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Adams Type Hybrid Block Methods Associated with Chebyshev Polynomial for the Solution of Ordinary Differential Equations

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Abstract

The new Hybrid Adams type Block Methods (HATBMs) for step length *k*=2,3 and 4 were developed for the solution of first order ordinary differential equations. Collocation and interpolation of Chebyshev polynomial approximation were adopted to derive some implicit linear multi-step methods at different values of k. Analysis of all the methods show that they were consistent, zero stable and convergent. All the newly constructed methods were demonstrated with numerical experiments to ascertain their level of convergence.

Keywords: Chebyshev polynomials, hybrid Adams type, block methods, ordinary differential equation.

1 Introduction

Numerical analysis is the area of mathematical and computer sciences that creates, analyses and implements algorithms for solving numerically the problems of continuous mathematics. Such problems originate from real world applications of algebra, geometry and calculus, these problems occur throughout the natural sciences, social sciences, engineering, medicine and business studies. Solutions to ordinary differential equations were derived using analytical or even exact methods but many often cannot be solved analytically because most real life problems are

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modeled into non linear equations involve complex shapes and processes. Hence there is a great need to develop an algorithm to cater for this type of differential equations.

Chebyshev polynomials are special group of polynomials whose properties and applications were discovered a century ago by the Russian mathematician Patnuty Lvovich Chebyshev. Their importance for practical computations was rediscovered 60 years ago by Comelious Lanczos the father of Numerical Analysis (see [1]).

A simple way of constructing Chebyshev polynomials relies on the recurrence relation of the form

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \qquad \forall\, n \geq 2\,.$$

However, $\boldsymbol{T_0(x)}$ and $\boldsymbol{T_1(x)}$ are constant terms with respect to the recurrence relation as 1 and $\boldsymbol{x}$ respectively.

Definition 1: Hybrid Linear Multistep method

The k step general linear multistep method can be written as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j} + h\beta_v f_{n+v}\,. \qquad (1)$$

where $\alpha_k \neq 0, \alpha_0^2 + \beta_0^2 > 0,\ u \notin \{0, k\}$

Definition 2: Zero Stable

The method in (1) is said to be zero stable if no root of the first characteristic polynomial $\rho(r) = \sum_{j=0}^{k} \alpha_j r^j$ has modulus greater than one and if every root with modulus one is simple

Definition 3: Order and Error Constants

A linear multistep method of the form

$$\sum_{j=0}^{k} \propto_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j}$$

$k \geq 2$ is said to be of order P if $C_0 = C_1 = C_2 = \cdots C_P = 0$

but $C_{P+1} \neq 0$ and C_{P+1} is called error constant (see [2]).

2 Background of the Study

Numerical methods for solving first order initial value problems often fall into two large categories, linear multistep and Runge-Kutta methods. A further division could be realised by dividing methods into those that are explicit and implicit methods. For example, an implicit linear multistep of Adams Moulton is of the form

$$y_{n+k} + y_{n+k-1} = h\sum_{i=0}^{k} \beta_i f_{n+i}\,. \qquad (2)$$

Most resent researchers have worked in this area among them are ([1-5]) to mention only a few, in their proposed methods the approximate solution ranges from power series, exponential functions, Chebyshev polynomials. The aim of this research work is to use the Chebyshev polynomials as our trial basis to develop some Adams type hybrid block methods at various values of step length k.

Theorem 1: Well-Posed Condition

Suppose that f and f_y , its first derivative with respect to y are continuous for $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$. Then the initial value problem $\boldsymbol{y'} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{a} \leq \boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{y}(\boldsymbol{a}) = \boldsymbol{\alpha}$ has a unique solution

$\boldsymbol{y}(\boldsymbol{x})\ \boldsymbol{for}\ \ \boldsymbol{a} \leq \boldsymbol{x} \leq\ \boldsymbol{b}$ and the problem is well posed (see [6]).

Theorem 2: Fundamental Theorem of Dahlquist

The necessary and sufficient condition for linear multistep method to be convergent are that it be consistent and zero- stable (see [6]).

3 Methodology

a) Two step Adams type hybrid block method at $(\boldsymbol{k} = \boldsymbol{2})$

Given an approximate solution of the form

$$\boldsymbol{P}(\boldsymbol{x}) = \sum_{\boldsymbol{n}=\boldsymbol{0}}^{\boldsymbol{m}} \boldsymbol{a_n T_n} \quad . \tag{3}$$

$$\boldsymbol{P}'(\boldsymbol{x}) = \sum_{\boldsymbol{n}=\boldsymbol{0}}^{\boldsymbol{m}} \boldsymbol{a_n T'_n} \quad . \tag{4}$$

From equation (3) and (4)

$$\sum_{\boldsymbol{n}=\boldsymbol{1}}^{\boldsymbol{m}} \boldsymbol{a_n T'_n} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \tag{5}$$

where $\boldsymbol{a_n}$ are the parameters to be determined and $\boldsymbol{T_n}$ are Chebyshev polynomial. We collocate (4) at $\boldsymbol{x} = \boldsymbol{x_{n+j}}, \boldsymbol{j} = (\boldsymbol{u}, \boldsymbol{1}, \boldsymbol{v}, \boldsymbol{2})$, specifically $\boldsymbol{m} = \boldsymbol{5}, \boldsymbol{u} = \frac{1}{2}, \boldsymbol{and}\ \boldsymbol{v} = \frac{3}{2}$ and interpolate (3) at $\boldsymbol{x} = \boldsymbol{x_{n+1}}$ to have the following non linear system of equations of the form

$$\sum_{\boldsymbol{n}=\boldsymbol{0}}^{\boldsymbol{5}} \boldsymbol{a_n T_n}(\boldsymbol{x}) = \boldsymbol{y_{n+1}} \quad .$$

$$\sum_{\boldsymbol{n}=\boldsymbol{1}}^{\boldsymbol{5}} \boldsymbol{a_n T'_n}(\boldsymbol{x}) = \boldsymbol{f_{n+j}}\,, \qquad \boldsymbol{j} = (\boldsymbol{u}, \boldsymbol{1}, \boldsymbol{v}, \boldsymbol{2}). \tag{6}$$

Our propose continuous formulation of the two step Adams type hybrid block method will be of the form

$$y(x)=\alpha_{n+1}(x)y_{n+1}+h\sum_{j=0}^{2}\beta_j(x)f_{n+j}+h\beta_u(x)f_{n+u}+h\beta_v(x)f_{n+v}\,. \qquad (7)$$

By using maple 17 mathematical software to determine the values of α *and* $\beta's$ and substituting to obtain the following continuous formula of the method as

$$y(x) = y_{n+1}+\left[-\frac{29}{180}h+(x-x_n)-\frac{25}{12}\frac{(x-x_n)^2}{h}+\frac{35}{18}\frac{(x-x_n)^3}{h^2}-\frac{5}{6}\frac{(x-x_n)^4}{h^3}+\frac{2}{15}\frac{(x-x_n)^5}{h^4}\right]f_n+\left[-\frac{31}{45}h+\frac{3(x-x_n)^4}{h^3}+\frac{4(x-x_n)^2}{h}-\frac{52}{9}\frac{(x-x_n)^3}{h^2}-\frac{8}{15}\frac{(x-x_n)^5}{h^4}\right]f_{n+\frac{1}{2}}+\left[-\frac{3(x-x_n)^2}{h}-\frac{19}{3}\frac{(x-x_n)^3}{h^2}-\frac{4(x-x_n)^4}{h^3}-\frac{2}{15}h+\frac{4}{5}\frac{(x-x_n)^5}{h^4}\right]f_{n+1}+\left[-\frac{4}{3}\frac{(x-x_n)^2}{h}-\frac{28}{9}\frac{(x-x_n)^3}{h^2}-\frac{18}{15}\frac{(x-x_n)^5}{h^4}-\frac{1}{14}h+\frac{7}{3}\frac{(x-x_n)^4}{h^3}\right]f_{n+\frac{3}{2}}+\left[-\frac{1}{4}\frac{(x-x_n)^2}{h}+\frac{1}{180}h+\frac{2}{15}\frac{(x-x_n)^5}{h^4}+\frac{11}{18}\frac{(x-x_n)^3}{h^2}-\frac{1}{2}\frac{(x-x_n)^4}{h^3}\right]f_{n+2}\,. \qquad (8)$$

Equation (8) is evaluated at $x=x_{n+j}, j=0,\frac{1}{2},\frac{3}{2},2$ to obtain the following discrete schemes as our two step Adams type hybrid methods (ATHBMs).

$$\begin{aligned}
y_{n+\frac{1}{2}}-y_{n+1}&=h\left[\frac{19}{1440}f_n-\frac{19}{60}f_{n+1}-\frac{11}{1440}f_{n+2}-\frac{173}{720}f_{n+\frac{1}{2}}+\frac{73}{720}f_{n+\frac{3}{2}}\right]\\
y_n-y_{n+1}&=h\left[-\frac{29}{180}f_n-\frac{2}{15}f_{n+1}+\frac{1}{180}f_{n+2}-\frac{31}{45}f_{n+\frac{1}{2}}-\frac{1}{45}f_{n+\frac{3}{2}}\right]\\
y_{n+\frac{3}{2}}-y_{n+1}&=h\left[\frac{11}{1440}f_n+\frac{19}{60}f_{n+1}-\frac{19}{1440}f_{n+2}-\frac{37}{720}f_{n+\frac{1}{2}}+\frac{173}{720}f_{n+\frac{3}{2}}\right]\\
y_{n+2}-y_{n+1}&=h\left[-\frac{1}{180}f_n+\frac{2}{15}f_{n+1}+\frac{29}{180}f_{n+2}+\frac{1}{45}f_{n+\frac{1}{2}}+\frac{31}{45}f_{n+\frac{3}{2}}\right]
\end{aligned} \qquad (9)$$

Equation (9) is of order $[5,5,5,5]^T$ with error constants of

$$[1.1194x10^{-4},-1.7361x10^{-4},1.1194x10^{-4},-1.7361x10^{-4}]^T$$

b) Schemes as our three step Adams type hybrid block methods (HTHBMs) at ($k=3$).

Equation (4) is collocated at $x=x_{n+j}, j=0,v,w,2$ *and* 3, specially $m=6, v=\frac{3}{2}, w=\frac{5}{2}$

Also equation (3) is interpolated at $x=x_{n+1}$ to obtain the following system of non linear equation of the form

$$\sum_{n=0}^{6}a_nT_n(x)=y_{n+1}\ .$$

$$\sum_{n=1}^{6}a_nT'_n(x)=f_{n+j}\,,\qquad j=(u,1,v,2,3)\,. \qquad (10)$$

Our propose continuous formula of three step Adams type hybrid block method is of the form

$$y(x)=\alpha_{n+1}(x)y_{n+1}+h\sum_{j=0}^{3}\beta_j(x)f_{n+j}+h\beta_v(x)f_{n+v}+h\beta_w(x)f_{n+w}\,. \qquad (11)$$

By using maple 17 mathematical software to determine the value of α *and* $\beta's$ and substituting in (11) to obtain the continuous formula of the method as

$$y(x) = y_{n+1} + \left[-\frac{11}{40}h + (\xi) - \frac{29}{20}\frac{(\xi)^2}{h} + \frac{29}{27}\frac{(\xi)^3}{h^2} - \frac{31}{72}\frac{(\xi)^4}{h^3} + \frac{4}{45}\frac{(\xi)^5}{h^4} - \frac{1}{135}\frac{(\xi)^6}{h^5}\right]f_n$$
$$+ \left[\frac{15}{2}\frac{(\xi)^2}{h} - \frac{19}{2}\frac{(\xi)^3}{h^2} + \frac{119}{24}\frac{(\xi)^4}{h^3} - \frac{673}{360}h - \frac{6}{5}\frac{(\xi)^5}{h^4} + \frac{1}{9}\frac{(\xi)^6}{h^5}\right]f_{n+1} + \left[-\frac{40}{3}\frac{(\xi)^2}{h} + \frac{136}{45}\frac{(\xi)^5}{h^4} + \frac{104}{45}h - \frac{104}{9}\frac{(\xi)^4}{h^3} + \frac{536}{27}\frac{(\xi)^3}{h^2} - \frac{8}{27}\frac{(\xi)^6}{h^5}\right]f_{n+\frac{3}{2}} + \left[-\frac{211}{120}h + \frac{45}{4}\frac{(\xi)^2}{h} - \frac{18(\xi)^3}{h^2} + \frac{91}{8}\frac{(\xi)^4}{h^3} - \frac{16}{5}\frac{(x-\xi)^5}{h^4} + \frac{1}{3}\frac{(x-\xi)^6}{h^5}\right]f_{n+2}$$
$$+ \left[\frac{32}{45}h - \frac{16}{3}\frac{(\xi)^4}{h^3} - \frac{24}{5}\frac{(\xi)^2}{h} - \frac{8}{45}\frac{(\xi)^6}{h^5} + \frac{8(\xi)^3}{h^2} + \frac{8}{5}\frac{(\xi)^5}{h^4}\right]f_{n+\frac{5}{2}} + \left[-\frac{43}{360}h + \frac{71}{72}\frac{(\xi)^4}{h^3} + \frac{1}{27}\frac{(\xi)^6}{h^5} - \frac{77}{54}\frac{(\xi)^3}{h^2} - \frac{14}{45}\frac{(\xi)^5}{h^4} + \frac{5}{6}\frac{(\xi)^2}{h}\right]f_{n+3}\,. \tag{12}$$

where $\xi = (x - x_n)$

Equation (12) is evaluated at $x = x_{n+j}, j = (0, \frac{3}{2}, 2, \frac{5}{2}, and\ 3)$ to obtain the following discrete schemes as our three Adams type hybrid block method (**HATBMs**)

$$y_n - y_{n+1} = h\left[-\frac{11}{40}f_n - \frac{673}{360}f_{n+1} + \frac{104}{45}f_{n+\frac{3}{2}} - \frac{211}{120}f_{n+2} + \frac{32}{45}f_{n+\frac{5}{2}} - \frac{43}{360}f_{n+3}\right]$$
$$y_{n+\frac{3}{2}} - y_{n+1} = h\left[-\frac{1}{640}f_n + \frac{1139}{5760}f_{n+1} + \frac{139}{360}f_{n+\frac{3}{2}} - \frac{217}{1920}f_{n+2} + \frac{13}{360}hf_{n+\frac{5}{2}} - \frac{31}{5760}f_{n+3}\right]$$
$$y_{n+2} - y_{n+1} = h\left[-\frac{1}{1080}f_n + \frac{7}{40}f_{n+1} + \frac{88}{135}f_{n+\frac{3}{2}} + \frac{7}{40}f_{n+2} - \frac{1}{1080}f_{n+3}\right]$$
$$y_{n+\frac{5}{2}} - y_{n+1} = h\left[-\frac{1}{640}f_n + \frac{123}{640}f_{n+1} + \frac{23}{40}f_{n+\frac{3}{2}} + \frac{333}{640}f_{n+2} + \frac{9}{40}h_{n+\frac{5}{2}} - \frac{7}{640}hf_{n+3}\right]$$
$$y_{n+3} - y_{n+1} = h\left[\frac{7}{45}f_{n+1} + \frac{32}{45}f_{n+\frac{3}{2}} + \frac{4}{15}f_{n+2} + \frac{32}{45}f_{n+\frac{5}{2}} + \frac{7}{45}f_{n+3}\right]\,. \tag{13}$$

Equation (13) is of order $[6,6,6,6,6]^T$ with error constant of $[1.9428(-03), 5.9420(-05), 2.4802(-05), 6.9754(-05), 6.6138(-05)]^T$

c) Schemes as our four step Adams type hybrid block method (HATBMs) at $(k = 4)$.

Equation (4) is collocated at $= x_{n+j}, j = (1, v, 2, w, 3, 4)$, specially $m = 7, v = \frac{3}{2}, and\ w = \frac{5}{2}$. Also (3) is interpolated at $x = x_{n+1}$ to obtain the following system of linear equation of the form

$$\sum_{n=0}^{7} a_n T_n(x_{n+1}) = y_{n+1}$$
$$\sum_{n=1}^{7} a_n T'_n(x_{n+j}) = f_{n+j}\,, \qquad j = (1, v, 2, w, 3, 4) \tag{14}$$

Our propose continuous formula of the four step Adams type hybrid block method is of the form

$$y(x) = y_{n+1} + h\sum_{n=1}^{4}\beta_j(x)f_{n+j} + h\beta_v(x)f_{n+v} + h\beta_w(x)f_{n+w} \tag{15}$$

By using maple 17 to determine the values of $\alpha\ and\ \beta's$ substituting them in (15) to obtain the continuous formula of the method as

$$\boldsymbol{y_n = y_{n+1} + \left[-\frac{63}{40}\frac{(x-x_n)^2}{h} + \frac{1421}{1080}\frac{(x-x_n)^3}{h^2} - \frac{91}{144}\frac{(x-x_n)^4}{h^3} - (x-x_n) + \frac{7}{40}\frac{(x-x_n)^5}{h^4} - \frac{7}{270}\frac{(x-x_n)^6}{h^5} - \frac{3923}{15120}h + \frac{1}{630}\frac{(x-x_n)^7}{h^6}\right]f_n + \left[\frac{10(x-x_n)^2}{h} - \frac{16483}{7560}h - \frac{263}{90}\frac{(x-x_n)^5}{h^4} - \frac{2}{63}\frac{(x-x_n)^7}{h^6} + \frac{647}{72}\frac{(x-x_n)^4}{h^3} - \frac{43}{3}\frac{(x-x_n)^3}{h^2} + \frac{13}{27}\frac{(x-x_n)^6}{h^5}\right]f_{n+1} + \left[\frac{128}{15}\frac{(x-x_n)^5}{h^4} + \frac{3124}{945}h - \frac{64}{3}\frac{(x-x_n)^2}{h} + \frac{32}{315}\frac{(x-x_n)^7}{h^6} - \frac{40}{27}\frac{(x-x_n)^6}{h^5} + \frac{4768}{135}\frac{(x-x_n)^3}{h^2} - \frac{220}{9}\frac{(x-x_n)^4}{h^3}\right]f_{n+\frac{3}{2}} + \left[\frac{59}{2}\frac{(x-x_n)^4}{h^3} - \frac{219}{20}\frac{(x-x_n)^5}{h^4} - \frac{1}{7}\frac{(x-x_n)^7}{h^6} + \frac{45}{2}\frac{(x-x_n)^2}{h} - \frac{159}{4}\frac{(x-x_n)^3}{h^2} + \frac{2(x-x_n)^6}{h^5} - \frac{221}{70}h\right]f_{n+2} + \left[-\frac{164}{9}\frac{(x-x_n)^4}{h^3} + \frac{64}{9}\frac{(x-x_n)^5}{h^4} + \frac{32}{315}\frac{(x-x_n)^7}{h^6} - \frac{64}{5}\frac{(x-x_n)^2}{h} + \frac{352}{15}\frac{(x-x_n)^3}{h^2} + \frac{1612}{945}h - \frac{184}{135}\frac{(x-x_n)^6}{h^5}\right]f_{n+\frac{5}{2}} + \left[\frac{361}{72}\frac{(x-x_n)^4}{h^3} + \frac{10}{3}\frac{(x-x_n)^2}{h} - \frac{2}{63}\frac{(x-x_n)^7}{h^6} - \frac{61}{30}\frac{(x-x_n)^5}{h^4} + \frac{11}{27}\frac{(x-x_n)^6}{h^5} - \frac{3253}{7560}h - \frac{169}{27}\frac{(x-x_n)^3}{h^2}\right]f_{n+3} + \left[-\frac{29}{144}\frac{(x-x_n)^4}{h^3} + \frac{31}{360}\frac{(x-x_n)^5}{h^4} - \frac{1}{8}\frac{(x-x_n)^2}{h} + \frac{47}{3024}h + \frac{1}{630}\frac{(x-x_n)^7}{h^6} - \frac{1}{54}\frac{(x-x_n)^6}{h^5} + \frac{29}{120}\frac{(x-x_n)^3}{h^2}\right]f_{n+4}} \tag{16}$$

Evaluating equation (16) at $\boldsymbol{x = x_{n+j}, j = (0, \frac{3}{2}, 2, \frac{5}{2}, 3\ and\ 4)}$ to have the following discrete schemes as our four steps Adams type hybrid block method

$$\boldsymbol{\begin{aligned}
y_{n+1} - y_n &= h\left[\frac{3923}{15120}f_n + \frac{16483}{7560}f_{n+1} - \frac{3124}{945}f_{n+\frac{3}{2}} + \frac{221}{70}f_{n+2} - \frac{1612}{945}hf_{n+\frac{5}{2}} + \frac{3253}{7560}f_{n+3} - \frac{47}{3024}f_{n+4}\right] \\
y_{n+\frac{3}{2}} - y_{n+1} &= h\left[-\frac{263}{241920}f_n + \frac{22769}{120690}f_{n+1} + \frac{3149}{7560}f_{n+\frac{3}{2}} - \frac{349}{2240}f_{n+2} + \frac{503}{7560}f_{n+\frac{5}{2}} - \frac{1801}{120960}f_{n+3} + \frac{23}{48384}f_{n+4}\right] \\
y_{n+2} - y_{n+1} &= h\left[-\frac{11}{15120}f_n + \frac{431}{25120}f_{n+1} + \frac{638}{945}f_{n+\frac{3}{2}} + \frac{11}{70}f_{n+2} + \frac{4}{315}f_{n+\frac{5}{2}} - \frac{37}{7560}f_{n+3} + \frac{1}{5040}f_{n+4}\right] \\
y_{n+\frac{5}{2}} - y_{n+1} &= h\left[-\frac{9}{8960}f_n + \frac{811}{4480}f_{n+1} + \frac{171}{280}f_{n+\frac{3}{2}} + \frac{1053}{2240}f_{n+2} + \frac{73}{280}f_{n+\frac{5}{2}} - \frac{99}{4480}f_{n+3} + \frac{1}{1792}f_{n+4}\right] \\
y_{n+3} - y_{n+1} &= h\left[-\frac{1}{1890}f_n + \frac{157}{945}f_{n+1} + \frac{128}{189}f_{n+\frac{3}{2}} + \frac{11}{35}f_{n+2} + \frac{128}{189}f_{n+\frac{5}{2}} + \frac{157}{945}f_{n+3} - \frac{1}{1890}f_{n+4}\right] \\
y_{n+4} - y_{n+1} &= h\left[-\frac{9}{560}f_n + \frac{167}{280}f_{n+1} - \frac{36}{35}f_{n+\frac{3}{2}} + \frac{243}{70}f_{n+2} - \frac{92}{35}f_{n+\frac{5}{2}} + \frac{657}{280}f_{n+3} + \frac{29}{112}f_{n+4}\right].
\end{aligned}} \tag{17}$$

Equation (17) is of order $[\mathbf{7,7,7,7,8,7}]^T$, with error constant

$$[\mathbf{1.0045(-03), -2.3542(-05), -1.2401(-05), 3.8508(-05), -1.0044(-03)}]^T$$

4 Convergence Analysis of the Two Step Hybrid Adams Type Block Method (HATBMs)

The two step hybrid Adams type block method is expressed in the form

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix} + \begin{bmatrix} \frac{-173}{720} & \frac{-19}{60} & \frac{37}{720} & \frac{-11}{1440} \\ \frac{31}{45} & \frac{2}{15} & \frac{1}{45} & \frac{-1}{180} \\ \frac{-37}{720} & \frac{19}{60} & \frac{173}{720} & \frac{-19}{1440} \\ \frac{1}{45} & \frac{2}{15} & \frac{31}{45} & \frac{29}{180} \end{bmatrix}\begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \frac{19}{1440} \\ 0 & 0 & 0 & \frac{29}{180} \\ 0 & 0 & 0 & \frac{11}{1440} \\ 0 & 0 & 0 & \frac{-1}{180} \end{bmatrix}\begin{bmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} \tag{18}$$

where $A^{(0)} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$B^{(0)} = \begin{bmatrix} \frac{-173}{720} & \frac{-19}{60} & \frac{37}{720} & \frac{-11}{1440} \\ \frac{31}{45} & \frac{2}{15} & \frac{1}{45} & \frac{-1}{180} \\ \frac{-37}{720} & \frac{19}{60} & \frac{173}{720} & \frac{-19}{1440} \\ \frac{1}{45} & \frac{2}{15} & \frac{31}{45} & \frac{29}{180} \end{bmatrix}, B^{(1)} = \begin{bmatrix} 0 & 0 & 0 & \frac{19}{1440} \\ 0 & 0 & 0 & \frac{29}{180} \\ 0 & 0 & 0 & \frac{11}{1440} \\ 0 & 0 & 0 & \frac{-1}{180} \end{bmatrix}$$

We normalized the block method (18) with the inverse of $A^{(0)}$ and applied the condition

$$\rho(\lambda) = \det\left[\lambda A^{(0)}(A^{(0)})^{-1} - A^{(1)}(A^{(0)})^{-1}\right] = 0$$

$$\rho(\lambda) = \left[\lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right] = 0$$

$$= \left[\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right] = 0$$

$$= det \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & (\lambda - 1) & 0 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} = 0$$

$=\lambda^3(\lambda - 1) = 0$

$=\lambda^3 = 0 \quad and \quad \lambda - 1 = 0$

Therefore, $\lambda_1 = \lambda_2 = \lambda_3 = 0, \; \lambda_4 = 1$.

From definition (2), the method (9) is zero stable and since the order of the method is $p \geq 1$, the method is consistent, thus convergent.

Similarly methods (13) and (17) were treated in the same manner and have shown their consistencies and convergent.

5 Numerical Experiments

All the three newly constructed Adams type hybrid block method at various values of $\boldsymbol{k} = \mathbf{2, 3}$ ***and*** $\mathbf{4}$, were demonstrated with three different linear and one non linear problems

Problem 1

$$y' = -y, \qquad y(0) = 1, \qquad 0 \leq x \leq 1, \qquad h = 0.1$$

Exact solution

$$y(x) = e^{-x}$$

Problem 2

$$y' = xy, \qquad y(0) = 1, \qquad 0 \le x \le 2, \qquad h = 0.1$$

Exact solution

$$y(x) = e^{\frac{x^2}{2}}$$

Problem 3

$$y' = y - x^2 + 1, \qquad y(0) = \frac{1}{2}, \qquad h = 0.1$$

Exact solution

$$y(x) = (x+1)^2 - \frac{1}{2}e^x$$

Problem 4

$$y' = 1 - y^2, \qquad y(0) = 0, \qquad h = 0.1$$

Exact solution

$$y(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$

Table 1. Approximate solutions to problem 1

x	Method of k=2	Method of k=3	Method of k=4
0.1000	0.904837417895975	0.904837417881202	0.904837418028227
0.2000	0.818730753110534	0.818730752939751	0.818730753071073
0.3000	0.740818220855484	0.740818220548903	0.740818220675358
0.4000	0.670320046323450	0.670320045918305	0.670320046037833
0.5000	0.606530660091209	0.606530659599218	0.606530659682672
0.6000	0.548811636543486	0.548811635994641	0.548811636211386
0.7000	0.496585304294834	0.496585303694640	0.496585303267860
0.8000	0.449328964660269	0.449328964033219	0.449328966406345
0.9000	0.406569660311169	0.406569659658082	0.406569649706824
1.0000	0.367879441759379	0.367879441100594	0.367879441171442

Table 2. Absolute errors of problem 1

x	Method (OR)	Error at k=2	Error at k=3	Error at k=4
0.1000	1.5490(-08)	1.3999(-10)	1.5476(-10)	7.7330(-12)
0.2000	9.3743(-09)	-3.2552(-11)	1.3823(-10)	6.9090(-12)
0.3000	1.6932(-08)	-1.7377(-10)	1.3282(-10)	6.3600(-12)
0.4000	1.6406(-08)	-2.8781(-10)	1.1733(-10)	-2.1940(-12)
0.5000	2.5228(-08)	-3.7858(-10)	1.1342(-10)	3.1970(-12)
0.6000	1.9716(-08)	-4.4946(-10)	9.9385(-11)	2.8320(-12)
0.7000	2.3504(-08)	-5.0342(-10)	9.6770(-11)	2.6370(-12)
0.8000	2.1992(-08)	-5.4305(-10)	8.4003(-11)	2.9400(-12)
0.9000	2.6862(-08)	-5.7057(-10)	8.2517(-11)	8.1400(-13)
1.0000	2.2220(-08)	-5.8794(-10)	7.0848(-11)	6.9600(-13)

Method (OR):- Method in Odekunleetal (2012)

Table 3. Approximate solutions to problem 2

x	Method k=2	Method k=3	Method k=4
0.1000	1.005012518167810	1.005012524024950	1.005012519628150
0.2000	1.020201340025330	1.020201343118400	1.020201338792430
0.3000	1.046027862950250	1.046027863348370	1.046027858627300
0.4000	1.083287074401830	1.083287071288450	1.083287067665950
0.5000	1.133148464484150	1.133148457147040	1.133148448352280
0.6000	1.197217380713540	1.197217367631370	1.197217379701250
0.7000	1.277621339108210	1.277621318512900	1.277621241114160
0.8000	1.377127803635090	1.377127770522030	1.377128070311000
0.9000	1.499302555226830	1.499302507651910	1.49930211675550
1.0000	1.648721347944550	1.648721280024100	1.648726712852990

Table 4. Absolute errors of problem 2

x	Method (OR)	Error at k=2	Error at k=3	Error at k=4
0.1000	3.1386(-07)	2.69159(-09)	3.16555(-09)	1.23125(-09)
0.2000	1.3364(-07)	1.43000(-12)	3.17164(-09)	1.23433(-09)
0.3000	3.1819(-07)	3.04153(-09)	3.43965(-09)	1.28142(-09)
0.4000	9.8972(-09)	6.72687(-09)	3.61349(-09)	9.01000(12)
0.5000	6.9521(-07)	1.14173(-08)	4.08021(-09)	3.3949(-09)
0.6000	4.0794(-07)	1.75917(-08)	4.50965(-09)	3.5428(-09)
0.7000	7.7319(-07)	2.59033(-08)	5.30801(-09)	3.8244(-09)
0.8000	7.0821(-07)	3.92991(-08)	6.18607(-09)	3.8275(-10)
0.9000	2.8682(-06)	5.51701(-08)	7.59514(-09)	1.3275(-08)
1.0000	2.0664(-06)	7.72444(-08)	9.32397(-09)	1.4427(-08)

Method (OR):- Method in Odekunleetal (2012)

Table 5. Approximate solutions to problem 3

x	Method k=2	Method k=3	Method k=4
0.1000	0.657414541069634	0.657414616647980	0.657414540968716
0.2000	0.829298620944187	0.829298629148387	0.829298620927051
0.3000	1.015070596134470	1.015070596059230	1.015070596217970
0.4000	1.214087650978350	1.214087651011410	1.214087651181800
0.5000	1.425639364300310	1.425639480369200	1.425639364675610
0.6000	1.648940599300000	1.648940599593540	1.648940599730980
0.7000	1.883123643700000	1.883123646024930	1.883123646607190
0.8000	2.127229525000000	2.127229535488470	2.127229534342820
0.9000	2.380198432346940	2.381019844412120	2.380198450381810
1.0000	2.640859072215840	2.640859085437560	2.640859060719960

Table 6. Absolute errors of problem 3

x	Error at k=2	Error at k=3	Error at k=4
0.1000	1.07459(-10)	1.22237(-10)	-6.54100(-12)
0.2000	2.42720(-11)	1.33398(-10)	-7.13600(-12)
0.3000	7.75300(-11)	1.52770(-10)	-7.97000(-12)
0.4000	2.01010(-10)	1.67950(-10)	-2.44000(-12)
0.5000	3.49630(-10)	1.91450(-10)	-2.56700(-11)
0.6000	5.27260(-10)	2.11200(-10)	7.37600(-11)
0.7000	7.38400(-10)	2.39830(-10)	-3.42430(-10)
0.8000	1.07538(-08)	2.65290(-10)	1.41094(-09)
0.9000	1.20746(-08)	3.00320(-10)	-5.96029(-09)
1.0000	1.35546(-08)	3.32920(-10)	2.50505(-08)

Table 7. Approximate solutions to problem 4

x	Method k=2	Method k=3	Method k=4
0.1000	0.099667998229271	0.099667958427121	0.099667983956123
0.2000	0.197375318495844	0.197375285530702	0.197375309998647
0.3000	0.291312602600503	0.291312578047386	0.291312602567245
0.4000	0.379948943860190	0.379948931365855	0.379948964538561
0.5000	0.462117132110030	0.462117128154898	0.462117113959080
0.6000	0.537049538236968	0.537049538236968	0.537049729661565
0.7000	0.604367748161445	0.604367754450766	0.604367039621352
0.8000	0.664036743935206	0.664036751802030	0.664039991411427
0.9000	0.716297848263363	0.716297853189866	0.716283642731265
1.0000	0.761594139111875	0.761594142508150	0.761657006565570

Table 8. Absolute errors of problem 4

x	Error at k=2	Error at k=3	Error at k=4
0.1000	-3.60432(-09)	3.61978(-08)	1.2312(-09)
0.2000	1.72906(-09)	3.46942(-08)	1.2343(-09)
0.3000	9.85108(-09)	3.44042(-08)	1.2814(-09)
0.4000	1.83950(-08)	3.08893(-08)	9.0100(-12)
0.5000	2.51500(-08)	2.91051(-08)	3.3949(-09)
0.6000	2.87611(-08)	2.46756(-08)	3.5428(-09)
0.7000	2.89557(-08)	2.26664(-08)	3.8244(-09)
0.8000	2.63326(-08)	1.84658(-08)	3.8293(-10)
0.9000	2.19357(-08)	1.70072(-08)	1.3275(-08)
1.0000	1.68439(-08)	1.34476(-08)	1.4427(-08)

6. Discussion of Results

Tables 1 shows the numerical results of problem 1while Table 2 displaces their absolute errors at various values of $\boldsymbol{k = 2, 3\ and\ 4}$. Tables 3 shows the numerical results of problem 2, while Table 4 displaces their absolute errors at various values of $\boldsymbol{k = 2, 3\ and\ 4}$. Tables 5 shows the numerical results of problem 3, while Table 6 displaces their absolute errors at various values of $\boldsymbol{k = 2, 3\ and\ 4}$. Finally Tables 7 shows the numerical results of problem 4, while Table 8 displaces their absolute errors at various values of $\boldsymbol{k = 2, 3\ and\ 4}$. We observed that in problem 1 and 2 which were equally solved by Method OR, the result obtained from various values of $\boldsymbol{k = 2, 3\ and\ 4}$ converges more accurately than the existing methods.

7 Conclusion

We conclude that this research paper demonstrates a successful derivation and implementation of Adam's type hybrid block linear multi-step method using Chebyshev polynomial as trial bases functions. Based on the problems solved all the methods at various values of $\boldsymbol{k = 2, 3\ and\ 4}$ performed better than the existing methods, particularly in problems 1 and 2, for example see Method (OR). All the methods are stable, consistent and convergent.

Competing Interests

Authors have declared that no competing interests exist.

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