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Covering a Regular Tetrahedron with Diminished Copies

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

Let T be a unit regular tetrahedron. A diminished copy of T is the image of T under a homothety with positive ratio smaller than 1. Let m be a positive integer and let $\gamma_m(T)$ be the smallest positive number r such that T can be covered by m translates of rT. Zong gave the results of $\gamma_4(T) = \dfrac{3}{4}$ and $\gamma_5(T) = \dfrac{9}{13}$. However, the values of $\gamma_6(T)$, $\gamma_7(T)$ and $\gamma_8(T)$ were not given then. In this article we give the upper bounds of $\gamma_6(T)$, $\gamma_7(T)$ and $\gamma_8(T)$.

Keywords: Covering; tetrahedron; Hadwiger's conjecture.

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1 Introduction

In n-dimensional Euclidean space E^n, let K be a convex body. We define int(K) as the interior of K and $c(K)$ as the smallest number of translates of int(K) that their union can cover K.

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In 1955, Levi [1] studied $c(K)$ for the two-dimensional convex domains and proved that:

$$c(K) = \begin{cases} 4, & if\ K\ is\ a\ parallelogram, \\ 3, & otherwise. \end{cases}$$

Let P denote an n-dimensional parallelepiped. It's easy to see that any translates of $\text{int}(P)$ can not cover two vertices of P. Therefore, it can be deduced that $c(P) = 2^n$.

Based on these results and some other observations, in 1975, Hadwiger [2] made the following conjecture: For every n-dimensional convex body K, we have

$$c(K) \leq 2^n,$$

where the equality holds if and only if K is a parallelepiped. Furthermore, 2^n homothetic copies are required only if the body is an affine n-cube.

This conjecture has been studied by many mathematicians. They have found some other problems which are relative to Hadwiger's conjecture such as the illumination problem and the separation problem [3,4,5,6]. For example, Lassak [7] proved this conjecture in the three-dimensional centrally symmetric case; Rogers and Zong [8] obtained

$$c(K) \leq \binom{2n}{n}(n \ln n + n \ln \ln n + 5n)$$

for general n-dimensional convex bodies, and

$$c(K) \leq 2^n(n \ln n + n \ln \ln n + 5n)$$

for centrally symmetric ones. Nevertheless, we are still far away from the solution of the conjecture, even the three-dimensional case.

Let T be a unit regular tetrahedron. A diminished copy of T is the image of T under a homothety with positive ratio smaller than 1. Let m be a positive integer and let $\gamma_m(T)$ be the smallest positive number r such that T can be covered by m translates of rT. Zong [9] gave the results of $\gamma_4(T) = \frac{3}{4}$ and $\gamma_5(T) = \frac{9}{13}$. However, the values of $\gamma_6(T)$, $\gamma_7(T)$ and $\gamma_8(T)$ were not given then. In this article we give the upper bounds of $\gamma_6(T)$, $\gamma_7(T)$ and $\gamma_8(T)$.

In the following proofs, we get the upper bounds of $\gamma_m(T)$ for $m = 6, 7, 8$ mainly depending on giving particular configurations. Assume tT is a diminished copy of T for some positive number $t(0 < t < 1)$, m is a positive integer, and T_1, T_2, T_3, T_4 are used to denote four translates of tT. Firstly, we put T_1, T_2, T_3, T_4 at four corners of T, satisfying one of $T_i's$ corners coincides with the corresponding corner of T for i = 1, 2, 3, 4. Then, we put $m-4$ translates of tT to cover the rest of T which is uncovered in the first step exactly right. Finally, we get an equation about t by analysing the structural properties of this configuration, such as the figure formed by the projections of all translates of tT in the configuration in one of $T's$ side face and the bottom faces of some of all translates of tT. Thus, we get a precise value for t in this given configuration, which is an upper bound of $\gamma_m(T)$.

2 Main Results

Theorem 2.1. $\gamma_6(T) \leq \frac{27}{40}$.

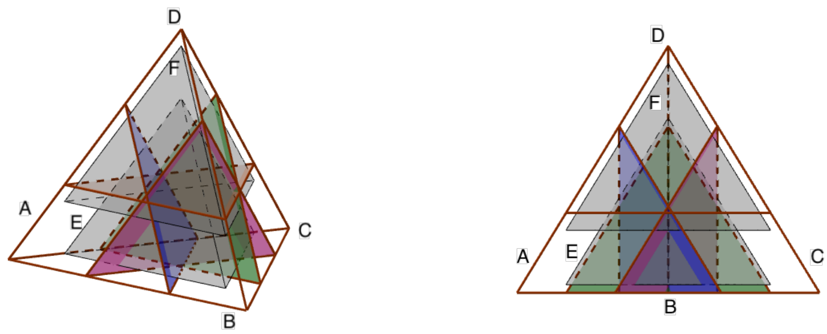

Fig. 1. case for m=6

Proof. We denote the six small congruent tetrahedra by A, B, C, D, E, F.
We put A, B, C, D at each corner of the unit regular tetrahedron, and assume the side length of A is t. We can see that there is a small tetrahedron K_1 formed by the intersection of A, B and C. Its side length is:

$$1 - 6 \times \frac{(1-t)}{2} = 3t - 2.$$

On each face of the unit regular tetrahedron, there is a small tetrahedron just the same size as K_1 on the center of the face respectively and we call them K_2, K_3 and K_4.

So, to give an appropriate configuration, we first put E above K_1 such that K_1's top vertex p is just on the bottom face of E and p is also the centroid of the bottom face of E. See Fig. 1(a).

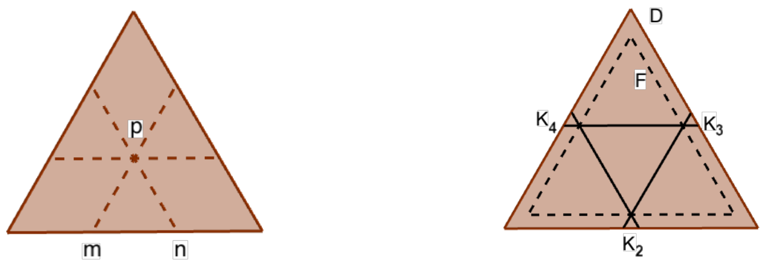

Fig. 2. case for m=6

Fig. 2(a) shows the bottom face of E. Since p is the centroid of the face, we can get that $mn = \frac{t}{3}$.

From our observation, the space that still be uncovered is three small tetrahedra L_1, L_2, L_3 which are cling to K_2, K_3 and K_4 respectively and its side length is:

$$(2t-1) - 2 \times (3t-2) - \frac{t}{3} = 3 - \frac{13}{3}t.$$

So the place of F must satisfy that 2 vertices of L_1 (L_2 and L_3) are just on the bottom face and the side face respectively. The picture of Fig. 2(b) shows the bottom face of D, the dotted regular triangle is the intersection part of F and the bottom face of D. Since the 3 small triangles are the faces of K_2, K_3 and K_4, we can get that the side length of the dotted regular triangle is:

$$t - 3 \times (3t-2) = 6 - 8t.$$

Finally, from the side faces of D and F ,we can get:

$$t-(6-8t)=3-\frac{13}{3}t$$
$$t=\frac{27}{40}.$$

So, A, B, C, D, E, F of side length $\frac{27}{40}$ can cover T by the configuration in Figure 1(a), then we have $\gamma_6(T)\le\frac{27}{40}$. □

Theorem 2.2. $\gamma_7(T)\le\frac{81}{121}$.

Proof. We denote the seven small congruent tetrahedra by A, B, C, D, E, F, G.

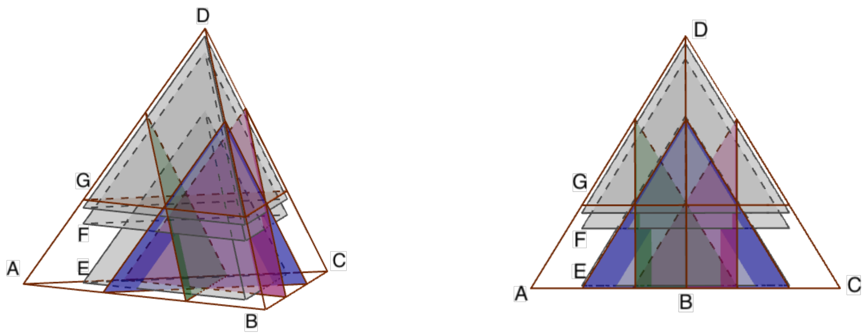

Fig. 3. case for m=7

Here, to give an appropriate configuration, the placements of A, B, C, D and E are the same as the case $m=6$. The difference is that, since $\gamma_7(T)\le\gamma_6(T)$, in the case for m=6, when t gets smaller, F can no more cover the three small regular tetrahedra L_1, L_2, L_3. We need another tetrahedron G to cover the uncovered space. In other words, in the case of $m=7$, F and G do the job just as F does in case for m = 6. Thus G covers three small regular tetrahedra of side length 9t-6, denoted by M_1, M_2, M_3, M_4 (See Fig. 3(a)). By observation and conventional calculation, we can get the part of a side of E which is covered by F but not by G in Fig. 3 has length $9-\frac{40}{3}t$.

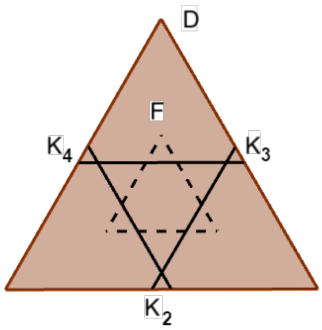

Fig. 4. case for m=7

Fig. 4 shows the bottom of D. The dotted regular triangle is the intersection part of F and the bottom face of D. Since the 3 small triangles are the faces of K_2, K_3 and K_4, we can get that the side length of the dotted regular triangle is $24 - 35t$.

Finally, from the side faces of D and F in Figure 3(b), we have:

$$24 - 35t = t - (9t - 6) - (9 - \frac{40}{3}t)$$
$$t = \frac{81}{121}.$$

Similar to Theorem 2.1, we get $\gamma_7(T) \leq \frac{81}{121}$. □

Theorem 2.3. $\gamma_8(T) \leq \frac{5}{8}$.

Proof. We denote the eight small congruent tetrahedra by A, B, C, D, E, F, G, H.

The case for $m = 8$ is different from the case when $m = 6$ or $m = 7$. Since when $m \leq 7$, there must be $t \geq \frac{2}{3}$.

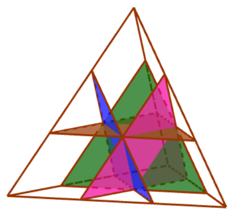

Fig. 5. case for m=4

Just as Fig. 5 shows: when $t = \frac{2}{3}$ on one face of T, the intersection of 3 side faces of different 3 small translates of T is just a point. So when $t < \frac{2}{3}$, there will be an uncovered small regular triangle on the middle of each face of T.

We first put A, B, C and D on each corner of T. Then we put E, F, G and H on each face of T. In Fig. 6(a), we just draw E instead of F, G and H.

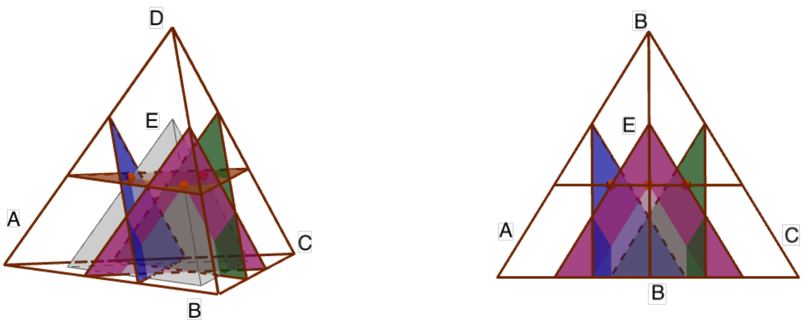

Fig. 6. case for m=8

Fig. 7. shows the bottom face of D. The dotted regular triangle is the intersection part of E and the bottom face of D. To give an appropriate configuration, we need to satisfy that the vertices p_1, p_2 and p_3 of the dotted triangle $p_1p_2p_3$ are just on m_1n_1, m_2n_2 and m_3n_3 respectively. So we can get the side length of $p_1p_2p_3$ is: $2 \times (2 - 3t) = 4 - 6t$.

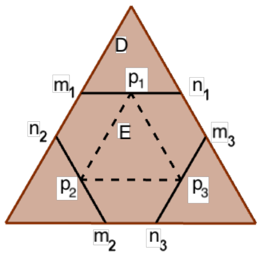

Fig. 7. case for m=8

Finally, from the side faces of D and E in Fig. 6(b), we have:

$$(4 - 6t) + (1 - t) = t$$
$$t = \frac{5}{8}.$$

Similar to Theorem 2.1, we get $\gamma_8(T) \leq \frac{5}{8}$. $\square$

3 Conclusions

In this paper, based on Zong's work of $\gamma_4(T) = \frac{3}{4}$ and $\gamma_5(T) = \frac{9}{13}$, we get the upper bounds of $\gamma_6(T)$, $\gamma_7(T)$ and $\gamma_8(T)$ by giving some particular configurations and analysing their structural properties. Nevertheless, we are still not sure of their exact values and try to look for a better way to solve this problem completely. Furthermore, according to the results of $\gamma_4(T)$ and $\gamma_8(T)$, we consider if there is a general formula for $\gamma_{2^m}(T)$ for all integers $m \geq 2$. We think it's a good question to think about in the future.

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Competing Interests

Authors have declared that no competing interests exist.

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