappa_{R^{-1}}(x) = (x_{R^{-1}}^\uparrow)_{R^{-1}}^\downarrow = R^{-1} \otimes (R^T \dot{\otimes} x)$ be the projector onto the set of φ -co-extents, an interior operator, and let a generic $y \in (\tilde{\mathcal{Y}}^\mu)^{-1}$. Consider a $\bar{\mathcal{K}}$ -combination of the columns of $R^{-1}, R^{-1} \otimes y$. We have: $\kappa_{R^{-1}}(x) = R^{-1} \otimes (R^T \dot{\otimes} (R^{-1} \otimes y)) = R^{-1} \otimes y$ by the properties of the matrix products, whence $\langle R^{-1} \rangle_{\bar{\mathcal{K}}} \subseteq \overline{\mathfrak{N}}_X^\gamma$. Now consider $c \in \overline{\mathfrak{N}}_X^\gamma$ whence $R^{-1} \otimes (R^T \dot{\otimes} c) = c$, but this precisely means that c is the combination of columns of R^{-1} , with coefficients $b = R^T \dot{\otimes} c$, whence $\overline{\mathfrak{N}}_X^\gamma \subseteq \langle R^{-1} \rangle_{\bar{\mathcal{K}}}$, as desired. Four-fold GC-dually for φ -extents, -intents, and -co-intents.

Note that we have also proven that there is a $b \in (\tilde{\mathcal{Y}}^\mu)^{-1}$ such that $b = R^T \dot{\otimes} c = c_{R^T}^\vee$ so that $c = R^{-1} \otimes b$. Which means that actually $b \in \underline{\mathfrak{B}}_Y^\mu$ are the coefficients of the combination above that generate $c \in \overline{\mathfrak{N}}_X^\gamma$ and vice-versa, so $\underline{\mathfrak{B}}_Y^\mu = \{c_{R^T}^\vee = R^T \dot{\otimes} c \mid c \in \overline{\mathfrak{N}}_X^\gamma\} \cong \{b_{R^T}^\exists = R^{-1} \otimes b \mid b \in \underline{\mathfrak{B}}_Y^\mu\} = \overline{\mathfrak{N}}_X^\gamma$. GC-dually, with the left adjunction we can prove: $\underline{\mathfrak{B}}_X^\gamma = \{d_R^\vee = R \dot{\otimes} d \mid d \in \overline{\mathfrak{N}}_Y^\mu\} \cong \{a_R^\exists = R^\otimes \otimes a \mid a \in \underline{\mathfrak{B}}_Y^\mu\} = \overline{\mathfrak{N}}_Y^\mu$. Likewise, with the Galois connection we generate $\underline{\mathfrak{B}}_X^\gamma \cong (\underline{\mathfrak{B}}_Y^\mu)^d$ and with the co-Galois connection, we generate $\overline{\mathfrak{N}}_X^\gamma \cong (\overline{\mathfrak{N}}_Y^\mu)^d$, whence statement 2 follows 4-fold GC-dually.

For statement 3, consider $\underline{\mathfrak{B}}_X^\gamma$. Since it is a complete $\bar{\mathcal{K}}$ -semimodule, it is a complete meet semilattice with the meet definition given by (44). For $\{a_i\}_{i \in J} \subseteq \underline{\mathfrak{B}}_X^\gamma$ we would like to define a join operation so that $a = \vee_{i \in J} a_i$, with $a \in \underline{\mathfrak{B}}_X^\gamma$. For that to be the case, since the left adjoint of (36) is a join morphism we must have: $a_R^\exists = (\vee_{i \in J} a_i)_R^\exists = \vee_{i \in J} (a_i)_R^\exists = \sum_{i \in J}^\bullet (a_i)_R^\exists$, the last step because $\overline{\mathfrak{B}}_Y^\mu$ is a $\bar{\mathcal{K}}$ -semimodule. Substituting the value of the adjunct and operating we get $a_R^\exists = \sum_{i \in J}^\bullet R^\otimes \otimes a_i = R^\otimes \otimes (\sum_{i \in J}^\bullet a_i) = (\sum_{i \in J}^\bullet a_i)_R^\exists$. If we close with the right adjoint we find: $a = (a_R^\exists)_R^\vee = ((\sum_{i \in J}^\bullet a_i)_R^\exists)_R^\vee = \pi_R(\sum_{i \in J}^\bullet a_i)$ which provides the operational form of the join. This is clearly an upper bound on the $\{a_i\}_{i \in J}$ since we $\sum_{i \in J}^\bullet a_i \geq a_i$ for $i \in J$, and we are applying to it a closure. It is the lowest upper bound since for $a' \in \underline{\mathfrak{B}}_X^\gamma$ with $\sum_{i \in J}^\bullet a_i \leq a' < a$, by applying the closure, which is monotone, we must have $\pi_R(\sum_{i \in J}^\bullet a_i) \leq \pi_R(a') \leq \pi_R(a)$ but since previously demonstrated $\pi_R(\sum_{i \in J}^\bullet a_i) = \pi_R(a)$ which entails $\pi_R(a') = \pi_R(a)$ and this entails $a' = a$ in $\underline{\mathfrak{B}}_X^\gamma$. Four-fold GC-dually for the other missing operations. \square

Note that the semimodule structures invoked by statement 1 in Proposition 3 involve a scalar multiplication that is not implied for the other operation in the lattice described in statement 3. In ([25], Section 3.1.3) a procedure similar to that used to construct the complementary latticial operation is invoked to provide the analogue of a “scalar product” in the dual semimodule structure, but we will not provide it here.

3.1.4. The Complements and the Tetrahedron Connection

It should be evident by now that the picture of the four-fold connection is completed by the existence of two more dual isomorphisms:

Lemma 3. *In the conditions of Theorem 4, there are two dual lattice isomorphisms:*

$$\begin{aligned} \cdot^c_X : \mathfrak{B}_X^\gamma &\rightarrow \overline{\mathfrak{N}}_X^\gamma & \cdot^c_Y : \mathfrak{B}_Y^\mu &\rightarrow \overline{\mathfrak{N}}_Y^\mu & (45) \\ a \mapsto a^c_X &= ((a^\uparrow_R)^\exists)^\downarrow_{R^T} = ((a)^\exists_R)^\downarrow_{R^{-1}} & b \mapsto b^c_Y &= ((b)^\downarrow_R)^\exists = ((b)^\downarrow_{R^{-1}})^\forall_{R^T} \end{aligned}$$

with inverses:

$$\begin{aligned} \cdot^X_c : \overline{\mathfrak{N}}_X^\gamma &\rightarrow \mathfrak{B}_X^\gamma & \cdot^Y_c : \overline{\mathfrak{N}}_Y^\mu &\rightarrow \mathfrak{B}_Y^\mu & (46) \\ c \mapsto c^X_c &= ((c)^\forall_{R^T})^\downarrow_R = ((c)^\uparrow_{R^{-1}})^\forall & d \mapsto d^Y_c &= ((d)^\downarrow_{R^{-1}})^\forall_{R^T} = ((d)^\forall_R)^\uparrow \end{aligned}$$

so that $a^c_X = c \leftrightarrow c^X_c = a$ and $b^c_Y = d \leftrightarrow d^Y_c = b$. On the lattices, these complements are just the inversion $a = c^{-1}$ and $b = d^{-1}$.

Proof. By composition of the polars in any two chains of generalized Galois connections in the four-fold connection: this involves one monotone and one antitone connections, whence the composition is antitone. Since they are compositions of isomorphisms when restricted to the images of the polars, they are also (antitone) isomorphisms.

Let $x \in \mathcal{X}$, with $\pi_R(x) = a \in \mathfrak{B}_X^\gamma$, $d = (x)^\exists_R = (a)^\exists_R$ and $c = (d)^\downarrow_{R^{-1}}$. Then $x^c_X = ((x)^\exists_R)^\downarrow_{R^{-1}} = (d)^\downarrow_{R^{-1}} = c$. Using the expressions of the polars, we have $x^c_X = ((x)^\exists_R)^\downarrow_{R^{-1}} = R^{-1} \otimes (R^T \dot{\otimes} x^{-1}) = (R \dot{\otimes} (R^\otimes \otimes x))^{-1} = \pi_R(x)^{-1} = a^{-1}$. So, in particular when $x = a$, we have $a^c_X = a^{-1} = c$.

Similarly, for $x \in \mathcal{X}$, with $\kappa_{R^{-1}}(x) = c \in \overline{\mathfrak{N}}_X^\gamma$, with $d = (x)^\uparrow_{R^{-1}}$, and $a = (d)^\forall_R$ we have: $x^X_c = ((x)^\uparrow_{R^{-1}})^\forall_R = (d)^\forall_R = a$. On the other hand, $((x)^\uparrow_{R^{-1}})^\forall_R = R \dot{\otimes} (R^\otimes \otimes x^{-1}) = (R^{-1} \otimes (R^T \dot{\otimes} x))^{-1} = (\kappa_{R^{-1}}(x))^{-1} = c^{-1}$, so, in particular when $x = c$, we have $c^X_c = c^{-1} = a$. The reasoning is row-column dual for the dual order isomorphism between \mathfrak{B}_Y^μ and $\overline{\mathfrak{N}}_Y^\mu$. \square

Recall that $(\cdot)^{-1} : \mathcal{X} \rightarrow \mathcal{X}, x \mapsto x^{-1}$ is a (dual order) isomorphism. We just proved it is also a (perfect) Galois connection between \mathfrak{B}_X^γ and $\overline{\mathfrak{N}}_X^\gamma$ just as Figure 3 shows.

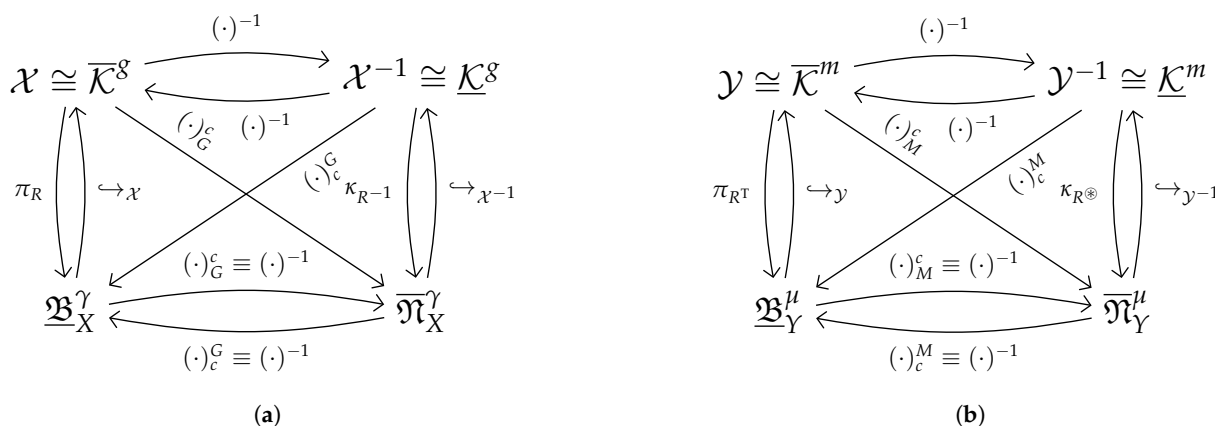


Figure 3. Complements between \mathfrak{B}_X^γ and $\overline{\mathfrak{N}}_X^\gamma$ and \mathfrak{B}_Y^μ and $\overline{\mathfrak{N}}_Y^\mu$. This diagram has to be overlaid on Figure 1. (a) The extent-co-extent complement, (b) the intent-co-intent complement.

Note that this means that we do not have to use the new notation for this “dual isomorphism” and we can use the inversion with this new understanding.

Furthermore, we have to complement our old version of the four-fold connection with the two new inversions, to create a sort of tetrahedral figure as in Figure 4.

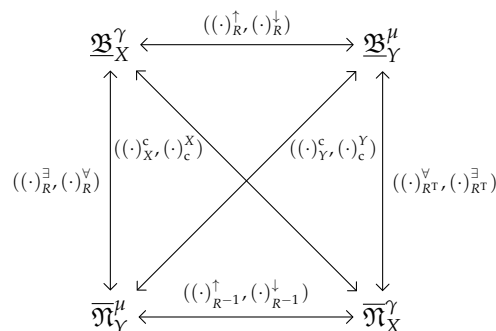


Figure 4. The tetrahedron of Galois connections of matrix $R \in \mathcal{K}^{g \times m}$.

3.1.5. Dual Exploration

It may seem strange that we started with $\overline{\mathcal{K}}$ but the scalar products leading to the definition of the different Galois connections operated in $\underline{\mathcal{K}}$. This is actually a requisite of the original work discovering the connections [14], but it also dovetails into previous, unrelated work [41,47].

We can do the opposite operation: start using $\underline{\mathcal{K}}$ what leads onto operations in $\overline{\mathcal{K}}$. Since the choice of semifield in which to interpret R represents a (choice of) modelling bias [23], using these results is what we call bias-duality. This accounts for a $\underline{\mathcal{K}}$ -dual of Theorem 4 and Proposition 3:

Theorem 5 (The $\underline{\mathcal{K}}$ -four-fold connection, invertible φ). *Let $\underline{\mathcal{K}}$ be a complete idempotent semifield, and consider the vector spaces $\mathcal{X} = \underline{\mathcal{K}}^g$ and $\mathcal{Y} = \underline{\mathcal{K}}^m$, a matrix $R \in \underline{\mathcal{K}}^{g \times m}$, an invertible element $\varphi = \gamma \otimes \mu \in \underline{\mathcal{K}}$ and the scaled spaces*

$$\begin{aligned} \tilde{\mathcal{X}}^\gamma &= \underline{\mathcal{K}}^g \dot{\otimes} \gamma & \tilde{\mathcal{Y}}^\mu &= \underline{\mathcal{K}}^m \dot{\otimes} \mu & (47) \\ (\tilde{\mathcal{Y}}^\mu)^{-1} &= \overline{\mathcal{K}}^m \dot{\otimes} \mu^{-1} = \widetilde{(\overline{\mathcal{K}}^m)}^{(\mu^{-1})} & (\tilde{\mathcal{X}}^\gamma)^{-1} &= \overline{\mathcal{K}}^g \dot{\otimes} \gamma^{-1} = \widetilde{(\overline{\mathcal{K}}^g)}^{(\gamma^{-1})} \end{aligned}$$

Then

1. The bracket $\langle x \mid R \mid y \rangle_{\underline{\mathcal{K}}}^{\text{OI}} = x^{\otimes} \otimes R \otimes y^{-1}$ induces a Galois connection $(\cdot^\uparrow_R, \cdot^\downarrow_R) : \tilde{\mathcal{X}}^\gamma \dashv \tilde{\mathcal{Y}}^\mu$ between the scaled spaces through the polars

$$x_R^\uparrow = R^T \dot{\otimes} x^{-1} \qquad y_R^\downarrow = R \dot{\otimes} y^{-1} \qquad (48)$$

whose composition generates closure operators:

$$\pi_R^d(x) = (x_R^\uparrow)^\downarrow = R \dot{\otimes} (R^{\otimes} \dot{\otimes} x) \qquad \pi_{R^T}^d(y) = (y_R^\downarrow)^\uparrow = R^T \dot{\otimes} (R^{-1} \dot{\otimes} y) \qquad (49)$$

which define two bijective sets, the system of φ -extents $\overline{\mathfrak{B}}_X^\gamma$ and the system of φ -intents $\overline{\mathfrak{B}}_Y^\mu$ over which the closures are the identities.

$$\overline{\mathfrak{B}}_X^\gamma = (\tilde{\mathcal{Y}}^\mu)_R^\downarrow = \pi_R^d(\tilde{\mathcal{X}}^\gamma) \qquad \overline{\mathfrak{B}}_Y^\mu = (\tilde{\mathcal{X}}^\gamma)_R^\uparrow = \pi_{R^T}^d(\tilde{\mathcal{Y}}^\mu) \qquad (50)$$

2. The bracket $\langle x \mid R \mid y \rangle_{\mathcal{K}}^{\text{IO}} = x^{\text{T}} \otimes R \otimes y$ induces a co-Galois connection $(\cdot^{\uparrow}_{R^{-1}}, \cdot^{\downarrow}_{R^{-1}}) : (\tilde{\mathcal{X}}^{\gamma})^{-1} \leftrightarrow (\tilde{\mathcal{Y}}^{\mu})^{-1}$ between the scaled spaces through the maps:

$$x^{\uparrow}_{R^{-1}} = R^{\otimes} \dot{\otimes} x^{-1} \qquad y^{\downarrow}_{R^{-1}} = R^{-1} \dot{\otimes} y^{-1} \qquad (51)$$

whose composition generate kernel (interior) operators:

$$\kappa_{R^{-1}}^{\text{d}}(x) = (x^{\uparrow}_{R^{-1}})^{\downarrow}_{R^{-1}} = R^{-1} \dot{\otimes} (R^{\text{T}} \otimes x) \quad \kappa_{R^{\otimes}}^{\text{d}}(y) = (y^{\downarrow}_{R^{-1}})^{\uparrow}_{R^{-1}} = R^{\otimes} \dot{\otimes} (R \otimes y) \qquad (52)$$

which define two bijective sets the systems of φ -neighbourhoods of objects or φ -co-extents $\underline{\mathfrak{N}}_X^{\gamma}$ and φ -neighbourhoods of attributes or φ -co-intents $\underline{\mathfrak{N}}_Y^{\mu}$ over which the kernel operators are the identities.

$$\underline{\mathfrak{N}}_X^{\gamma} = ((\tilde{\mathcal{Y}}^{\mu})^{-1})^{\downarrow}_{R^{-1}} = \kappa_{R^{-1}}^{\text{d}}((\tilde{\mathcal{X}}^{\gamma})^{-1}) \quad \underline{\mathfrak{N}}_Y^{\mu} = ((\tilde{\mathcal{X}}^{\gamma})^{-1})^{\uparrow}_{R^{-1}} = \kappa_{R^{\otimes}}^{\text{d}}((\tilde{\mathcal{Y}}^{\mu})^{-1}) \qquad (53)$$

3. The bracket $\langle x \mid R \mid y \rangle_{\mathcal{K}}^{\text{OO}} = x^{\otimes} \otimes R \otimes y$ induces a left adjunction $(\cdot^{\exists}_{R}, \cdot^{\forall}_{R}) : \tilde{\mathcal{X}}^{\gamma} \rightleftharpoons (\tilde{\mathcal{Y}}^{\mu})^{-1}$ between the scaled spaces through the left adjunct pair of maps:

$$x^{\exists}_{R} = R^{\otimes} \dot{\otimes} x \qquad y^{\forall}_{R} = R \otimes y \qquad (54)$$

whose compositions, are again, the closure of extents and interior of attributes:

$$\pi_{R}^{\text{d}}(x) = (x^{\exists}_{R})^{\forall}_{R} \qquad \kappa_{R^{\otimes}}^{\text{d}}(y) = (y^{\forall}_{R})^{\exists}_{R} \qquad (55)$$

which define another bijection between the systems of extents $\overline{\mathfrak{B}}_X^{\gamma}$ and neighbourhoods of attributes $\underline{\mathfrak{N}}_Y^{\mu}$:

$$\overline{\mathfrak{B}}_X^{\gamma} = ((\tilde{\mathcal{Y}}^{\mu})^{-1})^{\forall}_{R} \qquad \underline{\mathfrak{N}}_Y^{\mu} = (\tilde{\mathcal{X}}^{\gamma})^{\exists}_{R} \qquad (56)$$

4. The bracket $\langle x \mid R \mid y \rangle_{\mathcal{K}}^{\text{II}} = x^{\text{T}} \otimes R \otimes y^{-1}$ induces an adjunction on the right $(\cdot^{\forall}_{R^{\text{T}}}, \cdot^{\exists}_{R^{\text{T}}}) : (\tilde{\mathcal{X}}^{\gamma})^{-1} \rightleftharpoons \tilde{\mathcal{Y}}^{\mu}$ between the scaled spaces through the pair of adjunct maps:

$$x^{\forall}_{R^{\text{T}}} = R^{\text{T}} \otimes x \qquad y^{\exists}_{R^{\text{T}}} = R^{-1} \dot{\otimes} y \qquad (57)$$

whose compositions are, again, the interior of objects and the closure of attributes:

$$\kappa_{R^{-1}}^{\text{d}}(x) = (x^{\forall}_{R^{\text{T}}})^{\exists}_{R^{\text{T}}} \qquad \pi_{R^{\text{T}}}^{\text{d}}(y) = (y^{\exists}_{R^{\text{T}}})^{\forall}_{R^{\text{T}}} \qquad (58)$$

which define another bijection between the neighbourhood of objects $\underline{\mathfrak{N}}_X^{\gamma}$ and system of intents $\overline{\mathfrak{B}}_Y^{\mu}$:

$$\underline{\mathfrak{N}}_X^{\gamma} = (\tilde{\mathcal{Y}}^{\mu})^{\exists}_{R^{\text{T}}} \qquad \overline{\mathfrak{B}}_Y^{\mu} = ((\tilde{\mathcal{X}}^{\gamma})^{-1})^{\forall}_{R^{\text{T}}} \qquad (59)$$

5. The systems of φ -extents, -intents, -co-extents and -co-intents carry a $\overline{\mathcal{K}}$ - or $\underline{\mathcal{K}}$ -semimodule structure generated by the rows and columns of the matrix:

$$\begin{aligned} \overline{\mathfrak{B}}_X^{\gamma} &= \langle R \rangle_{\overline{\mathcal{K}}} & \overline{\mathfrak{B}}_Y^{\mu} &= \langle R^{\text{T}} \rangle_{\overline{\mathcal{K}}} \\ \underline{\mathfrak{N}}_Y^{\mu} &= \langle R^{\otimes} \rangle_{\underline{\mathcal{K}}} & \underline{\mathfrak{N}}_X^{\gamma} &= \langle R^{-1} \rangle_{\underline{\mathcal{K}}} \end{aligned} \qquad (60)$$

In particular, they are not sub-semimodules of the ambient spaces but their order duals, $\overline{\mathfrak{B}}_X^{\gamma} \subseteq (\tilde{\mathcal{X}}^{\gamma})^{-1}$, $\overline{\mathfrak{B}}_Y^{\mu} \subseteq (\tilde{\mathcal{Y}}^{\mu})^{-1}$, $\underline{\mathfrak{N}}_Y^{\mu} \subseteq \tilde{\mathcal{Y}}^{\mu}$, and $\underline{\mathfrak{N}}_X^{\gamma} \subseteq \tilde{\mathcal{X}}^{\gamma}$.

6. These structures are isomorphic or dually isomorphic:

$$\overline{\mathfrak{B}}_X^\gamma \cong (\overline{\mathfrak{B}}_Y^\mu)^d \cong \mathfrak{N}_Y^\mu \cong (\mathfrak{N}_X^\gamma)^d \tag{61}$$

7. In fact, they carry a complete lattice structure with operations:

$$\begin{aligned} a_i \in \overline{\mathfrak{B}}_X^\gamma, & \quad \bigwedge_{i \in I} a_i = \sum_{i \in I} \bullet a_i & \quad \bigvee_{i \in I} a_i = \pi_R^d(\sum_{i \in I} \bullet a_i) \\ b_i \in \overline{\mathfrak{B}}_Y^\mu, & \quad \bigwedge_{i \in I} b_i = \sum_{i \in I} \bullet b_i & \quad \bigvee_{i \in I} b_i = \pi_{R^T}^d(\sum_{i \in I} \bullet b_i) \\ d_i \in \mathfrak{N}_Y^\mu, & \quad \bigwedge_{i \in I} d_i = \kappa_{R^*}^d(\sum_{i \in I} \bullet d_i) & \quad \bigvee_{i \in I} d_i = \sum_{i \in I} \bullet d_i \\ c_i \in \mathfrak{N}_X^\gamma, & \quad \bigwedge_{i \in I} c_i = \kappa_{R^{-1}}^d(\sum_{i \in I} \bullet c_i) & \quad \bigvee_{i \in I} c_i = \sum_{i \in I} \bullet c_i \end{aligned} \tag{62}$$

Proof. Conventional approach: consider the bracket $\langle x \mid R \mid y \rangle_{\mathcal{K}}^{\text{OI}} = x \dot{\setminus} R / \dot{\setminus} y^T = x^* \otimes R \otimes y^{-1}$ using residuation in \mathcal{K} . Based on merely formal similarities to item 1 of Theorem 4, this is going to generate a Galois connection—in the order of \mathcal{K} which is dual to that of $\overline{\mathcal{K}}$. By following the proof in [24] we arrive at the new polars and thence, to the closures. Four-fold GC-dually.

Short approach: use bias-duality on the results of Theorem 4 and Proposition 3. \square

These results are summarized, again in a schematic diagram in Figure 5.

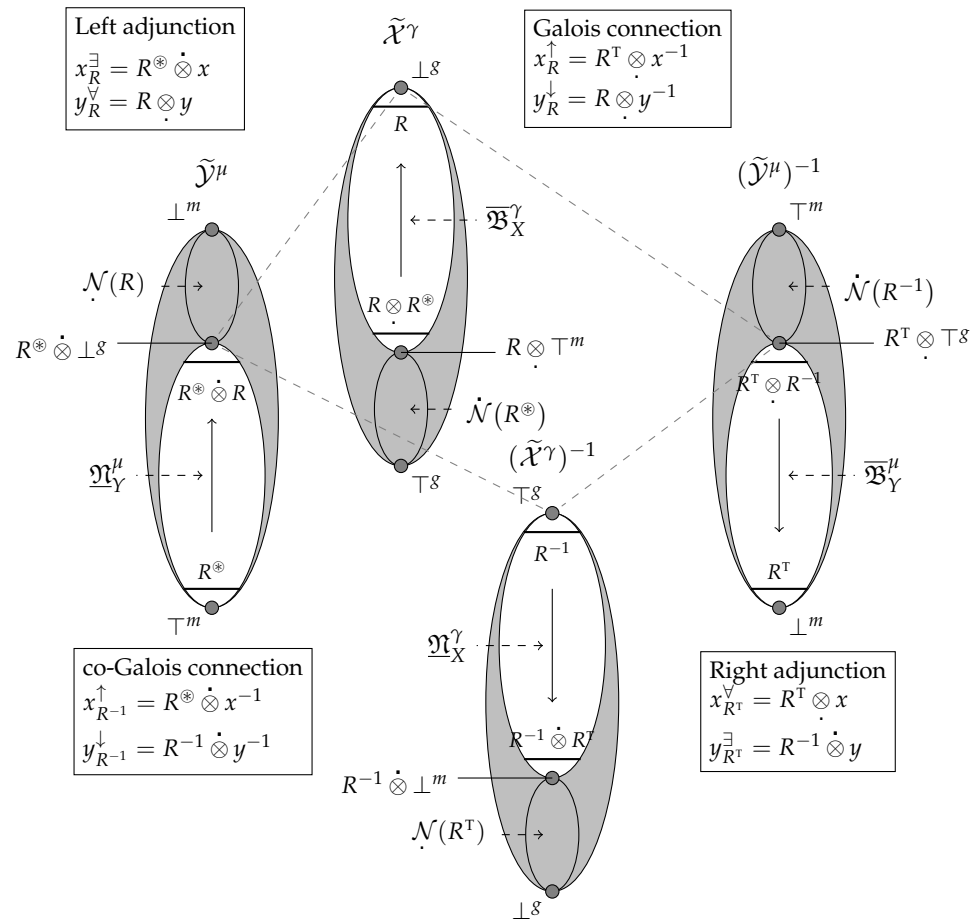


Figure 5. Schematics of the 4-fold Galois connection of R with \mathcal{K} bias, i.e., $\tilde{\mathcal{X}}^\gamma = \widetilde{(\mathcal{K}^s)}^\gamma$ (cfr. Figure 2). Note that the increasing sense of the arrows refers now to that of \mathcal{K} , dually-ordered to $\overline{\mathcal{K}}$.

Note that ([8], p. 70) already displays a less detailed version of Figures 2 and 5 side-by-side. Although the order-theoretic implications were not evident there, the following was:

Proposition 4. *Let K be the carrier set of both $\overline{\mathcal{K}}$ and $\underline{\mathcal{K}}$ and consider the matrix $R \in K^{s \times m}$, and an invertible element $\varphi = \gamma \otimes \mu \in K$. In general $\overline{\mathcal{K}}$ - and $\underline{\mathcal{K}}$ - $\overline{\mathcal{K}}$ -four-fold Formal Concept Analysis produce non-isomorphic lattices.*

Proof. Note that the fixpoint operators with $\underline{\mathcal{K}}$ bias in (49), (52), (55) or (58) are none of those with $\overline{\mathcal{K}}$ bias in (6), (11), (14) or (17). Table 1 shows how these two sets of operators relate (e.g., in bias duality).

Table 1. Comparison of closure and interior (kernel) operators obtained by changing the bias of Formal Concept Analysis (\mathcal{K} -FCA) from $\overline{\mathcal{K}}$ (left) to $\underline{\mathcal{K}}$ (right). Note how these are distinct but systematically related operators.

| Type of Fixpoint | with Bias in $\overline{\mathcal{K}}$ | with Bias in $\underline{\mathcal{K}}$ |
|------------------|---|---|
| extent | $\pi_R(x) = R \overset{\circ}{\otimes} (R^{\otimes} \overset{\circ}{\otimes} x)$ | $\pi_R^d(x) = R \overset{\circ}{\otimes} (R^{\otimes} \overset{\circ}{\otimes} x)$ |
| intent | $\pi_{R^T}(y) = R^T \overset{\circ}{\otimes} (R^{-1} \overset{\circ}{\otimes} y)$ | $\pi_{R^T}^d(y) = R^T \overset{\circ}{\otimes} (R^{-1} \overset{\circ}{\otimes} y)$ |
| co-extent | $\kappa_{R^{-1}}(x) = R^{-1} \overset{\circ}{\otimes} (R^T \overset{\circ}{\otimes} x)$ | $\kappa_{R^{-1}}^d(x) = R^{-1} \overset{\circ}{\otimes} (R^T \overset{\circ}{\otimes} x)$ |
| co-intent | $\kappa_{R^{\otimes}}(y) = R^{\otimes} \overset{\circ}{\otimes} (R \overset{\circ}{\otimes} y)$ | $\kappa_{R^{\otimes}}^d(y) = R^{\otimes} \overset{\circ}{\otimes} (R \overset{\circ}{\otimes} y)$ |

By a well-known result in order theory, different closure (kernel) operators generate different closure (interior) systems. \square

3.1.6. Generators and Bases

To find minimal representations for the lattices involved in Theorems 4 and 5 we introduce:

Lemma 4 (Block forms involving sets of independent vectors, $\underline{\mathcal{K}}$ bias). *Let K be the support set of a complete idempotent semifield $\underline{\mathcal{K}}$, a matrix $R \in \underline{\mathcal{K}}^{s \times m}$. Then,*

1. R can be reordered and factorized as

$$R = \begin{bmatrix} R_{IJ} & R_{I\bar{J}} \\ R_{\bar{I}J} & R_{\bar{I}\bar{J}} \end{bmatrix} \tag{63}$$

where I and J are the sets of indices of $\underline{\mathcal{K}}$ -independent rows and columns $|I| = l$ and $|J| = k$, respectively.

2. Alternatively, R can be factorized as

$$R = \begin{bmatrix} \underline{V} & \underline{V} \overset{\circ}{\otimes} \underline{W} \end{bmatrix} \qquad R = \begin{bmatrix} \underline{U} \\ \underline{Z} \overset{\circ}{\otimes} \underline{U} \end{bmatrix} \tag{64}$$

where \underline{V} and \underline{U} are the sets of $\underline{\mathcal{K}}$ -independent columns and rows of R , respectively, and \underline{W} and \underline{Z} are the coefficient matrices used to obtain the dependent columns and rows from them.

3. Furthermore, R can be factorized as

$$R = \begin{bmatrix} \overset{\circ}{E}_l \\ \underline{Z} \end{bmatrix} \overset{\circ}{\otimes} R_{IJ} \overset{\circ}{\otimes} \begin{bmatrix} \overset{\circ}{E}_k & \underline{W} \end{bmatrix} \quad \text{with} \quad \underline{V} = \begin{bmatrix} \overset{\circ}{E}_l \\ \underline{Z} \end{bmatrix} \overset{\circ}{\otimes} R_{IJ} \quad \text{and} \quad \underline{U} = R_{IJ} \overset{\circ}{\otimes} \begin{bmatrix} \overset{\circ}{E}_k & \underline{W} \end{bmatrix} \tag{65}$$

where $\underline{W} = R_{I\bar{J}}^{\otimes} \overset{\circ}{\otimes} R_{\bar{I}J}$ and $\underline{Z} = R_{\bar{I}J} \overset{\circ}{\otimes} R_{IJ}^{\otimes}$.

Proof. For the first statement, we use the \mathcal{K} version of the \mathcal{A} -test [6] (§ 2.1.3, Proposition 2) to detect a set of dependent columns with indices \bar{J} so that the independent indices are $J = M \setminus \bar{J}$, and likewise for the indices of the dependent rows \bar{I} and the independent ones $I = G \setminus \bar{I}$. From now on consider their cardinalities fixed as $0 < |J| = k \leq m$ and $0 < |I| = l \leq g$. Using these indices, we build a permutation matrix for rows P and another for columns Q as customary. By applying these to R' we obtain the required block structure on our target matrix $R = P^T \dot{\otimes} R' \dot{\otimes} Q$. Note that $R_{IJ} \in K^{l \times k}$ is the submatrix selected by the indices of \mathcal{K} -independent rows and columns.

To make (63) and (64) compatible, call $\underline{V} = [R_{IJ}^T \quad R_{\bar{I}\bar{J}}^T]^T$. Since \underline{V} are an independent set, they \mathcal{K} -generate the other columns, say as $\underline{V} \dot{\otimes} W = [R_{\bar{I}\bar{J}}^T \quad R_{\bar{I}\bar{J}}^T]^T$ whence $R = [\underline{V} \quad \underline{V} \dot{\otimes} W]$. Similarly, by transposing R and following the previous procedure we have $\underline{U}^T = [R_{IJ}^T \quad R_{\bar{I}\bar{J}}^T]^T$ we arrive at $R^T = [\underline{U}^T \quad \underline{U}^T \dot{\otimes} \underline{Z}^T]^T$ whence transposing obtains the second part of (64).

For the third statement, from the first part of (64) we have $\underline{U} = [R_{IJ} \quad R_{IJ} \dot{\otimes} \underline{W}]$ that we may factorize as $\underline{U} = R_{IJ} \dot{\otimes} [\dot{E}_k \quad \underline{W}]$. Dually, we can state $\underline{V}^T = [R_{\bar{I}\bar{J}}^T \quad \underline{W}^T \dot{\otimes} R_{\bar{I}\bar{J}}^T]^T = [\dot{E}_l \quad \underline{W}^T] \dot{\otimes} R_{\bar{I}\bar{J}}^T$. By applying the second part of (64) we find that $R = \begin{bmatrix} R_{IJ} & R_{IJ} \dot{\otimes} \underline{W} \\ \underline{Z} \dot{\otimes} R_{IJ} & \underline{Z} \dot{\otimes} R_{IJ} \dot{\otimes} \underline{W} \end{bmatrix}$ and factorizing left and right obtains the result. To find

\underline{W} we equate the last form and (63) so we get $R_{IJ} \dot{\otimes} \underline{W} = R_{\bar{I}\bar{J}}$. Since R_{IJ} is just a matrix, the previous equation means that in the GC of R_{IJ} , the intents W^{-1} get transformed as $(\underline{W}^{-1})_{R_{IJ}}^\downarrow = R_{IJ} \dot{\otimes} (\underline{W}^{-1})^{-1} = R_{\bar{I}\bar{J}}$ which are extents. Therefore these extents re-define \underline{W} as: $(R_{\bar{I}\bar{J}})_{R_{IJ}}^\uparrow = R_{IJ}^T \dot{\otimes} R_{\bar{I}\bar{J}}^{-1} = \underline{W}^{-1}$ whence $\underline{W} = R_{IJ}^{\otimes} \dot{\otimes} R_{\bar{I}\bar{J}}$. Row-column and GC-dually, we find $\underline{Z} = R_{\bar{I}\bar{J}} \dot{\otimes} R_{IJ}^{\otimes}$. \square

Note the following:

- The material in statement 2 was introduced in ([6], Definition 9). The novelty in the lemma is essentially parts 1 and 3.
- Note that the notation highlights the fact that we are using \mathcal{K} bias. The dual-bias proposition (implicit in the text) should change the notation accordingly to \bar{V}, \bar{U} , etc.

A reordering of rows (or columns) does not change the subspace being generated by the matrix. We introduce a slight complexification of the notation for the lattices to make evident which matrix they relate to:

Lemma 5. *In the conditions of Theorem 3 and the notation in Lemma 4:*

$$\mathfrak{B}_X^\gamma(R) = \mathfrak{B}_X^\gamma(V) \quad \mathfrak{B}_Y^\mu(R) = \mathfrak{B}_Y^\mu(U) \quad \bar{\mathfrak{N}}_Y^\mu(R) = \bar{\mathfrak{N}}_Y^\mu(U) \quad \bar{\mathfrak{N}}_X^\gamma(R) = \bar{\mathfrak{N}}_X^\gamma(V) \quad (66)$$

and in the conditions of Theorem 5 and the bias-dual of Lemma 4:

$$\bar{\mathfrak{B}}_X^\gamma(R) = \bar{\mathfrak{B}}_X^\gamma(\bar{V}) \quad \bar{\mathfrak{B}}_Y^\mu(R) = \bar{\mathfrak{B}}_Y^\mu(\bar{U}) \quad \mathfrak{N}_Y^\mu(R) = \mathfrak{N}_Y^\mu(\bar{U}) \quad \mathfrak{N}_X^\gamma(R) = \mathfrak{N}_X^\gamma(\bar{V}) \quad (67)$$

Proof. Recall that for a generic matrix $R \in \overline{\mathcal{K}}^{s \times m}$ we defined $\mathfrak{B}_X^\gamma(R) = \{a \in \overline{\mathcal{K}}^s \mid \pi_R(a) = a\}$ with $\pi_R(x) = R \dot{\otimes} (R^\otimes \dot{\otimes} x)$. To prove $\mathfrak{B}_X^\gamma(R) = \mathfrak{B}_X^\gamma(\underline{V})$, let $a \in \mathfrak{B}_X^\gamma(R) = \langle R \rangle_{\overline{\mathcal{K}}}$, so that $a = R \dot{\otimes} d$. Recall from (64) that $R = \begin{bmatrix} \underline{V} & \underline{V} \dot{\otimes} \underline{W} \end{bmatrix} \dot{\otimes} d$, therefore

$$\begin{aligned} \pi_{\underline{V}}(a) &= \underline{V} \dot{\otimes} (\underline{V}^\otimes \dot{\otimes} a) = \underline{V} \dot{\otimes} (\underline{V}^\otimes \dot{\otimes} (R \dot{\otimes} d)) = \underline{V} \dot{\otimes} (\underline{V}^\otimes \dot{\otimes} (\begin{bmatrix} \underline{V} & \underline{V} \dot{\otimes} \underline{W} \end{bmatrix} \dot{\otimes} d)) \\ &= \begin{bmatrix} \underline{V} \dot{\otimes} (\underline{V}^\otimes \dot{\otimes} (\underline{V} \dot{\otimes} d)) & \underline{V} \dot{\otimes} (\underline{V}^\otimes \dot{\otimes} (\underline{V} \dot{\otimes} \underline{W} \dot{\otimes} d)) \end{bmatrix} \end{aligned}$$

and by the properties of matrix products we finally have:

$$\pi_{\underline{V}}(a) = \begin{bmatrix} \underline{V} \dot{\otimes} d & \underline{V} \dot{\otimes} \underline{W} \dot{\otimes} d \end{bmatrix} = \begin{bmatrix} \underline{V} & \underline{V} \dot{\otimes} \underline{W} \end{bmatrix} \dot{\otimes} d = R \dot{\otimes} d = a$$

whence $\mathfrak{B}_X^\gamma(R) \subseteq \mathfrak{B}_X^\gamma(\underline{V})$. Now let $a \in \mathfrak{B}_X^\gamma(\underline{V}) = \langle \underline{V} \rangle_{\overline{\mathcal{K}}}$, so that $a = \underline{V} \dot{\otimes} d'$, whence

$$\begin{aligned} \pi_R(a) &= R \dot{\otimes} (R^\otimes \dot{\otimes} a) = \begin{bmatrix} \underline{V} & \underline{V} \dot{\otimes} \underline{W} \end{bmatrix} \dot{\otimes} (\begin{bmatrix} \underline{V} & \underline{V} \dot{\otimes} \underline{W} \end{bmatrix}^\otimes \dot{\otimes} (\underline{V} \dot{\otimes} d')) \\ &= \underline{V} \dot{\otimes} (\underline{V}^\otimes \dot{\otimes} (\underline{V} \dot{\otimes} d')) \dot{\oplus} (\underline{V} \dot{\otimes} \underline{W}) \dot{\otimes} ((\underline{V} \dot{\otimes} \underline{W})^\otimes \dot{\otimes} (\underline{V} \dot{\otimes} d')) \end{aligned}$$

Note that the second terms is the closure $\pi_{\underline{V} \dot{\otimes} \underline{W}}(\underline{V} \dot{\otimes} d')$ and that the first term can be reduced by the matrix products so

$$\pi_R(a) = (\underline{V} \dot{\otimes} d') \dot{\oplus} \pi_{\underline{V} \dot{\otimes} \underline{W}}(\underline{V} \dot{\otimes} d') = \underline{V} \dot{\otimes} d' = a$$

where the last but one equality comes from the closure property $\pi_{\underline{V} \dot{\otimes} \underline{W}}(\underline{V} \dot{\otimes} d') \geq \underline{V} \dot{\otimes} d'$ and the fact that $\dot{\oplus}$ selects the minimum. Therefore $\mathfrak{B}_X^\gamma(\underline{V}) \subseteq \mathfrak{B}_X^\gamma(R)$. Four-GC dually, and bias dually. \square

Some comments are in order:

- When referring to the lattices of different matrices, as is the case with R and \underline{V} , we cannot escape their appearing in the notation, as in $\mathfrak{B}_X^\gamma(R)$, since they are, in general, different lattices. The pertinence of the decomposition of R into its independent rows is precisely that the lattices remain the same.
- In $a = R \dot{\otimes} d = d_R^\forall$ the name is fixed since we know that $a = d_R^\forall \iff d = a_R^\exists \in \overline{\mathfrak{N}}_Y^\mu(R)$. Indeed, the 4-fold GC dual proves involve using $b = a_R^\uparrow = \mathfrak{B}_Y^\mu(R)$ and $c = b_{R^\dagger}^\exists \in \overline{\mathfrak{N}}_X^\gamma(R)$. However note that in $a = \underline{V} \dot{\otimes} d'$ in case $k \neq m$ we have $d' \in \overline{\mathcal{K}}^k$ and therefore $d' \notin \overline{\mathfrak{N}}_Y^\mu(R) \subseteq \overline{\mathcal{K}}^m$ (see below).
- We crucially triggered the simplification process for idempotent matrix products by introducing $a = R \dot{\otimes} d$. We also used the fact that products of matrices generate closure (or kernel) operators that allow us to discard the closed (or open) element being operated upon based on the properties of the addition $\dot{\oplus}$ (or \oplus). These two patterns reappear time an again in demonstrations with the blocked forms. With the preceding Lemma we have proven also the following corollaries.

Corollary 6. *In the conditions of Theorem 3 and the notation in Lemma 4:*

$$\mathfrak{B}_X^\gamma(R) = \langle \underline{V} \rangle_{\overline{\mathcal{K}}} \quad \mathfrak{B}_Y^\mu(R) = \langle \underline{U}^\dagger \rangle_{\overline{\mathcal{K}}} \quad \overline{\mathfrak{N}}_Y^\mu(R) = \langle \underline{U}^\otimes \rangle_{\overline{\mathcal{K}}} \quad \overline{\mathfrak{N}}_X^\gamma(R) = \langle \underline{V}^{-1} \rangle_{\overline{\mathcal{K}}} \quad (68)$$

and in the conditions of Theorem 5 and the bias-dual of Lemma 4

$$\overline{\mathfrak{B}}_X^\gamma(R) = \langle \overline{V} \rangle_{\overline{\mathcal{K}}} \quad \overline{\mathfrak{B}}_Y^\mu(R) = \langle \overline{U}^\top \rangle_{\overline{\mathcal{K}}} \quad \mathfrak{N}_Y^\mu(R) = \langle \overline{U}^\circ \rangle_{\mathcal{K}} \quad \mathfrak{N}_X^\gamma(R) = \langle \overline{V}^{-1} \rangle_{\mathcal{K}} \quad (69)$$

Proof. From the previous lemma we have $\mathfrak{B}_X^\gamma(R) = \mathfrak{B}_X^\gamma(V)$. From (42) we have $\mathfrak{B}_X^\gamma(V) = \langle \underline{V} \rangle_{\mathcal{K}}$. GC-dually for $\mathfrak{B}_Y^\mu(R)$. On the other hand, recall that $\mathfrak{B}_X^\gamma(R) = (\overline{\mathfrak{N}}_X^\gamma(R))^{-1} = \langle \underline{V} \rangle_{\mathcal{K}}$, whence $\overline{\mathfrak{N}}_X^\gamma(R) = (\langle \underline{V} \rangle_{\mathcal{K}})^{-1} = \langle \underline{V}^{-1} \rangle_{\overline{\mathcal{K}}}$, and co-GC-dually for $\overline{\mathfrak{N}}_Y^\mu(R)$. Bias-dually for (69). \square

Recall also that by [13] (Theorem 3.3.9) \underline{V} is a basis of this space, we have $\langle R \rangle_{\mathcal{K}} = \langle \underline{V} \rangle_{\mathcal{K}}$, another way to reach this result.

A simultaneous projection in both rows and columns, however, actually provides generators only for isomorphic spaces. We first prove a technical Lemma.

Lemma 6. In the conditions of Theorem 3, let R be factorized as in Lemma 4. Then the closure and interior operators of R_{IJ} can be expressed in terms of the sets of independent rows and columns of R , \underline{V} and \underline{U} as:

$$\begin{aligned} x \in \mathcal{K}^k, \pi_{\underline{U}}(x) &= \pi_{R_{IJ}}(x) & y \in \mathcal{K}^l, \pi_{\underline{V}^\top}(y) &= \pi_{R_{IJ}^\top}(y) \\ y \in (\mathcal{K}^l)^{-1}, \kappa_{\underline{V}^\circ}(y) &= \kappa_{R_{IJ}^\circ}(y) & x \in (\mathcal{K}^k)^{-1}, \kappa_{\underline{U}^{-1}}(x) &= \kappa_{R_{IJ}^{-1}}(x) \end{aligned} \quad (70)$$

Proof. For $y \in \mathcal{K}^l$, recall that $\pi_{\underline{V}^\top}(y) = \underline{V}^\top \dot{\otimes} (\underline{V}^{-1} \dot{\otimes} y)$ with $\underline{V}^\top = \left[R_{IJ}^\top \quad R_{IJ}^\top \dot{\otimes} \underline{Z}^\top \right]$ so that:

$$\begin{aligned} \pi_{\underline{V}^\top}(y) &= \left[R_{IJ}^\top \quad R_{IJ}^\top \dot{\otimes} \underline{Z} \right] \dot{\otimes} \left(\left[\begin{array}{c} R_{IJ}^{-1} \\ (\underline{Z}^\circ \dot{\otimes} R_{IJ}^{-1}) \end{array} \right] \right) \dot{\otimes} y = \left[R_{IJ}^\top \quad R_{IJ}^\top \dot{\otimes} \underline{Z} \right] \dot{\otimes} \left[\begin{array}{c} R_{IJ}^{-1} \dot{\otimes} y \\ (\underline{Z}^\circ \dot{\otimes} R_{IJ}^{-1}) \dot{\otimes} y \end{array} \right] = \\ &= R_{IJ}^\top \dot{\otimes} (R_{IJ}^{-1} \dot{\otimes} y) \dot{\oplus} (\underline{Z}^\top \dot{\otimes} R_{IJ})^\top \dot{\otimes} ((\underline{Z}^\top \dot{\otimes} R_{IJ})^{-1} \dot{\otimes} y) = R_{IJ}^\top \dot{\otimes} (R_{IJ}^{-1} \dot{\otimes} y) = \pi_{R_{IJ}^\top}(y) \end{aligned}$$

where the last but one step follows because $(\underline{Z}^\top \dot{\otimes} R_{IJ})^\top \dot{\otimes} ((\underline{Z}^\top \dot{\otimes} R_{IJ})^{-1} \dot{\otimes} y) = R_{IJ}^\top \dot{\otimes} \underline{Z} \dot{\otimes} (\underline{Z}^\circ \dot{\otimes} R_{IJ}^{-1} \dot{\otimes} y) \geq R_{IJ}^\top \dot{\otimes} (R_{IJ}^{-1} \dot{\otimes} y)$ since $\underline{Z} \otimes (\underline{Z}^\circ \otimes x) \geq x$ is a closure. The result follows because the upper addition selects the lowest term. For the other closure and interior operators we use $\underline{U} = \left[R_{IJ} \quad R_{IJ} \dot{\otimes} \underline{W} \right]$, \underline{U}^{-1} and \underline{V}° and find the results 4-fold GC-dually. \square

The previous lemma does not provide information unless both $k < m$ and $l < g$.

Proposition 5. In the conditions of Theorem 3, let R be factorized as in (65). Then, if $l < g$ and $k < m$,

$$\begin{aligned} \overline{\mathfrak{B}}_X^\gamma(R) &\cong \overline{\mathfrak{B}}_{X'}^\gamma(R_{IJ}) & \overline{\mathfrak{B}}_Y^\mu(R) &\cong \overline{\mathfrak{B}}_{Y'}^\mu(R_{IJ}) \\ \overline{\mathfrak{N}}_Y^\mu(R) &\cong \overline{\mathfrak{N}}_{Y'}^\mu(R_{IJ}) & \overline{\mathfrak{N}}_X^\gamma(R) &\cong \overline{\mathfrak{N}}_{X'}^\gamma(R_{IJ}) \end{aligned} \quad (71)$$

Proof. For the record, by Proposition 3 we have:

$$\begin{aligned} \mathfrak{B}_X^\gamma(R) &\cong (\mathfrak{B}_Y^\mu(R))^d \cong \overline{\mathfrak{N}}_{Y'}^\mu(R) \cong (\overline{\mathfrak{N}}_{X'}^\gamma(R))^d \\ \mathfrak{B}_X^\gamma(V) &\cong (\mathfrak{B}_{Y'}^\mu(V))^d \cong \overline{\mathfrak{N}}_{Y'}^\mu(V) \cong (\overline{\mathfrak{N}}_X^\gamma(V))^d \\ \mathfrak{B}_X^\gamma(U) &\cong (\mathfrak{B}_{Y'}^\mu(U))^d \cong \overline{\mathfrak{N}}_{Y'}^\mu(U) \cong (\overline{\mathfrak{N}}_X^\gamma(U))^d \\ \mathfrak{B}_{X'}^\gamma(R_{IJ}) &\cong (\mathfrak{B}_{Y'}^\mu(R_{IJ}))^d \cong \overline{\mathfrak{N}}_{Y'}^\mu(R_{IJ}) \cong (\overline{\mathfrak{N}}_{X'}^\gamma(R_{IJ}))^d. \end{aligned} \quad (72)$$

First we prove that if $k < m$ calling $\tilde{\mathcal{X}}^{\gamma} \equiv \bar{\mathcal{K}}^l$ then $\bar{\mathfrak{B}}_{\mathcal{X}}^{\gamma}(R) \cong \bar{\mathfrak{B}}_{\mathcal{X}'}^{\gamma}(R_{IJ})$ and $\bar{\mathfrak{N}}_{\mathcal{X}}^{\gamma}(R) \cong \bar{\mathfrak{N}}_{\mathcal{X}'}^{\gamma}(R_{IJ})$. Note that $\bar{\mathcal{K}}^m \equiv \tilde{\mathcal{Y}}^{\mu} \neq \tilde{\mathcal{Y}}^{\mu} \equiv \bar{\mathcal{K}}^k$ and we work with \underline{V} . Then, by Lemma 5, $\mathfrak{B}_{\mathcal{X}}^{\gamma}(R) \equiv \mathfrak{B}_{\mathcal{X}}^{\gamma}(\underline{V})$, and by Lemma 6 we have $\pi_{\underline{V}^T}(y) = \pi_{R_{IJ}^T}(y)$ that is, $\mathfrak{B}_{\mathcal{X}'}^{\mu}(\underline{V}) \equiv \mathfrak{B}_{\mathcal{X}'}^{\mu}(R_{IJ})$. Therefore by composition $\mathfrak{B}_{\mathcal{X}}^{\gamma}(R) \equiv \mathfrak{B}_{\mathcal{X}}^{\gamma}(\underline{V}) \cong (\mathfrak{B}_{\mathcal{X}'}^{\mu}(\underline{V}))^d \equiv (\mathfrak{B}_{\mathcal{X}'}^{\mu}(R_{IJ}))^d \cong \mathfrak{B}_{\mathcal{X}'}^{\gamma}(R_{IJ})$ where we have called $\tilde{\mathcal{X}}^{\gamma} \equiv \bar{\mathcal{K}}^l$, although in general we only know $l \leq g$. Of course, if $l = g$ we have $\tilde{\mathcal{X}}^{\gamma} \equiv \tilde{\mathcal{X}}^{\gamma}$ and $\mathfrak{B}_{\mathcal{X}}^{\gamma}(R) \equiv \mathfrak{B}_{\mathcal{X}}^{\gamma}(\underline{V}) \equiv \mathfrak{B}_{\mathcal{X}'}^{\gamma}(R_{IJ})$. Similarly, from $\bar{\mathfrak{N}}_{\mathcal{X}}^{\gamma}(R) \equiv \bar{\mathfrak{N}}_{\mathcal{X}}^{\gamma}(\underline{V})$ and $\kappa_{\underline{V}^{\otimes}}(y) = \kappa_{R_{IJ}^{\otimes}}(y)$ we prove $\bar{\mathfrak{N}}_{\mathcal{X}}^{\gamma}(R) \equiv \bar{\mathfrak{N}}_{\mathcal{X}}^{\gamma}(\underline{V}) \cong \bar{\mathfrak{N}}_{\mathcal{X}'}^{\gamma}(R_{IJ})$.

When $l < g$ with $\tilde{\mathcal{Y}}^{\mu} \equiv \bar{\mathcal{K}}^k$ we proceed row-column dually, using \bar{U} as the pivot and the third line of (72) to prove that $\bar{\mathfrak{B}}_{\mathcal{Y}}^{\mu}(R) \cong \bar{\mathfrak{B}}_{\mathcal{Y}'}^{\mu}(R_{IJ})$, and $\bar{\mathfrak{N}}_{\mathcal{Y}}^{\mu}(R) \cong \bar{\mathfrak{N}}_{\mathcal{Y}'}^{\mu}(R_{IJ})$.

If we demand that both $l < g$ and $k < m$ we obtain the desired results. □

Note that the bias duals of Lemma 6 and Proposition 5 demand that we use the $\bar{\mathcal{K}}$ -version of the test to detect $\bar{\mathcal{K}}$ -independent rows and columns I' and J' .

So when either $l < g$ or $k < m$ we can effectively reduce the computation of the original lattices of R to those of R_{IJ} , the reason why this procedure is called a reduction ([19], chap. 1).

Proposition 3 along with Lemma 6 and Proposition 5 suggest that we reify the results of (43) and (61) (see also [23]).

Definition 3 (The quadrilattice). Choose bias $\bar{\mathcal{K}}$ and an invertible $\varphi \in \bar{\mathcal{K}}$. For a matrix $R \in \bar{\mathcal{K}}^{g \times m}$, its quadrilattice $\bar{\mathcal{Q}}^{\varphi}(R)_{\bar{\mathcal{K}}}$ is the set of four-tuples (a, b, c, d) with $a \in \mathfrak{B}_{\mathcal{X}}^{\gamma}(R)$, $b \in \mathfrak{B}_{\mathcal{Y}}^{\mu}(R)$, $c \in \bar{\mathfrak{N}}_{\mathcal{X}}^{\gamma}(R)$, and $d \in \bar{\mathfrak{N}}_{\mathcal{Y}}^{\mu}(R)$ issued from Proposition 3 where $a_R^{\uparrow} = b \Leftrightarrow b_R^{\downarrow} = a \Leftrightarrow b_{R^T}^{\exists} = c \Leftrightarrow a_R^{\exists} = d$. Its bias-dual from Theorem 5 is $\bar{\mathcal{Q}}^{\varphi}(R)_{\bar{\mathcal{K}}}$.

Note that these are isomorphic or dually isomorphic lattices $\mathfrak{B}_{\mathcal{X}}^{\gamma}(R) \cong (\mathfrak{B}_{\mathcal{Y}}^{\mu}(R))^d \cong \bar{\mathfrak{N}}_{\mathcal{Y}}^{\mu}(R) \cong (\bar{\mathfrak{N}}_{\mathcal{X}}^{\gamma}(R))^d$, and further $\mathfrak{B}_{\mathcal{X}}^{\gamma}(R) = (\bar{\mathfrak{N}}_{\mathcal{X}}^{\gamma}(R))^{-1}$ and $\mathfrak{B}_{\mathcal{Y}}^{\mu}(R) = (\bar{\mathfrak{N}}_{\mathcal{Y}}^{\mu}(R))^{-1}$.

The following Proposition gives the promised form to the isomorphisms in Proposition 5.

Proposition 6. In the conditions of Theorem 3 and the notation in Lemma 4, let R be factorized as in (63) and consider the quadrilattices $\bar{\mathcal{Q}}^{\varphi}(R)_{\bar{\mathcal{K}}}$ and $\bar{\mathcal{Q}}^{\varphi}(R_{IJ})_{\bar{\mathcal{K}}}$. Then, the following mappings,

$$\begin{aligned} \phi : \bar{\mathcal{Q}}^{\varphi}(R)_{\bar{\mathcal{K}}} &\rightarrow \bar{\mathcal{Q}}^{\varphi}(R_{IJ})_{\bar{\mathcal{K}}} & (73) \\ (a, b, c, d) &\mapsto (a', b', c', d') = \left(\begin{bmatrix} \dot{E}_l & \dot{\mathcal{E}} \end{bmatrix} \dot{\otimes} a, \begin{bmatrix} \dot{E}_k & \dot{\mathcal{E}} \end{bmatrix} \dot{\otimes} b, \begin{bmatrix} \dot{E}_l & \dot{\mathcal{E}} \end{bmatrix} \dot{\otimes} c, \begin{bmatrix} \dot{E}_k & \dot{\mathcal{E}} \end{bmatrix} \dot{\otimes} d \right) \end{aligned}$$

and:

$$\begin{aligned} \phi^{-1} : \bar{\mathcal{Q}}^{\varphi}(R_{IJ})_{\bar{\mathcal{K}}} &\rightarrow \bar{\mathcal{Q}}^{\varphi}(R)_{\bar{\mathcal{K}}} & (74) \\ (a', b', c', d') &\mapsto \left(\begin{bmatrix} \dot{E}_l \\ \dot{Z} \end{bmatrix} \dot{\otimes} a', \begin{bmatrix} \dot{E}_k \\ \dot{W}^T \end{bmatrix} \dot{\otimes} b', \begin{bmatrix} \dot{E}_l \\ \dot{Z}^{-1} \end{bmatrix} \dot{\otimes} c', \begin{bmatrix} \dot{E}_k \\ \dot{W}^{\otimes} \end{bmatrix} \dot{\otimes} d' \right) \end{aligned}$$

are quadrilattice isomorphisms.

Proof. Note that the transformations for c and d have been chosen to ensure $a' = c'^{-1}$ and $b' = d'^{-1}$ what means we need only concern ourselves with the first two pairs of mappings.

The trivial case is if $k = g$ or $l = m$, when we simply have the identities. However, for $k < g$ we have $\tilde{\mathcal{X}}^{\gamma} \equiv \bar{\mathcal{K}}^g \neq \bar{\mathcal{K}}^k \equiv \tilde{\mathcal{X}}^{\gamma}$, and, similarly, for $l < m$ we have $\tilde{\mathcal{Y}}^{\mu} \equiv \bar{\mathcal{K}}^g \neq \bar{\mathcal{K}}^k \equiv \tilde{\mathcal{Y}}^{\mu}$, and this extends to their inverses, hence the ambient spaces of the lattices above are different.

First we prove that each pair of component mappings is actually a bijection. Let $a \in \mathfrak{B}_X^\gamma(R)$ then $a = \underline{V} \dot{\otimes} d'$, and the mappings acting on $\mathfrak{B}_X^\gamma(R)$ and $\mathfrak{B}_X^\gamma(R_{IJ})$ are:

$$\phi_a(a) = \begin{bmatrix} \dot{E}_l & \dot{E} \end{bmatrix} \dot{\otimes} \underline{V} \dot{\otimes} d' = \begin{bmatrix} \dot{E}_l & \dot{E} \end{bmatrix} \dot{\otimes} \begin{bmatrix} R_{IJ} \\ \underline{Z} \dot{\otimes} R_{IJ} \end{bmatrix} \dot{\otimes} d' = R_{IJ} \dot{\otimes} d' = d' \downarrow_{R_{IJ}} = a' \quad (75)$$

whereas in the inverse direction for $a \in \mathfrak{B}_{X'}^\gamma(R_{IJ})$, with $a = R_{IJ} \dot{\otimes} d'$:

$$\phi_a^{-1}(a') = \begin{bmatrix} \dot{E}_l \\ \underline{Z} \end{bmatrix} \dot{\otimes} R_{IJ} \dot{\otimes} d' = \begin{bmatrix} R_{IJ} \\ \underline{Z} \dot{\otimes} R_{IJ} \end{bmatrix} \dot{\otimes} d' = \underline{V} \otimes d' = a \quad (76)$$

Four-fold GC-dually for the other pairs of mappings. Note that by combining both derivations we see that the same d' appear in the definition of a and a' .

To see that the overall mappings respect the tuples in the quadrilattice, first consider $(a', b', c', d') \in \overline{\mathfrak{Q}}^\theta(R_{IJ})_{\overline{K}}$. We have to prove $a'_R \uparrow = b$: by definition $a' = \phi_a(a)$, and we know that $b' = (a') \uparrow_{R_{IJ}} = R_{IJ}^T \dot{\otimes} (a')^{-1}$, so

$$b = \begin{bmatrix} \dot{E}_k \\ \underline{W}^T \end{bmatrix} \dot{\otimes} b' = \begin{bmatrix} \dot{E}_k \\ \underline{W}^T \end{bmatrix} \dot{\otimes} R_{IJ}^T \dot{\otimes} (a')^{-1} = \underline{U}^T \dot{\otimes} (a')^{-1}$$

$$b'_R \downarrow = R \dot{\otimes} b^{-1} = \begin{bmatrix} \underline{U} \\ \underline{Z} \dot{\otimes} \underline{U} \end{bmatrix} \dot{\otimes} (\underline{U}^* \dot{\otimes} a') = \begin{bmatrix} \dot{E}_l \\ \underline{Z} \end{bmatrix} \dot{\otimes} \underline{U} \dot{\otimes} (\underline{U}^* \dot{\otimes} a')$$

but since $\pi_{\underline{U}}(a') = \pi_{R_{IJ}}(a') = a'$, we have $b'_R \downarrow = \begin{bmatrix} \dot{E}_l \\ \underline{Z} \end{bmatrix} \dot{\otimes} a' = a$. With a similar chain of equalities we can prove $a'_R \uparrow = b$. Note that the proof that $(a', b', c', d') = \phi(a, b, c, d) \in \overline{\mathfrak{Q}}^\theta(R_{IJ})_{\overline{K}}$ would use the same chains of proofs as the derivation above, but positing $b' \downarrow_{R_{IJ}} = a' \iff a'_R \uparrow = b'$ and trying to find whether $b'_R \downarrow = a \iff a'_R \uparrow = b$.

It is easy to see that the components are isomorphisms of complete lattices. For this proof let $i \in I$ by any set of indices. Then,

$$\phi_a(\wedge_{i \in I} a_i) = \begin{bmatrix} \dot{E}_l & \dot{E} \end{bmatrix} \dot{\otimes} (\sum_i \bullet a_i) = \sum_{i \in T} \bullet \begin{bmatrix} \dot{E}_l & \dot{E} \end{bmatrix} \dot{\otimes} a_i = \wedge_i \phi_a(a_i)$$

where we have used that $\mathfrak{B}_X^\gamma(R)$ and $\mathfrak{B}_X^\gamma(R_{IJ})$ are \mathcal{K} -semimodules whereby the meet is the semimodule addition. Four-fold GC dually for the meets in the other coordinate lattices.

To prove $\phi_a(\vee_{i \in I} a_i) = \vee_{i \in I} \phi_a(a_i)$, equivalently $\vee_{i \in I} a_i = \phi_a^{-1}(\vee_{i \in I} \phi_a(a_i))$, recall that for $\{a_i\}_{i \in I} \subseteq \mathfrak{B}_X^\gamma(R)$, the complete join is $\vee_{i \in I} a_i = \pi_R(\sum_{i \in I} \bullet a_i)$. We write:

$$\vee_{i \in I} a_i = \pi_R(\sum_{i \in I} \bullet a_i) = \pi_{\underline{V}}(\sum_{i \in I} \bullet a_i) = \underline{V} \dot{\otimes} (\underline{V}^* \dot{\otimes} (\sum_{i \in I} \bullet a_i))$$

Call $d'_i = \underline{V}^* \dot{\otimes} a_i$ so $a_i = \underline{V} \dot{\otimes} d'_i$ and $d' = \sum_{i \in I} \bullet d'_i$. Since $\kappa_{R^*}(d'_i) = \underline{V}^* \dot{\otimes} (\underline{V} \dot{\otimes} d'_i) = d'_i$ for all $i \in i$,

$$\vee_{i \in I} a_i = \underline{V} \dot{\otimes} (\underline{V}^* \dot{\otimes} (\sum_{i \in I} \bullet \underline{V} \dot{\otimes} d'_i)) = \underline{V} \dot{\otimes} (\sum_{i \in I} \bullet \underline{V}^* \dot{\otimes} (\underline{V} \dot{\otimes} d'_i)) = \underline{V} \dot{\otimes} (\sum_{i \in I} \bullet d'_i) = \underline{V} \dot{\otimes} d' = a$$

On the other hand, call $\phi_a(a_i) = a'_i$, so

$$\forall_{i \in I} \phi_a(a_i) = \pi_{R_{IJ}}(\sum_{i \in I} a'_i) = \pi_{\underline{U}}(\sum_{i \in I} a'_i) = \underline{U} \dot{\otimes} (\underline{U}^{\otimes} \dot{\otimes} (\sum_{i \in I} a'_i))$$

but since $\underline{U} = [R_{IJ} \quad R_{IJ} \dot{\otimes} \underline{W}] = R_{IJ} \dot{\otimes} [\dot{E}_k \quad \underline{W}]$ and $d'_i = R_{IJ}^{\otimes} \dot{\otimes} a'_i$, we write

$$\forall_{i \in I} \phi_a(a_i) = \underline{U} \dot{\otimes} \left(\left[\begin{matrix} E_k \\ \dot{W}^{\otimes} \end{matrix} \right] \dot{\otimes} (\sum_{i \in I} R_{IJ}^{\otimes} \dot{\otimes} a'_i) \right) = \underline{U} \dot{\otimes} \left(\left[\begin{matrix} E_k \\ \dot{W}^{\otimes} \end{matrix} \right] \dot{\otimes} (\sum_{i \in I} d'_i) \right) = \underline{U} \dot{\otimes} \left(\left[\begin{matrix} E_k \\ \dot{W}^{\otimes} \end{matrix} \right] \dot{\otimes} d' \right) = \underline{U} \dot{\otimes} d$$

with $\phi_d(d) = \left[\begin{matrix} E_k \\ \dot{\mathcal{E}} \end{matrix} \right] \dot{\otimes} (\sum_{i \in I} d_i) = \sum_{i \in I} \left[\begin{matrix} E_k \\ \dot{\mathcal{E}} \end{matrix} \right] \dot{\otimes} d_i = \sum_{i \in I} d'_i = d'$, whence

$$\phi_a^{-1}(\forall_{i \in I} \phi_a(a_i)) = \left[\begin{matrix} \dot{E}_l \\ \underline{Z} \end{matrix} \right] \dot{\otimes} ([R_{IJ} \quad R_{IJ} \dot{\otimes} \underline{W}] \dot{\otimes} d) = R \dot{\otimes} d = a$$

□

3.2. The Basics of $\overline{\mathcal{K}}$ -Four-Fold Formal Concept Analysis

In this section we transform the results on the 4-fold Galois connection in Section 3.1 as a basis for a $\overline{\mathcal{K}}$ -based four-fold FCA ($\overline{\mathcal{K}}$ -4FCA).

3.2.1. Framing the Analysis

First note that in order to use the results of Theorems 4 and 5—including those of Proposition 3—we have to provide the following items of information, a process we call framing (the analysis):

1. **BIASING.** Choosing a non-empty support K and the actual complete idempotent semifield whether $\overline{\mathcal{K}}$ or $\underline{\mathcal{K}}$.
2. **CONTEXTUALIZATION.** Providing $R \in K^{g \times m}$ and $G \neq \emptyset \neq M$, the sets of objects and attributes, respectively, with $|G| = g$ and $|M| = m$ to build the formal context $(G, M, R)_{\overline{\mathcal{K}}}$, since we use $\overline{\mathcal{K}}$ -bias.
3. **SCALING.** Providing invertible $\varphi = \gamma \otimes \mu$ used to scale the left and right semimodules.

Note that:

- An alternate way to carry out framing that does not alter its result is to first contextualize the analysis and then provide the bias.
- Yet some other work casts the systematic variation of φ as an exploration of the hidden regularities of the formal context [30,32,48].
- The corner cases for scaling $\varphi \in \{\perp, \top\}$ have limited interest, due to Proposition 2.
- In previous work, we have defined a fourth step of framing: COLOURING [23] the choice of which type of connection to do an analysis with. $\overline{\mathcal{K}}$ -4FCA makes this redundant.

3.2.2. 4-Fold Formal (φ) -Concepts

Contextualization allows us to use traditional FCA notation and write the name of the lattices in Proposition 3 by referring to the object G and attribute sets M as:

$$\begin{aligned} \mathfrak{B}_G^\gamma(G, M, R)_{\overline{\mathcal{K}}} &= \mathfrak{B}_X^\gamma(G, M, R)_{\overline{\mathcal{K}}} & \mathfrak{B}_M^\mu(G, M, R)_{\overline{\mathcal{K}}} &= \mathfrak{B}_Y^\mu(G, M, R)_{\overline{\mathcal{K}}} & (77) \\ \overline{\mathfrak{N}}_M^\mu(G, M, R)_{\overline{\mathcal{K}}} &= \overline{\mathfrak{N}}_Y^\mu(G, M, R)_{\overline{\mathcal{K}}} & \overline{\mathfrak{N}}_G^\gamma(G, M, R)_{\overline{\mathcal{K}}} &= \overline{\mathfrak{N}}_X^\gamma(G, M, R)_{\overline{\mathcal{K}}} \end{aligned}$$

Therefore, with the backing of (43) (and (61)) it makes sense to define the following [23]:

Definition 4. Let $(G, M, R)_{\overline{\mathcal{K}}}$ be a biased context and $\varphi = \gamma \otimes \mu$ invertible values. Then a generalized 4-fold formal φ -concept or quadriconcept is a 4-tuple (a, b, c, d) $a \in \mathfrak{B}_G^\gamma(G, M, R)_{\overline{\mathcal{K}}}$, $b \in \mathfrak{B}_M^\mu(G, M, R)_{\overline{\mathcal{K}}}$, $c \in \overline{\mathfrak{N}}_G^\gamma(G, M, R)_{\overline{\mathcal{K}}}$, $d \in \overline{\mathfrak{N}}_M^\mu(G, M, R)_{\overline{\mathcal{K}}}$, such that the following relations hold:

$$\begin{aligned} a &= (b)_{\downarrow R} = (d)_{\downarrow R} & b &= (a)_{\uparrow R} = (c)_{\downarrow R^\top} \\ d &= (c)_{\uparrow R^{-1}} = (a)_{\downarrow R} & c &= (d)_{\downarrow R^{-1}} = (b)_{\downarrow R^\top} \end{aligned} \tag{78}$$

The set of such quadriconcepts is $\Omega^\varphi(G, M, R)_{\overline{\mathcal{K}}}$. Notice that as a corollary of Lemma 3 and Definition 3 for (a, b, c, d) a generalized concept

$$a = c^{-1} \quad c = a^{-1} \quad b = d^{-1} \quad d = b^{-1} \tag{79}$$

and also that quadriconcepts are partially ordered with the hierarchical order

$$(a_1, b_1, c_1, d_1) \leq (a_2, b_2, c_2, d_2) \Leftrightarrow a_1 \leq a_2 \Leftrightarrow b_1 \geq b_2 \Leftrightarrow c_1 \geq c_2 \Leftrightarrow d_1 \leq d_2. \tag{80}$$

Due to the linear algebraic properties of the adjoints and Definition 4, we may easily build multiple four-fold concepts at the same time.

Corollary 7 (The concept-forming functions). *In the conditions of Definition 4, given a matrix $X \in \mathcal{K}^{g \times k}$ whose columns are \mathcal{K} -valued sets of objects we can build all of their four-fold concepts $\overline{\gamma}_4(X) = (A, B, C, D)$, where every co-indexed 4-tuple of columns is a four-fold concept, as*

$$B = X_{\uparrow R} \quad A = B_{\downarrow R} \quad C = A^{-1} \quad D = B^{-1}$$

Dually for matrices $Y \in \mathcal{K}^{m \times k}$ with \mathcal{K} -valued sets of attributes for columns with $\overline{\mu}_4(Y) = (A, B, C, D)$ as

$$A = Y_{\downarrow R} \quad B = A_{\uparrow R} \quad C = A^{-1} \quad D = B^{-1}$$

Proof. Just gather all those results as $\overline{\gamma}_4(X) = (A, B, C, D)$ coindexing extents, intents, co-extents and co-intents on the same generalized concept. Row-column dually for $\overline{\mu}_4(Y)$. \square

It is always interesting to know the top and bottom elements of the isomorphic lattices.

Corollary 8 (4-fold top and bottom). *In the conditions of Definition 4,*

$$\top_4 = \overline{\gamma}_4(\top^g) = (\top^g, R^\top \dot{\otimes} \perp^g, \perp^g, R^\otimes \dot{\otimes} \top^g) \quad \perp_4 = \overline{\mu}_4(\top^m) = (R \dot{\otimes} \perp^m, \top^m, R^{-1} \dot{\otimes} \top^m, \perp^m)$$

Note how the order of extents and intents are dual. These concepts have important special cases:

Lemma 7. *In the conditions of Definition 4,*

1. $\top_4 = (\top^g, \perp^m, \perp^g, \top^m)$ if and only if R has no full columns.
2. $\perp_4 = (\perp^g, \top^m, \top^g, \perp^m)$ if and only if R has no full rows.

Proof. Note that $R \dot{\otimes} \perp^m = \perp^g \iff R^{-1} \dot{\otimes} \top^m = \top^g$, so we only have to consider either extents or co-extents. Suppose $R \dot{\otimes} \perp^m = \perp^g$. Then, if $R_i = \top^{1 \times m}$, then $R_i \dot{\otimes} \perp^m = \top$, whence $R \dot{\otimes} \perp^m \neq \perp^g$, a contradiction. For the other direction, suppose for a row i there is a non-saturated column k , then $R_i \dot{\otimes} \perp^m = (\sum_{j \neq k} R_{ij} \otimes \perp) \dot{\oplus} \perp = \perp$. The result follows if this holds for every row. The other results follows column-row dually. \square

For concept lattices it is always important to obtain the object- and attribute-concepts, since they have special properties.

Lemma 8. *In the conditions of Definition 4, the object-concepts $\bar{\gamma}_4(E_g)$ and attribute-concepts $\bar{\mu}_4(E_m)$ are:*

$$\bar{\gamma}_4(E_g) = (R \dot{\otimes} R^{\otimes}, R^{\top}, R^{-1} \dot{\otimes} R^{\top}, R^{\otimes}) \quad \bar{\mu}_4(E_m) = (R, R^{\top} \dot{\otimes} R^{-1}, R^{-1}, R^{\otimes} \dot{\otimes} R) \quad (81)$$

These are join-dense and meet dense on the quadrilattice, respectively.

Proof. Since the columns of $X \equiv E_g$ are the singleton sets of objects, we have:

$$B = R^{\top} \dot{\otimes} (E_g)^{-1} = R^{\top} \quad A = R \dot{\otimes} R^{\otimes} \quad C = R^{-1} \dot{\otimes} R^{\top} \quad D = R^{\otimes} \dot{\otimes} (R \dot{\otimes} R^{\otimes}) = R^{\otimes}$$

Similarly, the columns of $Y \equiv E_m$ are the singleton sets of attributes, whence:

$$A = R \dot{\otimes} (E_m)^{-1} = R \quad B = R^{\top} \dot{\otimes} R^{-1} \quad C = R^{-1} \dot{\otimes} (R^{\top} \dot{\otimes} R^{-1}) = R^{-1} \quad D = R^{\otimes} \dot{\otimes} R$$

Note that the intents of $\bar{\gamma}_4(E_g)$ are meet-dense on \mathfrak{B}_M^{φ} , hence join-dense in $(\mathfrak{B}_M^{\varphi})^d$, and so $\bar{\gamma}_4(E_g)$ are join-dense in $\bar{\Omega}^{\varphi}(G, M, R)_{\bar{\mathcal{K}}}$. Dually for $\bar{\mu}_4(E_m)$, the meet-dense. \square

These four special elements are captured in Table 2, on the left, and also drawn in Figures 2 and 5.

Table 2. Special elements of the 4-fold lattices with biases in $\bar{\mathcal{K}}$ (left column) and $\underline{\mathcal{K}}$ (right column) (see Section 3.1.5).

| Special Element | Bias in $\bar{\mathcal{K}}$ | Bias in $\underline{\mathcal{K}}$ |
|-----------------|--|---|
| top | $\bar{\gamma}_4(\top^g) = (\top^g, R^{\top} \dot{\otimes} \perp^g, \perp^g, R^{\otimes} \dot{\otimes} \top^g)$ | $\bar{\gamma}_4(\perp^g) = (\perp^g, R^{\top} \dot{\otimes} \top^g, \top^g, R^{\otimes} \dot{\otimes} \perp^g)$ |
| meet dense | $\bar{\mu}_4(E_m) = (R, R^{\top} \dot{\otimes} R^{-1}, R^{-1}, R^{\otimes} \dot{\otimes} R)$ | $\bar{\mu}_4(\dot{E}_m) = (R, R^{\top} \dot{\otimes} R^{-1}, R^{-1}, R^{\otimes} \dot{\otimes} R)$ |
| join dense | $\bar{\gamma}_4(E_g) = (R \dot{\otimes} R^{\otimes}, R^{\top}, R^{-1} \dot{\otimes} R^{\top}, R^{\otimes})$ | $\bar{\gamma}_4(\dot{E}_g) = (R \dot{\otimes} R^{\otimes}, R^{\top}, R^{-1} \dot{\otimes} R^{\top}, R^{\otimes})$ |
| bottom | $\bar{\mu}_4(\top^m) = (R \dot{\otimes} \perp^m, \top^m, R^{-1} \dot{\otimes} \top^m, \perp^m)$ | $\bar{\mu}_4(\perp^m) = (R \dot{\otimes} \top^m, \perp^m, R^{-1} \dot{\otimes} \perp^m, \top^m)$ |

3.2.3. A Theorem of $\bar{\mathcal{K}}$ -Four-Fold Formal Concept Analysis

Theorem 6 (Basic theorem of $\bar{\mathcal{K}}$ -4FCA). *Let G be a set of formal objects, M a set of formal attributes, $(G, M, R)_{\bar{\mathcal{K}}}$ a $\bar{\mathcal{K}}$ -biased context, and invertible values $\varphi = \gamma \otimes \mu$. Then:*

1. **The context analysis phase.** Consider the scaled spaces of (29).
 - (a) The polar and co-polar operators, left and right adjuncts of (30), (33), (36), and (39) form a quadruple Galois connection as in Figures 2 and 4 whose generalized or 4-fold formal φ -concepts are the quadriconcepts (a, b, c, d) of Definition 4 whence the set of generalized formal concepts is $\Omega^{\varphi}(G, M, R)_{\bar{\mathcal{K}}}$.
 - (b) The set of quadriconcepts with the hierarchical order of Definition 4 is a complete lattice called the quadrilattice $\bar{\Omega}^{\varphi}(G, M, R)_{\bar{\mathcal{K}}} = \langle \Omega^{\varphi}(G, M, R)_{\bar{\mathcal{K}}}, \leq \rangle$ with infima and suprema given componentwise from (44) as:

$$\bigwedge_{t \in T} (a_t, b_t, c_t, d_t) = \left(\bigwedge_{t \in T} a_t, \bigvee_{t \in T} b_t, \bigvee_{t \in T} c_t, \bigwedge_{t \in T} d_t \right) \quad \bigvee_{t \in T} (a_t, b_t, c_t, d_t) = \left(\bigvee_{t \in T} a_t, \bigwedge_{t \in T} b_t, \bigwedge_{t \in T} c_t, \bigvee_{t \in T} d_t \right) \quad (82)$$

(c) The concept-forming functions $\bar{\gamma}_4(\cdot) : G \rightarrow \bar{\mathcal{Q}}^\varphi(G, M, R)_{\bar{K}}$ and $\bar{\mu}_4(\cdot) : M \rightarrow \bar{\mathcal{Q}}^\varphi(G, M, R)_{\bar{K}}$

$$g \mapsto \bar{\gamma}_4(g) = \bar{\gamma}_4\left(\left(\dot{E}_G\right)_g\right) \quad m \mapsto \bar{\mu}_4(m) = \bar{\mu}_4\left(\left(\dot{E}_M\right)_m\right) \quad (83)$$

—where $e_g = (\dot{E}_G)_g$ and $e_m = (\dot{E}_M)_m$ select the vectors representing the singletons $\{g\}$ and $\{m\}$, respectively, and \dot{E}_G and \dot{E}_M are the matrices of unitary vectors for objects and attributes, respectively—are mappings such that $\bar{\gamma}_4(G)$ is supremum-dense in $\bar{\mathcal{Q}}^\varphi(G, M, R)_{\bar{K}}$, and $\bar{\mu}_4(M)$ is infimum-dense in $\bar{\mathcal{Q}}^\varphi(G, M, R)_{\bar{K}}$.

2. The context synthesis phase.

(a) A complete lattice $\mathbb{L} = \langle L, \leq \rangle$ is isomorphic to (read “can be built as”) the quadrilattice $\bar{\mathcal{Q}}^\varphi(G, M, R)_{\bar{K}}$ if and only if there are mappings $\bar{\gamma} : G \rightarrow L$ and $\bar{\mu} : M \rightarrow L$ such that

- $\bar{\gamma}(G)$ is supremum-dense in \mathbb{L} , $\bar{\mu}(M)$ is infimum-dense in \mathbb{L} , and
- $R(g, m) \geq \varphi$ is equivalent to $\bar{\gamma}(g) \leq \bar{\mu}(m)$ for all $g \in G$ and all $m \in M$.

(b) In particular, consider the doubling context of \mathbb{L} , $K(\mathbb{L}) = (L, L, R_{\leq}^\varphi)$, and the standard context of \mathbb{L} , $S(\mathbb{L}) = (J(\mathbb{L}), M(\mathbb{L}), R_{\leq}^\varphi)$ where $J(\mathbb{L})$ and $M(\mathbb{L})$ are the sets of join- and meet-irreducibles, respectively, of \mathbb{L} , and

$$R_{\leq}^\varphi = \begin{cases} \varphi & \text{if } \bar{\gamma}(g) \leq \bar{\mu}(m) \\ \perp & \text{otherwise} \end{cases} \quad (84)$$

then

$$\bar{\mathcal{Q}}^\varphi(L, L, R_{\leq}^\varphi)_{\bar{K}} \cong \mathbb{L} \cong \bar{\mathcal{Q}}^\varphi(J(\mathbb{L}), M(\mathbb{L}), R_{\leq}^\varphi)_{\bar{K}}.$$

Proof. For the analysis part, statement (1.a) comes from Theorem 4, Proposition 3.2 and the definition of quadrilattice. For statement (1.b) note that the quadruple order of (80) is just the gathering of all the orderings of the connections. The component-wise infima and maxima come from Proposition 3.3. Finally, (1.c) are just object and attribute interfaces to the concept-forming functions of Corollary 7 based on the unitary vectors which obtain the join- and meet-irreducibles by Lemma 8, that is,

$$\bar{\gamma}_4(G) = \bar{\gamma}_4\left(\left(\dot{E}_G\right)\right) \quad \bar{\mu}_4(M) = \bar{\mu}_4\left(\left(\dot{E}_M\right)\right). \quad (85)$$

For the synthesis part, we apply Theorem 1 twice: first on the lattice \mathbb{L} to create and isomorphic copy of it as a concept lattice, $\mathfrak{B}(G, M, I)_1$, then once again on a quadrilattice and $\mathfrak{B}(G, M, I)_2$ and we require that both concept lattices are the same. Here are the details.

For the first application, we have that $\mathbb{L} \cong \mathfrak{B}(G, M, I)_1$ iff there are mappings $\bar{\gamma} : G \rightarrow L$ and $\bar{\mu} : M \rightarrow L$ so that $\bar{\gamma}(G)$ and $\bar{\mu}(M)$ are join- and meet dense in \mathbb{L} ; $gI_1m \iff \bar{\gamma}(g) \leq \bar{\mu}(m)$ for all $g \in G, m \in M$; and ([20], Ch. 3), provides an explicit form for the isomorphic pair of functions:

$$\begin{aligned} \phi_1 : \mathfrak{B}_G &\rightarrow L & \psi_1 : L &\rightarrow \mathfrak{B}_G & (86) \\ A \mapsto \phi_1(A) &= \bigvee_{g \in A} \bar{\gamma}(g) & x \mapsto \psi_1(x) &= \{g \in G \mid \bar{\gamma}(g) \leq x\} \triangleq A_x \end{aligned}$$

Note that the isomorphism is only actually with the lattice of extents instead of the whole concept lattice. Taking this hint for the second application we concentrate on the lattice of extents instead of the whole quadrilattice. We have that $\mathfrak{B}(G, M, I)_2 \cong \mathfrak{B}_G^\varphi(G, M, R)_{\overline{\mathcal{K}}}$ when there are mappings $\overline{\gamma}_4() : G \rightarrow \mathfrak{B}_G^\varphi(G, M, R)_{\overline{\mathcal{K}}}$ and $\overline{\mu}_4() : M \rightarrow \mathfrak{B}_G^\varphi(G, M, R)_{\overline{\mathcal{K}}}$ so that $\overline{\gamma}_4(G)$ and $\overline{\mu}_4(M)$ are join- and meet dense in $\overline{\mathfrak{Q}}^\varphi(G, M, R)_{\overline{\mathcal{K}}}$, respectively, and on the other hand, $gI_2m \iff \overline{\gamma}_4(g) \leq \overline{\mu}_4(m)$ for all $g \in G, m \in M$. We develop the last part of the equation:

$$\begin{aligned} \overline{\gamma}_4(g) \leq \overline{\mu}_4(m) &\iff \overline{\gamma}_4(e_g) \leq \overline{\mu}_4(e_m) && \text{(Definition)} \\ &\iff \pi_R(e_g) \leq e_m \downarrow_R && \text{(Focus on extents)} \\ &\iff R \dot{\otimes} (R^\otimes \otimes e_g) \leq R \dot{\otimes} (e_m)^{-1} && \text{(Definition of adjuncts)} \\ &\iff (e_g)^\otimes \dot{\otimes} R \geq (e_m)^\top && \text{(Compatibility of products, conjugation)} \\ &\iff (e_g)^\otimes \dot{\otimes} R \dot{\otimes} (e_m)^{-1} \geq e && \text{(Vector residuation)} \end{aligned}$$

If we restitute the expressions for the scalings in the ambient spaces:

$$\gamma^{-1} \otimes (e_g)^\otimes \dot{\otimes} R \dot{\otimes} (e_m)^{-1} \otimes \mu^{-1} \geq e \iff (e_g)^\otimes \dot{\otimes} R \dot{\otimes} (e_m)^{-1} \geq \gamma \otimes \mu = \varphi$$

whence

$$gI_2m \iff R(g, m) \geq \varphi \tag{87}$$

We can also provide an explicit form for the isomorphism, again onto the lattice of extents on both accounts using $\pi_R(g) = \pi_R(e_g)$:

$$\begin{aligned} \phi_4 : \mathfrak{B}_G &\rightarrow \mathfrak{B}_G^\varphi(G, M, R)_{\overline{\mathcal{K}}} & \psi_4 : \mathfrak{B}_G^\varphi(G, M, R)_{\overline{\mathcal{K}}} &\rightarrow \mathfrak{B}_G \\ A \mapsto \phi_4(A) &= \bigvee_{g \in A} \pi_R(g) & a \mapsto \psi_4(a) = A_a &= \{g \in G \mid \pi_R(g) \leq a\} \end{aligned} \tag{88}$$

To complete the proof, we just match these two isomorphisms: First, recall that the concept forming functions of part (1.c) obtain the join- and meet-dense as in (85). Notice how the G and M for $\mathfrak{B}(G, M, I)_2$ are those of $(G, M, R)_{\overline{\mathcal{K}}}$. When we equate it to $\mathfrak{B}(G, M, I)_1$ the same G and M are those to be injected into \mathbb{L} as dense elements. Second, if we equate both concept lattices, we also equate their induced contexts whence $\overline{\gamma}(g) \leq \overline{\mu}(m) \iff gIm \iff R(g, m) \geq \varphi$.

Third, since the composition of isomorphisms (of orders) is also an isomorphism (of orders), we get the full expression of the overarching isomorphism:

$$\begin{aligned} \phi : \mathfrak{B}_G^\varphi(G, M, R)_{\overline{\mathcal{K}}} &\rightarrow L & \psi : L &\rightarrow \mathfrak{B}_G^\varphi(G, M, R)_{\overline{\mathcal{K}}} \\ a \mapsto \phi(a) = \phi_1(\psi_4(a)) &= \bigvee_{g \in A_a} \overline{\gamma}(g) & x \mapsto \psi(x) = \phi_4(\psi_1(x)) &= \phi_4(A_x) = \bigvee_{g \in A_x} \pi_R(e_g) \end{aligned} \tag{89}$$

Then the statements in (2.b) follow by either considering $G \equiv L$ and $M \equiv L$ and the relation defined by (84) for the “double context” $(L, L, R_{\leq}^\varphi)_{\overline{\mathcal{K}}}$ or analogously concentrating on the join- and meet reducibles $G = \mathcal{J}(\mathbb{L})$ and $M = \mathcal{M}(\mathbb{L})$ for the “small context” $(\mathcal{J}(\mathbb{L}), \mathcal{M}(\mathbb{L}), R_{\leq}^\varphi)_{\overline{\mathcal{K}}}$. Of course, instead of just inserting the lattice of extents, we synthesize the whole quadrilattice from it, e.g. by using the context building function on the φ extents $\overline{\mathfrak{Q}}^\varphi(G, M, R)_{\overline{\mathcal{K}}} = \overline{\gamma}_4(\mathfrak{B}_G^\varphi(G, M, R)_{\overline{\mathcal{K}}})$. \square

Example 3 (FCA as $\overline{\mathcal{K}}$ -4FCA). *The column spaces of all Galois connections between idempotent spaces are complete lattices, and the synthesis part of the Fundamental Theorem 1 ensures that these*

can be expressed as the concept lattice of a particular context. This is what the proof of Theorem 6 is based upon.

However, “Standard” FCA can be implemented by \mathfrak{B} -4FCA using the default scaling $\varphi = \gamma = \mu = e$ (actually, any value other than \perp), synthesizing $\mathbb{L} \equiv \mathfrak{B}(G, M, I)$ as $\mathfrak{Q}(G, M, I) \cong \overline{\mathfrak{Q}}^\varphi(G, M, R_1^\varphi)_{\overline{\mathcal{K}}}$ and colouring on the first two components (cfr. [25] which uses only $\overline{\mathcal{K}}$ -FCA [3] for a similar purpose).

Since these steps use only elements from \mathfrak{B} , an isomorphic copy of which is embedded in every complete (idempotent) semifield, essentially choosing any of the latter allows for an implementation of FCA, e.g., computing in the lattice $\overline{\mathbb{R}}_{\max,+}$. To maintain the order embedding with the boolean semiring— $\mathfrak{B} \supseteq \mathbb{B}$ with $0 < 1$ —those completed semifields where $\perp < \top$ should be chosen as bias.

3.2.4. On the Relation between $\overline{\mathcal{K}}$ -4FCA and $\underline{\mathcal{K}}$ -4FCA

The order-dual of Theorem 6 is readily seen by dualizing all of the constructions and objects used in it, so we will not write it here. The relationship between their results, however, is not merely about “turning orders upside-down”—that would be the case if only \oplus and $\dot{\oplus}$ were involved in expressions—but reflects a more profound duality of which we present some examples below.

Example 4 (Utilities and costs as biases). *The bias plays an important role in applications: For instance, applications that maximize utility functions are naturally thought of with bias in $\overline{\mathbb{R}}_{\max,+}$, while applications that minimize cost functions are naturally thought of with bias in $\overline{\mathbb{R}}_{\min,+} = (\overline{\mathbb{R}}_{\max,+})^{-1}$. The theory sketched above tell us exactly how to pass from thinking in terms of (additively accumulating, idempotent) utilities to thinking in terms of (additively accumulating, idempotent) costs. The complete semifield notation, further, allows us to mix them properly.*

Example 5 (Resistances and conductances). *The same kind of duality is found when adding resistances (series addition) and conductances (parallel addition of resistances) in circuit analysis. The upper and lower operators simply model different operations on the same underlying phenomenon (resistivity) that has two manifestations (resistance/conductance). Note that this duality is manifested in all completed positive semifields, not just in idempotent ones [49].*

A quick investigation shows that we may relate the change of bias to the original bias in $\overline{\mathcal{K}}$.

Lemma 9. *Let (G, M, R) be a formal context, $\varphi = \gamma \otimes \mu$ invertible scalars and $\underline{\mathcal{K}}$ the bias. Then*

$$\begin{aligned} \overline{\mathfrak{B}}_G^\gamma(G, M, R)_{\underline{\mathcal{K}}} &= \overline{\mathfrak{N}}_G^\gamma(G, M, R^{-1})_{\overline{\mathcal{K}}} & \overline{\mathfrak{B}}_M^\mu(G, M, R)_{\underline{\mathcal{K}}} &= \overline{\mathfrak{N}}_M^\mu(G, M, R^{-1})_{\overline{\mathcal{K}}} & (90) \\ \overline{\mathfrak{N}}_M^\mu(G, M, R)_{\underline{\mathcal{K}}} &= \overline{\mathfrak{B}}_M^\mu(G, M, R^{-1})_{\overline{\mathcal{K}}} & \overline{\mathfrak{N}}_G^\gamma(G, M, R)_{\underline{\mathcal{K}}} &= \overline{\mathfrak{B}}_G^\gamma(G, M, R^{-1})_{\overline{\mathcal{K}}} \end{aligned}$$

Proof. The lattice-defining closures of $(G, M, R)_{\underline{\mathcal{K}}}$ are those of the right column of Table 1. If we substitute R for R^{-1} we realise that these are the interior operators of $(G, M, R^{-1})_{\overline{\mathcal{K}}}$. Furthermore, we then realize that the polars in (48) are those in (10) for the inverted incidence R^{-1} . By the unicity of closure and interior operators and their lattices the result follows for $\overline{\mathfrak{B}}_G^\gamma(G, M, R)_{\underline{\mathcal{K}}} = \overline{\mathfrak{N}}_G^\gamma(G, M, R^{-1})_{\overline{\mathcal{K}}}$ and $\overline{\mathfrak{B}}_M^\mu(G, M, R)_{\underline{\mathcal{K}}} = \overline{\mathfrak{N}}_M^\mu(G, M, R^{-1})_{\overline{\mathcal{K}}}$. By using the bracket $\langle x \mid R \mid y \rangle_{\underline{\mathcal{K}}}^{IO} = x^{-1} \setminus R \dot{\setminus} y^\otimes = x^\top \otimes R \otimes y$ that generates the co-Galois connection with bias in $\underline{\mathcal{K}}$ we obtain the second set of results. \square

Proposition 7. *Let (G, M, R) be a formal context, $\varphi = \gamma \otimes \mu$ invertible scalars and $\underline{\mathcal{K}}$ the bias. Then*

$$\underline{\mathfrak{Q}}^\varphi(G, M, R)_{\underline{\mathcal{K}}} = (\overline{\mathfrak{Q}}^\varphi(G, M, R^{-1})_{\overline{\mathcal{K}}})^{-1} \tag{91}$$

Proof. Note that $\underline{\Omega}^\varphi(G, M, R)_{\mathcal{K}}$ has four “component” lattices

$$\underline{\Omega}^\varphi(G, M, R)_{\mathcal{K}} = (\underline{\mathfrak{B}}_G^\gamma(G, M, R)_{\mathcal{K}}, \underline{\mathfrak{B}}_M^\mu(G, M, R)_{\mathcal{K}}, \overline{\mathfrak{A}}_G^\gamma(G, M, R)_{\mathcal{K}}, \overline{\mathfrak{A}}_M^\mu(G, M, R)_{\mathcal{K}})$$

By using the result in (90) we may write

$$\underline{\Omega}^\varphi(G, M, R)_{\mathcal{K}} = (\underline{\mathfrak{A}}_G^\gamma(G, M, R^{-1})_{\overline{\mathcal{K}}}, \underline{\mathfrak{A}}_M^\mu(G, M, R^{-1})_{\overline{\mathcal{K}}}, \overline{\mathfrak{B}}_G^\gamma(G, M, R^{-1})_{\overline{\mathcal{K}}}, \overline{\mathfrak{B}}_M^\mu(G, M, R^{-1})_{\overline{\mathcal{K}}})$$

Recalling that $(c, d, a, b) = (a^{-1}, b^{-1}, c^{-1}, d^{-1})$, we apply the inversion operator to the quadrilattice as a subspace.

$$\begin{aligned} (\underline{\Omega}^\varphi(G, M, R)_{\mathcal{K}})^{-1} &= (\underline{\mathfrak{A}}_G^\gamma(G, M, R^{-1})_{\overline{\mathcal{K}}}, \underline{\mathfrak{A}}_M^\mu(G, M, R^{-1})_{\overline{\mathcal{K}}}, \overline{\mathfrak{B}}_G^\gamma(G, M, R^{-1})_{\overline{\mathcal{K}}}, \overline{\mathfrak{B}}_M^\mu(G, M, R^{-1})_{\overline{\mathcal{K}}})^{-1} \\ &= (\underline{\mathfrak{A}}_G^\gamma(G, M, R^{-1})_{\overline{\mathcal{K}}}^{-1}, \underline{\mathfrak{A}}_M^\mu(G, M, R^{-1})_{\overline{\mathcal{K}}}^{-1}, \overline{\mathfrak{B}}_G^\gamma(G, M, R^{-1})_{\overline{\mathcal{K}}}^{-1}, \overline{\mathfrak{B}}_M^\mu(G, M, R^{-1})_{\overline{\mathcal{K}}}^{-1}) \\ &= (\overline{\mathfrak{B}}_G^\gamma(G, M, R^{-1})_{\overline{\mathcal{K}}}, \overline{\mathfrak{B}}_M^\mu(G, M, R^{-1})_{\overline{\mathcal{K}}}, \underline{\mathfrak{A}}_G^\gamma(G, M, R^{-1})_{\overline{\mathcal{K}}}, \underline{\mathfrak{A}}_M^\mu(G, M, R^{-1})_{\overline{\mathcal{K}}}) \\ &= \overline{\Omega}^\varphi(G, M, R^{-1})_{\overline{\mathcal{K}}} \end{aligned}$$

Since the inversion is an involution, the result follows. \square

Note that, we have suggested in passing the following results, important for transposition and conjugation contexts:

Corollary 9. *If (a, b, c, d) is a φ -quadriconcept of $\overline{\Omega}^\varphi(G, M, R)_{\overline{\mathcal{K}}}$, then (d, c, b, a) is a φ -quadriconcept of $\underline{\Omega}^\varphi(M, G, R^{\otimes})_{\mathcal{K}}$, and vice-versa.*

3.2.5. Applications: Context Reduction and Reconstruction

We have prepared two operations on contexts and their quadrilattices: reduction and reconstruction. Reduction is just a corollary of previous results.

Corollary 10 (context reduction in $\overline{\mathcal{K}}$ -4FCA). *In the conditions of Definition 4 for context $(G, M, R)_{\overline{\mathcal{K}}}$, if I is the index of independent rows and J that of independent columns of R , and R_{IJ} the submatrix of R selected by such indices, then $\overline{\Omega}^\varphi(G, M, R)_{\overline{\mathcal{K}}} \cong \overline{\Omega}^\varphi(I, J, R_{IJ})_{\overline{\mathcal{K}}}$.*

Proof. This is a corollary of Propositions 5 and 6 in the new setting of Definition 4. \square

It has already been proven that formal concepts allow us to approximate their incidence ([6], Section 3). The following result is a generalization of this.

Lemma 10. *Let $R \in K^{s \times m}$ be a matrix over a completed idempotent semifield \mathcal{K} . Then:*

1. *Every generalized φ -quadriconcept $(\bar{a}, \bar{b}, \bar{c}, \bar{d})$ of the generalized concept lattice $\overline{\Omega}^\varphi(G, M, R)_{\overline{\mathcal{K}}}$ generates a lower bound in the reconstruction of the matrix,*

$$R \geq \bar{a} \dot{\otimes} \bar{b}^T = \bar{a} \dot{\otimes} \bar{d}^{\otimes} = \bar{c}^{-1} \dot{\otimes} \bar{b}^T = \bar{c}^{-1} \dot{\otimes} \bar{d}^{\otimes} \tag{92}$$

2. *Every generalized φ -quadriconcept $(\underline{a}, \underline{b}, \underline{c}, \underline{d})$ of the generalized concept lattice $\underline{\Omega}^\varphi(G, M, R)_{\mathcal{K}}$ generates an upper bound in the reconstruction of the matrix,*

$$R \leq \underline{a} \dot{\otimes} \underline{b}^T = \underline{a} \dot{\otimes} \underline{d}^{\otimes} = \underline{c}^{-1} \dot{\otimes} \underline{b}^T = \underline{c}^{-1} \dot{\otimes} \underline{d}^{\otimes} \tag{93}$$

Proof. We prove this for the left adjunction. Since $\bar{a}_R^{\bar{c}} = \bar{d}$ and $\bar{d}_R^{\bar{c}} = \bar{a}$ we have that $\bar{a}^{\otimes} \dot{\otimes} R \dot{\otimes} \bar{d} \geq e$, whence:

$$R \geq \bar{a}^{\otimes} \dot{\setminus} e / \bar{d} \iff R \geq \bar{a} \dot{\otimes} \bar{d}^{\otimes}$$

If we try to tighten the bound as in: $R \geq \bar{a} \otimes \bar{b}^T \oplus \bar{a} \otimes \bar{d}^{\otimes} \oplus \bar{c}^{-1} \otimes \bar{b}^T \oplus \bar{c}^{-1} \otimes \bar{d}^{\otimes}$ we may compact this as $R \geq (\bar{a} \oplus \bar{c}^{-1}) \otimes (\bar{b}^T \oplus \bar{d}^{\otimes})$ but we know from (79) that $\bar{c} = \bar{a}^{-1}$ and $\bar{d} = \bar{b}^{-1}$ whence $R \geq (a \oplus a) \otimes (d^{\otimes} \oplus d^{\otimes}) = a \otimes d^{\otimes}$, the last step, by the idempotency of the addition. Bias-dually we obtain the upper bound. \square

The rest of a theory of context reconstruction for quadrilattices can be easily developed from the results in [6] for a single Galois connection.

4. Discussion

4.1. FCA, \mathcal{K} -Formal Concept Analysis and $\overline{\mathcal{K}}$ -4FCA

In essence, the formal analysis of a context $(G, M, R)_{\overline{\mathcal{K}}}$ is based in a Galois connection induced by linear operations in the dual semifield $\underline{\mathcal{K}}$ and a transformed matrix. Thus while FCA is eminently about computing with lattices, \mathcal{K} -FCA is computing with lattices in lattices.

While \mathcal{K} -FCA (so, indeed, FCA) favours the use of antitone pairs of mutual quasi-inverses—that is, Galois connections—Linear Algebra favours pairs of monotonic quasi-inverses—that is, left or right adjunctions (Appendix A).

Reasons of symmetry and generalization suggest that any attempt to define \mathcal{K} -FCA in terms of an adjunction would also make the absence of the Galois connections evident and that is the reason why [23] first suggested that the proper study of \mathcal{K} -FCA should not be a single GC or adjunction, but the four connections at once. This paper is a better justification and an exploration of that idea and why it is an alternative to just considering simple linear functions, i.e., adjunctions, as the basic construction in Linear Algebra over complete idempotent semifields.

4.2. The Case for $\overline{\mathcal{K}}$ -Four-Fold Formal Concept Analysis

Perhaps the most important result that we are providing is Theorem 6. By comparing the results of Sections 3.1 and 3.2 we may ask ourselves whether $\overline{\mathcal{K}}$ -four-fold Formal Concept Analysis warrants the reification carried in it.

Of course, one strong reason for it is the synthesis part of Theorem 6: the notions of independently motivated sets of objects G and attributes M that generate supersets of the join- and meet-irreducibles by means of the quadriconcept-forming functions suggest that the conceptualization step of Section 3.2.1 is important for the universal-representation capabilities of quadrilattices (and concept lattices in general).

This issue is related to Wille’s initial goal to make Lattice Theory more “actionable” [18]. Concepts like the “index sets” used in the reduction process in Section 3.1 would do for purely algebraic constructs. However, the cognitive implications imposed on concept lattices by the annotation of object- and attribute-concepts with their respective object and attribute names provide a lot of the affordances of concept lattices—and by extension quadrilattices—for data visualization and manipulation [31,32].

Another reason is the existence of the inverses of extents and intents described by the tetrahedral connection depicted in Figure 4. The relation between extents and co-extents, intents and co-intents is as intimate—in the existence of the perfect Galois connections represented by the inversions—as that of extents and intents—in the existence of the Galois connection of (5). Truly, extent and intents refer to two modes of existence as related by the matrix, but our point here is that extents and co-extents are two “polarities” of that existence. Since the choice of bias— $\overline{\mathcal{K}}$ or $\underline{\mathcal{K}}$ —highlights each of those polarities by the results in Section 3.2.4 its seems important to have the four-types of objects available both for mathematical manipulation and for cognitive focusing.

Yet, this very straightforward relationship between extents and co-extents, intents and co-intents, instantiating a very straightforward inversion relationship—or negation from the point of view of logics—may induce the belief that two of these domains is “more” basic, vital or irreducible than the others. If so, which? If from the point of view of $\overline{\mathcal{K}}$ -four-fold

Formal Concept Analysis we take these to be extents and intents, the results in Section 3.2.4 suggest that from the point of view of \mathcal{K} -four-fold Formal Concept Analysis, co-extents and -intents would reasonably be the important ones for the same—dualized—reasons, and vice-versa. Section 3.2.4, perhaps it is wise to carry all of them so as to maintain all of their affordances at hand. With the reification we put forward the idea that it is the quadrilattice the gestalt that captures the better notion of “concept”.

Furthermore, it is well-known that a (concept) lattice does not have, in general, a complement for each of its elements whereas the existence of complements is necessary for many applications. To wit, measure theory demands that complementary—not just incompatible—events be available for every event. Quadrilattices show that such elements may be available even if none of the lattices in the tetrahedron is boolean—hence, a fortiori, a sigma-algebra. Ways to profit by the wealth of complements available in the quadrilattices construct should be explored in the future.

The result in Section 3.2.5 on reconstructing the relationship by means of upper and lower bounds built with the $\bar{\mathcal{K}}$ - or \mathcal{K} -quadriconcepts may seem to suggest that only two of the isomorphic lattices are important, since one two coordinates of the quadriconcepts are needed for it. Then again, precisely knowing which parts are being considered is mandatory for a successful reconstruction, what sustains the notion that the quadriconcepts are gestalten whose parts refer to each other in precisely the manner prescribed by the four-fold connection.

Although the concept of quadrilattices is developed around invertible scaling, easier to manipulate algebraically, it is also evident in the results of Section 3.1.2 that it is possible to define a quadrilattice for $\varphi \in \{\perp, \top\}$, although not very representative of what the content of relation R is: most $\bar{\mathcal{K}}$ -4FCA using the extremal values on any relation would return the same $\mathbb{1}$ or $\mathbb{2}$ lattices.

As in many previous mathematical explorations, perhaps we should wait until the applications catch up with the theory for a final ruling in this matter. The paradigm of computing in (and with) lattices is a promising ground for such endeavours.

Author Contributions: conceptualization, F.J.V.-A. and C.P.-M.; formal analysis, F.J.V.-A. and C.P.-M.; funding acquisition, C.P.-M.; investigation, F.J.V.-A. and C.P.-M.; methodology, F.J.V.-A. and C.P.-M.; writing—original draft, F.J.V.-A.; writing—review and editing, F.J.V.-A. and C.P.-M. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Spanish Government-MinECo project TEC2017-84395-P and the Dept. of Research and Innovation of Madrid Regional Authority project EMPATIA-CM (Y2018/TCS-5046).

Acknowledgments: This paper evolved from conference papers presented by the authors at ICFCFA 2017 and FUZZ-IEEE 2018. We would like to acknowledge the reviewers of previous versions of this paper for their timely criticism and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

| | |
|---------------------------|--|
| FCA | Formal Concept Analysis |
| $\bar{\mathcal{K}}$ -FCA | $\bar{\mathcal{K}}$ semifield-valued Formal Concept Analysis |
| $\bar{\mathcal{K}}$ -4FCA | four-fold $\bar{\mathcal{K}}$ -FCA |

Appendix A. Residuated Maps, Adjunctions and Galois Connections

This section follows [20]. Let $\mathcal{P} = \langle P, \leq_P \rangle$ and $\mathcal{Q} = \langle Q, \leq_Q \rangle$ be partially ordered sets. We have:

- A map $f : P \rightarrow Q$ is residuated if inverse images of principal (order) ideals of Q under f are again principal ideals. Its residual map or simply residual, $f^\# : Q \rightarrow P$ is $f^\#(q) = \max\{p \in P \mid f(p) \leq_Q q\}$.

- A map $g : Q \rightarrow P$ is dually residuated if the inverse images of principal dual (order) ideals under g are again dual ideals. Its dual residual map or simply dual residual, $g^b : P \rightarrow Q$ is $g^b(p) = \min\{q \in Q \mid p \leq_P g(q)\}$.

This duality of concepts is fortunately simplified by a well-known theorem stating that residual maps are dually residuated, while dual residual maps are residuated, hence we may maintain only the two notions of residuated maps and their residuals.

In fact, the two notions are so entwined that we give a name to them: an adjoint pair of maps (λ, ρ) is a pair $(\lambda : P \rightarrow Q, \rho : Q \rightarrow P)$ between two ordered sets such that $\forall p \in P, q \in Q, p \leq_P \rho(q) \iff \lambda(p) \leq_Q q$, equivalently, $p \leq_P \rho(\lambda(p))$ and $\lambda(\rho(q)) \leq_Q q$.

If the order relation is actually partial the lower or left adjoint, λ is uniquely determined by its right or upper adjoint, ρ , and conversely. The characterization theorem for adjoint maps states that (λ, ρ) are adjoint if and only if, λ is residuated with residual ρ , or equivalently, ρ is dually residuated with λ its dual residual.

Now consider the orders $\mathcal{P} = \langle P, \leq_P \rangle$ and $\mathcal{Q} = \langle Q, \leq_Q \rangle$ and their order duals $\mathcal{P}^d = \langle P, \geq_P \rangle$ and $\mathcal{Q}^d = \langle Q, \geq_Q \rangle$, to obtain two adjoint and two dually adjoint pairs:

Definition A1. (Four different types of Galois connections and adjunctions)

1. (λ, ρ) is an adjunction on the left or simply a left adjunction, and we write $(\lambda, \rho) : \mathcal{P} \rightleftharpoons \mathcal{Q}$ iff: $\forall p \in P, q \in Q \lambda(p) \leq_Q q \iff p \leq_P \rho(q)$, that is, the functions are covariant, and we say that λ is the lower or left adjoint while ρ is the upper or right adjoint.
2. $(\rho, \lambda) : \mathcal{P} \rightleftharpoons \mathcal{Q}$ is an adjunction on the right or simply a right adjunction iff: $\forall p \in P, q \in Q \rho(p) \geq_Q q \iff p \leq_P \lambda(q)$, both functions are covariant, ρ is the upper adjoint, and λ the lower adjoint.
3. (φ, ψ) is a Galois Connection (proper), of two dual adjoints $(\varphi, \psi) : \mathcal{P} \dashv \mathcal{Q}$ iff: $\forall p \in P, q \in Q \varphi(p) \geq_Q q \iff p \leq_P \psi(q)$, that is, both functions are contravariant. For that reason they are sometimes named contravariant or symmetric adjunctions on the right.
4. (Δ, Δ') is a co-Galois connection, of dual adjoints $(\Delta, \Delta') : \mathcal{P} \dashv \mathcal{Q}$ if: $\forall p \in P, q \in Q \Delta(p) \leq_Q q \iff p \geq_P \Delta'(q)$, that is, both functions are contravariant. For that reason they are sometimes named contravariant or symmetric adjunctions on the left. (Δ', Δ) is also a co-Galois connection.

Furthermore, as a sort of graphical summary, we introduce the diagram to the upper left-hand corner of Figure A1 as the pattern that carries the structures described in [17].

We illustrate how to read it with the diagram at the top left, which has:

- A closure system, $\rho(Q) = \overline{P}$, the closure range of the right adjoint (see below).
- An interior system, $\lambda(P) = \underline{Q}$, the kernel range of the left adjoint (see below).
- A closure function [37] [suggest “closure operator”] $\gamma_P = \rho \circ \lambda \geq_P I_P$, from P to the closure range $\overline{P} = \rho(Q)$, with adjoint inclusion map \hookrightarrow_P , where I_P denotes the identity over P .
- A kernel function [37] [also “interior operator”, “kernel operator”] $\kappa_P = \lambda \circ \rho \leq_Q I_Q$, from Q to the range of $\underline{Q} = \lambda(P)$, with adjoint inclusion map \hookrightarrow_Q , where I_Q denotes the identity over Q .
- a perfect adjunction $(\tilde{\lambda}, \tilde{\rho}) : \overline{P} \rightleftharpoons \underline{Q}$, that is, a dual order isomorphism between the closure and kernel ranges \overline{P} and \underline{Q} .

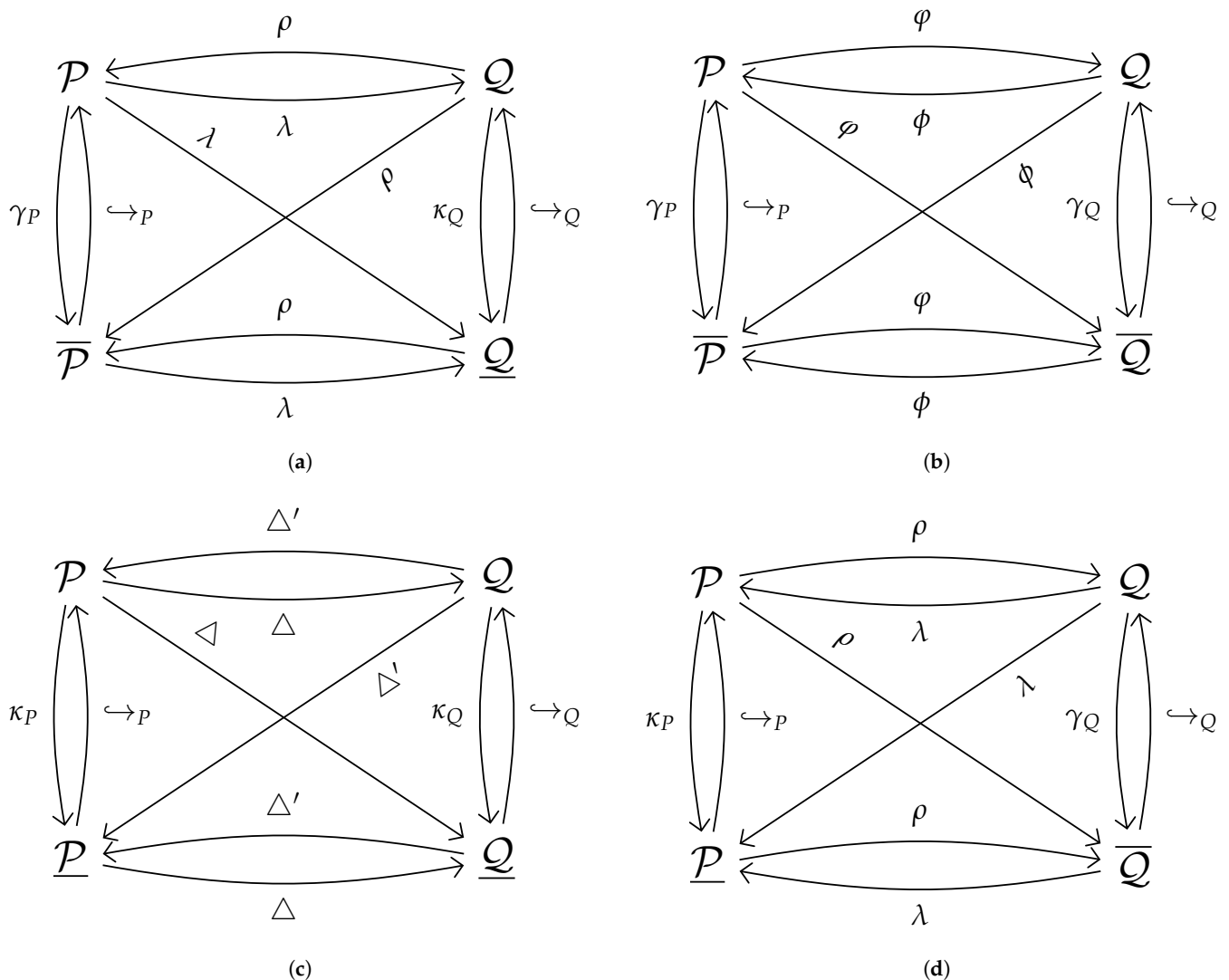


Figure A1. Diagrams visually depicting the maps and structures involved in the adjunction on the left $(\lambda, \rho) : P \rightleftarrows Q$ (top left), Galois connection $(\varphi, \psi) : P \rightleftarrows Q$ (top right), the co-Galois connection $(\Delta, \Delta') : P \rightleftarrows Q$ (bottom left) and the adjunction on the right $(\rho, \lambda) : P \rightleftarrows Q$ (bottom right) between two partially ordered sets (adapted from [17]). Closure operators are denoted by γ_P, γ_Q , interior (kernel) operators by κ_P, κ_Q , closure systems by $\overline{P}, \overline{Q}$ and interior (kernel) systems by $\underline{P}, \underline{Q}$. (a) A left adjunction, (b) A Galois connection, (c) A co-Galois connection, (d) A right adjunction.

Compare the mathematical objects above with those in a Galois connection proper seen in the top right of Figure A1: the ranges are both closure systems and both compositions closure operators due to the dualisation of the second set (we write γ_Q for the new closure operator), resulting in the well-known perfect Galois connection, $(\tilde{\varphi}, \tilde{\psi}) : \overline{P} \rightleftarrows \overline{Q}$, the pair of dual order-isomorphic closure ranges lying at the heart of Formal Concept Analysis. The diagrams in the bottom left and right of Figure A1 show analogue structures for co-Galois connections and right adjunctions respectively.

The different monotonicity conditions account for different properties of the adjoint maps [20]:

- if (λ, ρ) form a left adjunction, then λ is residuated, preserves existing least upper bounds (for lattices, joins) and ρ preserves existing greatest lower bounds (for lattices, meets).
- if (φ, ψ) form a Galois connection, then both φ and ψ invert existing least upper bounds (for lattices, they transform joins into meets).

- if (ρ, λ) form a right Galois connection, then ρ preserves existing greatest lower bounds (meets for lattices) and λ is residuated, preserves existing least upper bounds (joins for lattices).
- if (Δ, Δ') form a co-Galois connection, then both Δ and Δ' invert existing greatest lower bounds (for lattices, they transform meets into joins).

Table A1 summarises the main properties of all types of Galois connections.

Table A1. Summary of Galois connections and their properties, for P, Q posets.

| Left Adjunction (TYPE OO): $(\lambda, \rho) : \mathcal{P} \rightleftharpoons \mathcal{Q}$ | Galois Connection (TYPE OI): $(\varphi, \psi) : \mathcal{P} \leftrightarrow \mathcal{Q}$ |
|---|---|
| $\forall p \in P, q \in Q \quad \lambda(p) \leq_{\mathcal{Q}} q \Leftrightarrow p \leq_{\mathcal{P}} \rho(q)$ $I_{\mathcal{P}} \leq \rho \circ \lambda$ and $I_{\mathcal{Q}} \geq \lambda \circ \rho$ $\lambda = \lambda \circ \rho \circ \lambda$ and $\rho = \rho \circ \lambda \circ \rho$ λ monotone, residuated ρ monotone, residual λ join-preserving, ρ meet-preserving | $\forall p \in P, q \in Q \quad \varphi(p) \geq_{\mathcal{Q}} q \Leftrightarrow p \leq_{\mathcal{P}} \psi(q)$ $I_{\mathcal{P}} \leq \psi \circ \varphi$ and $I_{\mathcal{Q}} \leq \varphi \circ \psi$ $\varphi = \varphi \circ \psi \circ \varphi$ and $\psi = \psi \circ \varphi \circ \psi$ φ antitone ψ antitone φ join-inverting, ψ join-inverting |
| co-Galois connection (TYPE IO): $(\Delta, \Delta') : \mathcal{P} \rightharpoonup \mathcal{Q}$ | Right Adjunction (TYPE II): $(\rho, \lambda) : \mathcal{P} \rightleftharpoons \mathcal{Q}$ |
| $\forall p \in P, q \in Q \quad \Delta(p) \leq_{\mathcal{Q}} q \Leftrightarrow p \geq_{\mathcal{P}} \Delta'(q)$ $I_{\mathcal{P}} \geq \Delta' \circ \Delta$ and $I_{\mathcal{Q}} \geq \Delta \circ \Delta'$ $\Delta = \Delta \circ \Delta' \circ \Delta$ and $\Delta' = \Delta' \circ \Delta \circ \Delta'$ Δ antitone Δ' antitone Δ meet-inverting, Δ' meet-inverting | $\forall p \in P, q \in Q \quad \rho(p) \geq_{\mathcal{Q}} q \Leftrightarrow p \leq_{\mathcal{P}} \lambda(q)$ $I_{\mathcal{P}} \geq \lambda \circ \rho$ and $I_{\mathcal{Q}} \leq \rho \circ \lambda$ $\rho = \rho \circ \lambda \circ \rho$ and $\lambda = \lambda \circ \rho \circ \lambda$ ρ monotone, residual λ monotone, residuated ρ meet-preserving, λ join-preserving |

See [38] for a revision of the genesis and importance of Galois Connections and adjunctions, as well as a discussion of the different notation and nomenclatures for these concepts. An early tutorial with mathematical applications in mind is [36].

A Naming Convention for Galois Connections

The following naming convention was put forward in [17]. It stresses the composition with order- and dual order-isomorphisms and relates to the original names as annotated in Figure A1.

- We take the TYPE OO Galois connection to be a basic adjunction composed with an even number of anti-isomorphism on the domain and range orders.
- To obtain a TYPE OI Galois connection compose a basic adjunction with an odd number of anti-isomorphism on the range.
- To get a a TYPE IO Galois connection we compose a basic adjunction with an odd number of anti-isomorphisms on the domain.
- Finally, a TYPE II Galois connection, is a basic adjunction with an odd number of anti-isomorphisms composed on both the domain and range.

References

1. Kaburlasos, V.G. The Lattice Computing (LC) Paradigm. In Proceedings of the 15th International Conference on Concept Lattices and their Applications CLA, Tallinn, Estonia, 29 June–1 July 2020; Valverde-Albacete, F.J., Trnecka, M., Eds.; Tallinn University of Technology: Tallinn, Estonia, 2020; pp. 1–7.
2. Kaburlasos, V.G.; Ritter, G.X. (Eds.) *Computational Intelligence Based on Lattice Theory*; Springer: Berlin, Germany, 2010.
3. Platzer, A. *Logical Foundations of Cyber-Physical Systems*; Springer: Berlin, Germany, 2018.
4. Golan, J.S. *Semirings and Their applications*; Kluwer Academic: Dordrecht, The Netherlands, 1999.
5. Gondran, M.; Minoux, M. *Graphs, Dioids and Semirings. New Models and Algorithms*; Operations Research Computer Science Interfaces Series; Springer Science+Business Media: New York, NY, USA, 2008
6. Valverde-Albacete, F.J.; Peláez-Moreno, C. The Singular Valued Decomposition over completed idempotent semifields. *Mathematics* **2020**, *8*, 1577. [CrossRef]
7. Moreau, J.J. Inf-convolution, Sous-additivité, convexité des fonctions Numériques. *J. Math. Pures et Appl.* **1970**, *49*, 109–154.

8. Cuninghame-Green, R. *Minimax Algebra*; Number 166 in Lecture Notes in Economics and Mathematical Systems; Springer: Berlin, Germany, 1979.
9. Baccelli, F.; Cohen, G.; Olsder, G.; Quadrat, J. *Synchronization and Linearity*; Wiley: New York, NY, USA, 1992.
10. Ritter, G.X.; Sussner, P. An Introduction to Morphological Neural Networks. In Proceedings of the 13th International Conference on Pattern Recognition (ICPR'96), Vienna, Austria, 25–29 August 1996; pp. 709–717.
11. Maragos, P. Dynamical systems on weighted lattices: General theory. *Math. Control. Signals Syst.* **2017**, *29*, 1–49. [[CrossRef](#)]
12. Butkovic, P. Max-algebra: The linear algebra of combinatorics? *Linear Algebra Appl.* **2003**, *367*, 313–335. [[CrossRef](#)]
13. Butkovič, P. *Max-linear Systems. Theory and Algorithms*; Monographs in Mathematics; Springer-Verlag London Limited: London, UK, 2010.
14. Cohen, G.; Gaubert, S.; Quadrat, J.P. Duality and separation theorems in idempotent semimodules. *Linear Algebra Its Appl.* **2004**, *379*, 395–422. [[CrossRef](#)]
15. Valverde-Albacete, F.J.; Peláez-Moreno, C. Towards a Generalisation of Formal Concept Analysis for Data Mining Purposes. Concept Lattices. In Proceedings of the International Conference on Formal Concept Analysis (ICFCA 06), Dresden, Germany, 13–17 February 2006; Volume LNAI 3874, pp. 161–176.
16. Valverde-Albacete, F.J.; Peláez-Moreno, C. Galois Connections between Semimodules and Applications in Data Mining. Formal Concept Analysis. In Proceedings of the 5th International Conference on Formal Concept Analysis, ICFCA 2007, Clermont-Ferrand, France, 12–16 February 2007; Kusnetzov, S., Schmidt, S., Eds.; Springer: Berlin, Germany, 2007; pp. 181–196.
17. Valverde-Albacete, F.J.; Peláez-Moreno, C. Extending conceptualisation modes for generalised Formal Concept Analysis. *Inf. Sci.* **2011**, *181*, 1888–1909. [[CrossRef](#)]
18. Wille, R. Restructuring lattice theory: An approach based on hierarchies of concepts. In *Ordered Sets (Banff, Alta., 1981)*; Reidel: Dordrecht, The Netherlands; Reidel: Boston, MA, USA, 1982; pp. 445–470.
19. Ganter, B.; Wille, R. *Formal Concept Analysis: Mathematical Foundations*; Springer: Berlin/Heidelberg, Germany, 1999.
20. Davey, B.; Priestley, H. *Introduction to Lattices and Order*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2002.
21. Carpineto, C.; Romano, G. *Concept Data Analysis; Theory and Applications*, John Wiley & Sons, Ltd.: Chichester, UK, 2005.
22. Garg, V.K. *Introduction to Lattice Theory with Computer Science Applications*; John Wiley & Sons: Hoboken, NJ, USA, 2016.
23. Valverde-Albacete, F.J.; Peláez-Moreno, C. The Linear Algebra in Extended Formal Concept Analysis Over Idempotent Semifields. In *Formal Concept Analysis*; Bertet, K., Borchmann, D., Cellier, P., Ferré, S., Eds.; Springer: Berlin/Heidelberg, Germany; Springer: Rennes, France, 2017; pp. 211–227.
24. Valverde-Albacete, F.J.; Peláez-Moreno, C. The Linear Algebra in Formal Concept Analysis over Idempotent Semifields. In *Formal Concept Analysis*; Number 9113 in LNAI; Springer: Berlin/Heidelberg, Germany, 2015; pp. 97–113.
25. Valverde-Albacete, F.J.; Peláez-Moreno, C. K-Formal Concept Analysis as linear algebra over idempotent semifields. *Inf. Sci.* **2018**, *467*, 579–603. [[CrossRef](#)]
26. Gaubert, S. *Théorie des Systèmes Linéaires Dans les Dioïdes*. Ph.D. Thesis, École des Mines de Paris, Paris, France, 1992.
27. Cohen, G.; Gaubert, S.; Quadrat, J. Kernels, images and projections in dioids. In Proceedings of the Workshop on Discrete Event Systems (WODES), Scotland, UK, 19–21 August 1996; pp. 1–8.
28. Cohen, G.; Gaubert, S.; Quadrat, J.P. Projection and aggregation in maxplus algebra. In *Current Trends in Nonlinear Systems and Control*; Birkhäuser Boston: Boston, MA, USA, 2006; pp. 443–454.
29. Di Loreto, M.; Gaubert, S.; Katz, R.D.; Loiseau, J.J. Duality Between Invariant Spaces for Max-Plus Linear Discrete Event Systems. *SIAM J. Control. Optim.* **2010**, *48*, 5606–5628. [[CrossRef](#)]
30. González-Calabozo, J.M.; Valverde-Albacete, F.J.; Peláez-Moreno, C. Interactive knowledge discovery and data mining on genomic expression data with numeric formal concept analysis. *BMC Bioinform.* **2016**, *17*, 374. [[CrossRef](#)] [[PubMed](#)]
31. Peláez-Moreno, C.; García-Moral, A.I.; Valverde-Albacete, F.J. Analyzing phonetic confusions using Formal Concept Analysis. *J. Acoust. Soc. Am.* **2010**, *128*, 1377–1390. [[CrossRef](#)] [[PubMed](#)]
32. Valverde-Albacete, F.J.; González-Calabozo, J.M.; Peñas, A.; Peláez-Moreno, C. Supporting scientific knowledge discovery with extended, generalized Formal Concept Analysis. *Expert Syst. Appl.* **2016**, *44*, 198–216. [[CrossRef](#)]
33. Burusco, A.; Fuentes-González, R. The study of the L-fuzzy concept lattice. *Mathw. Soft Comput.* **1994**, *3*, 209–218.
34. Pollandt, S. *Fuzzy-Begriffe: Formale Begriffsanalyse Unscharfer Daten*; Springer: Berlin, Germany, 1996.
35. Belohlavek, R. *Fuzzy Relational Systems; Foundations and Principles*; Springer Science & Business Media: Boston, MA, USA, 2002.
36. Erné, M.; Koslowski, J.; Melton, A.; Strecker, G. A primer on Galois Connections. *Annals of the New York Academy of Sciences*, **1993**, *704*, 1, 103–125. [[CrossRef](#)]
37. Düntsch, I.; Gediga, G. Approximation operators in qualitative data analysis. In *Theory and Applications of Relational Structures as Knowledge Instruments*; de Swart, H., Ed.; Springer: Berlin, Germany, 2003; Volume 2929 in LNCS, pp. 214–230.
38. Erné, M. Adjunctions and Galois connections: Origins, History and Development. In *Galois Connections and Applications*; Denecke, K., Erné, M., Wismath, S.L., Eds.; Kluwer Academic: Dordrecht, The Netherlands, 2004; pp. 1–138.
39. Doignon, J.P.; Falmagne, J.C. *Knowledge Spaces*; Springer: Berlin, Germany, 1998.
40. Falmagne, J.C.; Doignon, J.P. *Learning Spaces. Interdisciplinary Applied Mathematics*; Springer: Berlin, Germany, 2011.

41. Valverde-Albacete, F.J.; Peláez-Moreno, C.; Cabrera, I.P.; Cordero, P.; Ojeda-Aciego, M. Formal Independence Analysis. In *Information Processing and Management of Uncertainty in Knowledge-Based Systems: Applications (IPMU 2018)*; Medina, J., Ojeda-Aciego, M., Verdegay, J., Perfilieva, I., Bouchon-Meunier, B., Yager, R., Eds.; Number 853 in CCIS; Springer International Publishing: Cham, Switzerland, 2018; pp. 596–608.
42. Valverde-Albacete, F.J.; Peláez-Moreno, C. Further Galois connections between Semimodules over Idempotent Semirings. In *Proceedings of the 4th Conference on Concept Lattices and Applications (CLA 07)*, Montpellier, France, 24–26 October 2007; Diatta, J., Eklund, P., Liquière, M., Eds.; 331 of CEUR WS: 2007; pp. 199–212.
43. Valverde-Albacete, F.J.; Peláez-Moreno, C. On the Relation between Semifield-Valued FCA and the Idempotent Singular Value Decomposition. In *Proceedings of the IEEE International Conference on Fuzzy Systems (FUZZ IEEE 2018)*, Rio de Janeiro, Brazil, 8–13 July 2018; pp. 1–8.
44. Burris, S.; Sankappanavar, H. *A Course in Universal Algebra*, the Millenium ed.; Self-Published: Washingtonville, NY, USA, 1999.
45. Cohen, G.; Gaubert, S.; Quadrat, J. Linear projectors in the max-plus algebra. In *Proceedings of the 5th IEEE Mediterranean Conference on Control and Systems*, Paphos, Greece, 24 January 1997; pp. 1–6.
46. Gaubert, S.; Katz, R.D. The tropical analogue of polar cones. *Linear Algebra Its Appl.* **2009**, *431*, 608–625. [[CrossRef](#)]
47. Valverde-Albacete, F.J.; Peláez-Moreno, C.; Cabrera, I.P.; Cordero, P.; Ojeda-Aciego, M. A Data Analysis Application of Formal Independence Analysis. In *Concept Lattices and their Applications (CLA 2018)*; Ignatov, D.I., Nourine, L., Eds.; Department of Computer Science, Palacky University Olomouc: Olomouc, Czech Republic, 2018; pp. 117–118.
48. Valverde-Albacete, F.J.; Peláez-Moreno, C. A Framework for Supervised Classification Performance Analysis with Information-Theoretic Methods. *IEEE Trans. Knowl. Data Eng.* **2020**, *32*, 2075–2087. [[CrossRef](#)]
49. Valverde-Albacete, J.F.; Peláez-Moreno, C. The Rényi Entropies Operate in Positive Semifields. *Entropy* **2019**, *21*, 780. [[CrossRef](#)] [[PubMed](#)]