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Evaluation of Some Convolution Sums and Representation Numbers of Quadratic Forms of Discriminant -135

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Abstract

We evaluate the convolution sums $\sum_{l+135m=n} \sigma(l)\sigma(m)$, $\sum_{3l+45m=n} \sigma(l)\sigma(m)$, $\sum_{5l+27m=n} \sigma(l)\sigma(m)$, $\sum_{9l+15m=n} \sigma(l)\sigma(m)$, $\sum_{l+45m=n} \sigma(l)\sigma(m)$, $\sum_{5l+9m=n} \sigma(l)\sigma(m)$, and $\sum_{3l+15m=n} \sigma(l)\sigma(m)$ for all nϵN, using the theory of quasimodular forms and determine the number of representations of a positive integer n by some direct sum of the forms $x_1^2+x_1x_2+34x_2^2, 5x_1^2+5x_1x_2+8x_2^2, 4x_1^2 \pm 3x_1x_2+9x_2^2$, $2x_1^2 \pm x_1x_2+17x_2^2$ of discriminant -135 by modular forms.
Moreover, we explain how to determine the number of representations of $Q+tP, t>1, t|135$,for various sums Q, P of the above quadratic forms.

Keywords: Quasimodular forms, divisor functions, convolution sums, representation number.

2010 Mathematics Subject Classification: 11A25; 11F11; 11F25; 11F20

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1 Introduction

Let $\sigma(m)$ be the sum of positive divisors of a positive integer m. It is well known that divisor function σ appears in a number of remarkable identities, including relationship on the Riemann zeta function and the Eisenstein series of modular forms. It was studied by Ramanujan [1], who has found a number of important congruences and identities. It was also used in the counting of the number of nonisomorphic branched coverings of surfaces of genus g with a given ramification type σ, and in the orbitwise counting of $\mathfrak{H}$(2), see [2]. On the other hand, the work on representation number r(Q,n) of quadratic forms has been started by Fermat in 1640 on $Q = x^2 + y^2$. Later the formula $r_Q(n) = 4\left(\sum_{d|n \text{ d is odd}} (-1)^{\frac{d-1}{2}}\right)$ has been proved by Euler. Afterwards it was advanced by Jacobi, see [3] with the proof of

$$r_Q(n) = 8\left(\sum_{d|n \ 4\nmid d} d\right), Q = x^2 + y^2 + z^2 + t^2.$$

It would be nice to obtain such simple formulas for other positive definite quadratic forms so that we would be able to understand the number of solutions of the equation $Q = n$ for any positive integer.We will do it in section 3 by means of modular forms of some direct sum following positive definite quadratic forms

$$F = x_1^2 + x_1x_2 + 34x_2^2, \Lambda = 5x_1^2 + 5x_1x_2 + 8x_2^2,$$
$$\Phi = 4x_1^2 + 3x_1x_2 + 9x_2^2, \Phi^{-1} = 4x_1^2 - 3x_1x_2 + 9x_2^2,$$
$$\Psi = 2x_1^2 + x_1x_2 + 17x_2^2, \Psi^{-1} = 2x_1^2 - x_1x_2 + 17x_2^2.$$

Here, we also want to study some convolutions of divisor functions

$$W_N(n) := \sum_{\substack{m<n/N \\ m,n\in\mathbb{N}}} \sigma(m)\sigma(n - Nm),$$

$$W_{a,b}(n) := \sum_{\substack{al+bm=n \\ l,m\in\mathbb{N}}} \sigma(l)\sigma(m),$$

where N, a, b positive integers. Some of them were calculated as early as 19th century. For example, $W_1(n)$ was evaluated by [4], [5], and [1]. The convolution sums $W_N(n)$ (for $1 \le N \le 24$ with a few exceptions) and $W_{a,b}(n)$ for $(a, b) \in \{(2, 3), (3, 4), (3, 8), (2, 9)\}$ have been evaluated by using either elementary methods or analytic methods (see [4], [5], [1], [6], [7], [8], [9], [10], [11]], [12], [13], [14], [15], [16], [17], [18], [19]).

A different important algebraic method by means of quasimodular forms was introduced by Royer in [20] and it was applied in [21], in order to get $W_{15,} W_{3,5}$. The quasimodular forms was defined in [22]. The number of representations of integers by certain quadratic forms.

(see [9], [14], [12], [23], [18], [19], [21], [24]) have been found by evaluation of these convolution sums. In this article, following the method of Royer [21], and using Magma for the calculations, we evaluate the following convolution sums

$$W_{135}(n), W_{3,45}(n), W_{5,27}(n), W_{9,15}(n)$$

by using the theory of quasimodular forms.

Although, in general it may be a challenging problem to describe the integer solutions to a polynomial equation of several variables, by using the above convolution sums we will describe the formulas for the number of representations of integers of $Q + tP$ in the last section.

1.1 ***Notations.***

Let α, β be roots of $x^3 - x^2 - 23x + 6$ and δ, γ be roots of $x^3 + 5x^2 - 8x - 30.$

Let η be a root of $x^2 + x - 3, \omega$ be a root of $x^2 - 18.$

Let $\sigma_k(n) := \sum_{\substack{d|n \\ d>0}} d^k, k, n \in \mathbb{N}$ and write σ for σ_1.

Let ψ be the Dirichlet character mod 3 sending 2 to -1 and

$$\sigma_k^{\psi,\psi}(n) := \sum_{\substack{d|n \\ d>0}} \psi\left(\frac{n}{d}\right) \psi(d)\, d^k \text{ for k=1,3.}$$

Let $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1, c \equiv 0 mod N, N \in \mathbb{N} \right\}.$

Let $M_k[\Gamma_0(N)]$ resp., $S_k[\Gamma_0(N)]$ be the space of modular forms resp., cusp forms of weight k with respect to $\Gamma_0(N)$.

Let

$$\begin{aligned}
E_4(z) &: = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, q^n, E_4^{\psi,\psi}(z) = \sum_{n=1}^{+\infty} \sigma_3^{\psi,\psi}(n) q^n \\
E_2(z) &: = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)\, q^n, E_2^{\psi,\psi}(z) = \sum_{n=1}^{+\infty} \sigma^{\psi,\psi}(n) q^n, \\
\Phi_{1,N} &: = \frac{1}{N-1}\left(N E_2(Nz) - E_2(z)\right), N \in \mathbb{N} \\
q &: = e^{2\pi i z}, Im z > 0.
\end{aligned}$$

Let $\Delta_{k,N} = \sum_{n=1}^{\infty} \tau_{k,N}(n)$ be the unique newform in $S_k[\Gamma_0(N)]$ if dim$S_k[\Gamma_0(N)] = 1$,and

$$\Delta_{k,N,1} = \sum_{n=1}^{\infty} \tau_{k,N,1}(n), \Delta_{k,N,2} = \sum_{n=1}^{\infty} \tau_{k,N,2}(n), ..., \Delta_{k,N,m} = \sum_{n=1}^{\infty} \tau_{k,N,m}(n)$$

be the basis of newforms in $S_k[\Gamma_0(N)]$ if dim$S_k[\Gamma_0(N)] = m.$

Let $\widetilde{M_4^{\leq 2}}[\Gamma_0(N)]$ be the space of quasimodular forms of weight 4 with depth 2, see [21].

2 Evaluation of $W_{135}(n)$ $W_{3,45}(n), W_{5,27}(n), W_{9,15}(n), W_{45}(n)$ $W_{3,15}(n), W_{5,9}(n)$

The vector space $M_4[\Gamma_0(135)]$ is 60 dimensional and spanned by the 12 linearly independent Eisen-

stein forms

$$E_4, E_4(3z), E_4(5z), E_4(9z), E_4(15z), E_4(27z), E_4(45z), E_4(135z),$$

$$E_4^{\psi,\psi}(z) = q - 9q^2 + \cdots + 0q^{135} + O\left(q^{136}\right) \in M_4[\Gamma_0(9)],$$

$$E_4^{\psi,\psi}(3z), E_4^{\psi,\psi}(5z), E_4^{\psi,\psi}(15z),$$

32 oldforms

$$\Delta_{4,5}(z), \Delta_{4,5}(5z), \Delta_{4,5}(9z), \Delta_{4,5}(27z),$$
$$\Delta_{4,9}(z), \Delta_{4,9}(3z), \Delta_{4,9}(5z), \Delta_{4,9}(15z),$$
$$\Delta_{4,15,1}(z), \Delta_{4,15,1}(3z), \Delta_{4,15,1}(9z), \Delta_{4,15,2}(z), \Delta_{4,15,2}(3z), \Delta_{4,15,2}(9z),$$
$$\Delta_{4,27,1}(z), \Delta_{4,27,1}(5z), \Delta_{4,27,2}(z), \Delta_{4,27,2}(5z),$$
$\Delta_{4,27,3}(z), \Delta_{4,27,3}(5z), \Delta_{4,27,4}(z), \Delta_{4,27,4}(5z)$, the coefficients are in $\mathbb{Q}[x]/<x^2-18>$, with roots ω,-ω,
$$\Delta_{4,45,1}(z), \Delta_{4,45,1}(3z), \Delta_{4,45,2}(z), \Delta_{4,45,2}(3z), \Delta_{4,45,3}(z), \Delta_{4,45,3}(3z),$$
$$\Delta_{4,45,4}(z), \Delta_{4,45,4}(3z), \Delta_{4,45,5}(z), \Delta_{4,45,5}(3z),$$
$$\Delta_{4,135,1}(z), \Delta_{4,135,2}(z), \Delta_{4,135,1}(z), \Delta_{4,135,1}(z), \Delta_{4,135,1}(z), \Delta_{4,135,1}(z),$$
$$\Delta_{4,135,1}(z), \Delta_{4,135,1}(z),$$

and 16 newforms, see theorem 5.8.3 in [25],

$$\Delta_{4,135,1}(z), \Delta_{4,135,2}(z), \Delta_{4,135,3}(z), \Delta_{4,135,4}(z),$$
$\Delta_{4,135,5}(z), \Delta_{4,135,6}(z), \Delta_{4,135,7}(z)$, the coefficients are in $\mathbb{Q}[x]/<x^3-x^2-23x+6>$ with roots $\alpha, \beta, 1-\alpha-\beta$
$$\Delta_{4,135,8}(z), \Delta_{4,135,9}(z), \Delta_{4,135,10}(z),$$
$\Delta_{4,135,11}(z), \Delta_{4,135,12}(z), \Delta_{4,135,13}(z)$, the coefficients are in $\mathbb{Q}[x]/<x^3+5x^2-8x-30>$ with roots $\gamma, \delta, -5-\gamma-\delta$
$$\Delta_{4,135,14}(z), \Delta_{4,135,15}(z), \Delta_{4,135,16}(z).$$

The vector space $M_2[\Gamma_0(135)]$ is 24 dimensional (see theorem 4.6.2 in [25]) and spanned by 11 Eisenstein series

$$\Phi_{1,3}, \Phi_{1,5}, \Phi_{1,9}, \Phi_{1,15}, \Phi_{1,27}, \Phi_{1,45}, \Phi_{1,135},$$

$$E_2^{\psi,\psi}(z) = q-3q^2+\cdots+0q^{135}+O(q^{136}) \in M_2[\Gamma_0(9)],$$
$$E_2^{\psi,\psi}(3z), E_2^{\psi,\psi}(5z), E_2^{\psi,\psi}(15z)$$

and oldforms

$$\Delta_{2,15}(z), \Delta_{2,15}(3z), \Delta_{2,15}(9z), \Delta_{2,27}(z),$$
$$\Delta_{2,27}(5z), \Delta_{2,45}(z), \Delta_{2,45}(3z),$$

and 6 newforms (see theorem 5.8.3 in [25])

$$\Delta_{2,135,1}(z), \Delta_{2,135,2}(z),$$
$\Delta_{2,135,3}(z), \Delta_{2,135,4}(z)$, the coefficients are in $\mathbb{Q}[x]/<x^2+x-3>$, with roots $\eta, -1-\eta$
$$\Delta_{2,135,5}(z), \Delta_{2,135,6}(z).$$

Now it is known that, see [21],

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(N)] = M_4[\Gamma_0(N)] \oplus DM_2[\Gamma_0(N)] \oplus \mathbb{C}DE_2,$$

where the derivation D is defined by D:=$\frac{1}{2\pi i}\frac{d}{dz}$. Since $E_2(135z)E_2(z)$ is a quasimodular form, it can be written as linear combination of basis elements of $M_4[\Gamma_0(N)]$, derivative of basis elements of $M_2[\Gamma_0(N)]$ and derivative of E_2.In order to find the coefficients of this linear combination of 85 elements we have to determine the powers of q^i for which the corresponding 85 parts is linearly independent. For instance, in this case we can take the powers as

$$1, 2, \ldots, 74, 76, 78, 82, 88, 91, 92, 96, 103, 106, 109, 136.$$

On the other hand, since

$$E_2(135z)E_2(z) = 1 - \sum_{n=1}^{+\infty}(24\left(\sigma(n) + \sigma(\frac{n}{135})\right) + 576W_{135}(n))q^n,$$

we get the following Theorem.

$$\begin{aligned}
W_{135}(n) =& -1/540n\sigma(n) + 1/24\sigma(n) - 1/4n\sigma\left(\frac{n}{135}\right) + 1/24\sigma\left(\frac{n}{135}\right) + 1/50544\sigma_3(n) \\
&+ 1/6318\sigma_3\left(\frac{n}{3}\right) + 25/50544\sigma_3\left(\frac{n}{5}\right) + 1/702\sigma_3\left(\frac{n}{9}\right) + 25/6318\sigma_3\left(\frac{n}{15}\right) \\
&+ 3/208\sigma_3\left(\frac{n}{27}\right) + 25/702\sigma_3\left(\frac{n}{45}\right) + 75/208\sigma_3\left(\frac{n}{135}\right) - 2/5265\tau_{4,5}(n) \\
&- 25/7371\tau_{4,5}\left(\frac{n}{3}\right) - 25/819\tau_{4,5}\left(\frac{n}{9}\right) - 18/65\tau_{4,5}\left(\frac{n}{27}\right) - 2/1215\tau_{4,9}(n) \\
&- 2/135\tau_{4,9}\left(\frac{n}{3}\right) - 10/243\tau_{4,9}\left(\frac{n}{5}\right) - 10/27\tau_{4,9}\left(\frac{n}{15}\right) - 1/720\tau_{4,15,1}(n) \\
&- 1/80\tau_{4,15,1}\left(\frac{n}{3}\right) - 9/80\tau_{4,15,1}\left(\frac{n}{9}\right) - 1/324\tau_{4,15,2}(n) - 31/756\tau_{4,15,2}\left(\frac{n}{3}\right) \\
&- 1/4\tau_{4,15,2}\left(\frac{n}{9}\right) - 2/1215\tau_{4,27,1}(n) - 10/243\tau_{4,27,1}\left(\frac{n}{5}\right) \\
&+ (-1/1836\omega - 7/2754)\tau_{4,27,3}(n) + (-25/1836\omega - 175/2754)\tau_{4,27,3}\left(\frac{n}{5}\right) \\
&- 1/1134\tau_{4,45,2}(n) - 1/126\tau_{4,45,2}\left(\frac{n}{3}\right) - 1/189\tau_{4,45,3}(n) \\
&- 1/21\tau_{4,45,3}\left(\frac{n}{3}\right) - 1/405\tau_{4,45,4}(n) - 1/45\tau_{4,45,4}\left(\frac{n}{3}\right) - 1/918\tau_{4,135,2}(n) \\
&+ 1/4565970(32\alpha^2 - 1206\alpha - 6989)\tau_{4,135,5}(n) + (1/2282985(-16\alpha - 587)\beta \\
&+ 1/4565970(-32\alpha^2 + 32\alpha - 6253))\tau_{4,135,6}(n) + (1/2282985(16\alpha + 587)\beta \\
&+ 1/4565970(1174\alpha - 7427))\tau_{4,135,7}(n) \\
&+ 1/304398(-10\alpha^2 + 142\alpha - 517)\tau_{4,135,8}(n) + (1/152199(5\alpha + 66)\beta \\
&+ 1/304398(10\alpha^2 - 10\alpha - 747))\tau_{4,135,9}(n) + (1/152199(-5\alpha - 66)\beta \\
&+ 1/101466(-44\alpha - 205))\tau_{4,135,10}(n) + (1/11961\gamma^2 - 17/35883\gamma \\
&- 173/71766)\tau_{4,135,11}(n) + ((-1/11961\gamma - 2/35883)\delta + (-1/11961\gamma^2 \\
&+ 5/11961\gamma - 125/71766))\tau_{4,135,12}(n) + ((1/11961\gamma + 2/35883)\delta \\
&+ (2/35883\gamma - 145/71766))\tau_{4,135,13}(n).
\end{aligned}$$

Similarly, we prove

$$
\begin{aligned}
W_{3,45}(n) = &- 1/180n\sigma\left(\frac{n}{3}\right) + 1/24\sigma\left(\frac{n}{3}\right) - 7/180n\sigma\left(\frac{n}{45}\right) + 1/24\sigma\left(\frac{n}{45}\right) \\
&+ 1/45n\sigma\left(\frac{n}{135}\right) + 1/624\sigma_3\left(\frac{n}{3}\right) + 3/208\sigma_3\left(\frac{n}{9}\right) + 25/624\sigma_3\left(\frac{n}{15}\right) \\
&- 133/208\sigma_3\left(\frac{n}{45}\right) + \sigma_3\left(\frac{n}{135}\right) - 1/455\tau_{4,5}\left(\frac{n}{3}\right) - 9/455\tau_{4,5}\left(\frac{n}{9}\right) \\
&- 1/80\tau_{4,15,1}\left(\frac{n}{3}\right) - 1/84\tau_{4,15,2}\left(\frac{n}{3}\right)
\end{aligned}
$$

$$
\begin{aligned}
W_{5,27}(n) = &- 1/108n\sigma(\frac{n}{5}) + 1/24\sigma(\frac{n}{5}) - 1/20n\sigma(\frac{n}{27}) + 1/24\sigma(\frac{n}{27}) + 1/50544\sigma_3(n) \\
&+ 1/6318\sigma_3(\frac{n}{3}) + 25/50544\sigma_3(\frac{n}{5}) + 1/702\sigma_3(\frac{n}{9}) + 25/6318\sigma_3(\frac{n}{15}) \\
&+ 3/208\sigma_3(\frac{n}{27}) + 25/702\sigma_3(\frac{n}{45}) + 75/208\sigma_3(\frac{n}{135}) - 2/5265\tau_{4,5}(n) \\
&- 25/7371\tau_{4,5}\left(\frac{n}{3}\right) - 25/819\tau_{4,5}\left(\frac{n}{9}\right) - 18/65\tau_{4,5}\left(\frac{n}{27}\right) - 2/1215\tau_{4,9}(n) \\
&- 2/135\tau_{4,9}\left(\frac{n}{3}\right) - 10/243\tau_{4,9}\left(\frac{n}{5}\right) \\
&- 10/27\tau_{4,9}\left(\frac{n}{15}\right) + 1/720\tau_{4,15,1}(n) + 1/80\tau_{4,15,1}\left(\frac{n}{3}\right) \\
&+ 9/80\tau_{4,15,1}\left(\frac{n}{9}\right) - 1/324\tau_{4,15,2}(n) - 31/756\tau_{4,15,2}\left(\frac{n}{3}\right) - 1/4\tau_{4,15,2}\left(\frac{n}{9}\right) \\
&- 2/1215\tau_{4,27,1}(n) - 10/243\tau_{4,27,1}\left(\frac{n}{5}\right) + (-1/1836\omega - 7/2754)\tau_{4,27,3}(n) + \\
&(-25/1836\omega - 175/2754)\tau_{4,27,3}\left(\frac{n}{5}\right) + (1/1836\omega - 7/2754)\tau_{4,27,4}(n) \\
&+ (25/1836\omega - 175/2754)\tau_{4,27,4}\left(\frac{n}{5}\right) + 1/1134\tau_{4,45,2}(n) \\
&+ 1/126\tau_{4,45,2}\left(\frac{n}{3}\right) + 1/189\tau_{4,45,3}(n) + 1/21\tau_{4,45,3}\left(\frac{n}{3}\right) - 1/405\tau_{4,45,4}(n) \\
&- 1/45\tau_{4,45,4}\left(\frac{n}{3}\right) - 1/918\tau_{4,135,2}(n) + 1/4565970(32\alpha^2 - 1206\alpha \\
&- 6989)\tau_{4,135,5}(n) + (1/2282985(-16\alpha - 587)\beta \\
&+ 1/4565970(-32\alpha^2 + 32\alpha - 6253))\tau_{4,135,6}(n) + (1/2282985(16\alpha + 587)\beta \\
&+ 1/4565970(1174\alpha - 7427))\tau_{4,135,7}(n) + 1/304398(10\alpha^2 \\
&- 142\alpha + 517)\tau_{4,135,8}(n) + (1/152199(-5\alpha - 66)\beta + 1/304398(-10\alpha^2 + 10\alpha \\
&+ 747))\tau_{4,135,9}(n) + (1/152199(5\alpha + 66)\beta + 1/101466(44\alpha + 205))\tau_{4,135,10}(n) \\
&+ (-1/11961\gamma^2 + 17/35883\gamma + 173/71766)\tau_{4,135,11}(n) \\
&+ ((1/11961\gamma + 2/35883)\delta + (1/11961\gamma^2 - 5/11961\gamma \\
&+ 125/71766))\tau_{4,135,12}(n) + ((-1/11961\gamma - 2/35883)\delta \\
&+ (-2/35883\gamma + 145/71766))\tau_{4,135,13}(n)
\end{aligned}
$$

$$
\begin{aligned}
W_{9,15}(n) = &- 1/60n\sigma(\frac{n}{9}) + 1/24\sigma(\frac{n}{9}) - 1/36n\sigma(\frac{n}{15}) + 1/24\sigma(\frac{n}{15}) + 1/624\sigma_3(\frac{n}{3}) \\
&+ 3/208\sigma_3(\frac{n}{9}) + 25/624\sigma_3(\frac{n}{15}) + 75/208\sigma_3(\frac{n}{45}) - 1/455\tau_{4,5}\left(\frac{n}{3}\right) \\
&- 9/455\tau_{4,5}\left(\frac{n}{9}\right) + 1/80\tau_{4,15,1}\left(\frac{n}{3}\right) - 1/84\tau_{4,15,2}\left(\frac{n}{3}\right)
\end{aligned}
$$

$$\begin{aligned}W_{9,15}\left(n\right)=&-1/60n\sigma(\frac{n}{9})+1/24\sigma(\frac{n}{9})-1/36n\sigma(\frac{n}{15})+1/24\sigma(\frac{n}{15})+1/624\sigma_3(\frac{n}{3})+3/208\sigma_3(\frac{n}{9})\\&+25/624\sigma_3(\frac{n}{15})+75/208\sigma_3(\frac{n}{45})-1/455\tau_{4,5}\left(\frac{n}{3}\right)-9/455\tau_{4,5}\left(\frac{n}{9}\right)\\&+1/80\tau_{4,15,1}\left(\frac{n}{3}\right)-1/84\tau_{4,15,2}\left(\frac{n}{3}\right)\end{aligned}$$

$$\begin{aligned}W_{9,15}\left(n\right)=&-1/60n\sigma(\frac{n}{9})+1/24\sigma(\frac{n}{9})-1/36n\sigma(\frac{n}{15})+1/24\sigma(\frac{n}{15})+1/624\sigma_3(\frac{n}{3})\\&+3/208\sigma_3(\frac{n}{9})+25/624\sigma_3(\frac{n}{15})+75/208\sigma_3(\frac{n}{45})-1/455\tau_{4,5}\left(\frac{n}{3}\right)\\&-9/455\tau_{4,5}\left(\frac{n}{9}\right)+1/80\tau_{4,15,1}\left(\frac{n}{3}\right)-1/84\tau_{4,15,2}\left(\frac{n}{3}\right)\end{aligned}$$

$$\begin{aligned}W_{45}=&-1/180n\sigma(n)+1/24\sigma(n)-1/4n\sigma(\frac{n}{45})+1/24\sigma(\frac{n}{45})+1/5616\sigma_3(n)\\&+1/702\sigma_3(\frac{n}{3})+25/5616\sigma_3(\frac{n}{5})+3/208\sigma_3(\frac{n}{9})+25/702\sigma_3(\frac{n}{15})+75/208\sigma_3(\frac{n}{45})\\&-17/16380\tau_{4,5}\left(n\right)-11/1638\tau_{4,5}\left(\frac{n}{3}\right)-153/1820\tau_{4,5}\left(\frac{n}{9}\right)-1/270\tau_{4,9}\left(n\right)\\&-5/54\tau_{4,9}\left(\frac{n}{5}\right)-1/240\tau_{4,15,1}\left(n\right)-3/80\tau_{4,15,1}\left(\frac{n}{3}\right)-1/126\tau_{4,15,2}\left(n\right)\\&-1/14\tau_{4,15,2}\left(\frac{n}{3}\right)-1/504\tau_{4,45,2}\left(n\right)-1/84\tau_{4,45,3}\left(n\right)-1/180\tau_{4,45,4}\left(n\right)\end{aligned}$$

$$\begin{aligned}W_{3,15}=&-1/60n\sigma(\frac{n}{3})+1/24\sigma(\frac{n}{3})-1/12n\sigma(\frac{n}{15})+1/24\sigma(\frac{n}{15})+5/312\sigma_3(\frac{n}{3})\\&+125/312\sigma_3(\frac{n}{15})-1/130\tau_{4,5}\left(\frac{n}{3}\right)\end{aligned}$$

$$\begin{aligned}W_{5,9}=&-1/36n\sigma(\frac{n}{5})+1/24\sigma(\frac{n}{5})-1/20n\sigma(\frac{n}{9})+1/24\sigma(\frac{n}{9})+1/5616\sigma_3(n)+1/702\sigma_3(\frac{n}{3})\\&+25/5616\sigma_3(\frac{n}{5})+3/208\sigma_3(\frac{n}{9})+25/702\sigma_3(\frac{n}{15})+75/208\sigma_3(\frac{n}{45})-17/16380\tau_{4,5}\left(n\right)\\&-11/1638\tau_{4,5}\left(\frac{n}{3}\right)-153/1820\tau_{4,5}\left(\frac{n}{9}\right)-1/270\tau_{4,9}\left(n\right)-5/54\tau_{4,9}\left(\frac{n}{5}\right)\\&+1/240\tau_{4,15,1}\left(n\right)+3/80\tau_{4,15,1}\left(\frac{n}{3}\right)-1/126\tau_{4,15,2}\left(n\right)-1/14\tau_{4,15,2}\left(\frac{n}{3}\right)\\&+1/504\tau_{4,45,2}\left(n\right)+1/84\tau_{4,45,3}\left(n\right)-1/180\tau_{4,45,4}\left(n\right).\end{aligned}$$

Obviously, the formula for $W_{3,45}\left(n\right)$ and $W_{9,15}\left(n\right)$ can also be obtained by the formula of $W_{15}\left(n\right), W_{3,5}\left(n\right)$ (see [21]) since

$$W_{3,45}\left(n\right)=W_{15}\left(\frac{n}{3}\right), W_{9,15}\left(n\right)=W_{3,5}\left(\frac{n}{3}\right).$$

3 The Quadratic forms of Discriminant -135

Let

$$\begin{gathered}F=x_1^2+x_1x_2+34x_2^2, \Lambda=5x_1^2+5x_1x_2+8x_2^2,\\ \Phi=4x_1^2+3x_1x_2+9x_2^2, \Phi^{-1}=4x_1^2-3x_1x_2+9x_2^2,\\ \Psi=2x_1^2+x_1x_2+17x_2^2, \Psi^{-1}=2x_1^2-x_1x_2+17x_2^2\end{gathered}$$

be the reduced forms of binary quadratic forms of discriminant -135.

Consider the following 21 quadratic forms of 4 variables

$$F_2, \Lambda_2, \Phi_2, \Psi_2, F \oplus \Lambda, F \oplus \Phi, F \oplus \Psi, \Lambda \oplus \Phi, \Lambda \oplus \Psi, \Phi \oplus \Psi,$$

where $F_2 := F \oplus F, \Phi_2 := \Phi \oplus \Phi, \Psi_2 := \Psi \oplus \Psi, \Lambda_2 := \Lambda \oplus \Lambda$.Their theta functions are in $M_2(\Gamma_0(135))$,(see Hecke Theorem 2.1 in [26]) so we can determine the representation numbers by the following Theorem

$$\begin{aligned} r_{F_2}(n) =& 2/9\sigma(n) - 8/9\sigma\left(\frac{n}{3}\right) + 10/9\sigma\left(\frac{n}{5}\right) + 8/3\sigma\left(\frac{n}{9}\right) && (3.1)\\ & - 40/9\sigma\left(\frac{n}{15}\right) - 6\sigma\left(\frac{n}{27}\right) + 40/3\sigma\left(\frac{n}{45}\right) - 30\sigma\left(\frac{n}{135}\right) && (3.2)\\ & + 2/9\tau_{2,15}(n) + 2/9\tau_{2,15}\left(\frac{n}{3}\right) + 2\tau_{2,15}\left(\frac{n}{9}\right) + 8/9\tau_{2,27}(n) + 40/9\tau_{2,27}\left(\frac{n}{5}\right)\\ & + 8/9\tau_{2,135,2}(n) + (16/117\eta + 20/39)\tau_{2,135,3}(n) + (-16/117\eta + 44/117)\tau_{2,135,4}(n)\\ & + (-16/117\eta + 44/117)\tau_{2,135,5}(n) + (16/117\eta + 20/39)\tau_{2,135,6}(n). \end{aligned}$$

$$\begin{aligned} r_{\Lambda_2}(n) =& 2/9\sigma(n) - 8/9\sigma\left(\frac{n}{3}\right) + 10/9\sigma\left(\frac{n}{5}\right) + 8/3\sigma\left(\frac{n}{9}\right)\\ & - 40/9\sigma\left(\frac{n}{15}\right) - 6\sigma\left(\frac{n}{27}\right) + 40/3\sigma\left(\frac{n}{45}\right) - 30\sigma\left(\frac{n}{135}\right) - 2/9\tau_{2,15}(n)\\ & - 2/9\tau_{2,15}\left(\frac{n}{3}\right) - 2\tau_{2,15}\left(\frac{n}{9}\right) + 8/9\tau_{2,27}(n) + 40/9\tau_{2,27}\left(\frac{n}{5}\right)\\ & - 8/9\tau_{2,135}\left(\frac{n}{2}\right) + (-16/117\eta - 20/39)\tau_{2,135,3}(n) + (16/117\eta - 44/117)\tau_{2,135,4}(n)\\ & + (-16/117\eta + 44/117)\tau_{2,135,5}(n) + (16/117\eta + 20/39)\tau_{2,135,6}(n) \end{aligned}$$

$$\begin{aligned} r_{\Phi_2}(n) =& 2/9\sigma(n) - 8/9\sigma\left(\frac{n}{3}\right) + 10/9\sigma\left(\frac{n}{5}\right) + 8/3\sigma\left(\frac{n}{9}\right)\\ & - 40/9\sigma\left(\frac{n}{15}\right) - 6\sigma\left(\frac{n}{27}\right) + 40/3\sigma\left(\frac{n}{45}\right) - 30\sigma\left(\frac{n}{135}\right) + 2/9\tau_{2,15}(n)\\ & + 2/9\tau_{2,15}\left(\frac{n}{3}\right) + 2\tau_{2,15}\left(\frac{n}{9}\right) - 4/9\tau_{2,27}(n) - 20/9\tau_{2,27}\left(\frac{n}{5}\right) - 4/9\tau_{2,135,2}(n)\\ & + (-14/117\eta + 2/39)\tau_{2,135,3}(n) + (14/117\eta + 20/117)\tau_{2,135,4}(n) + (-22/117\eta\\ & + 2/117)\tau_{2,135,5}(n) + (22/117\eta + 8/39)\tau_{2,135,6}(n) \end{aligned}$$

$$\begin{aligned} r_{\psi_2}(n) =& 2/9\sigma(n) - 8/9\sigma\left(\frac{n}{3}\right) + 10/9\sigma\left(\frac{n}{5}\right) + 8/3\sigma\left(\frac{n}{9}\right)\\ & - 40/9\sigma\left(\frac{n}{15}\right) - 6\sigma\left(\frac{n}{27}\right) + 40/3\sigma\left(\frac{n}{45}\right) - 30\sigma\left(\frac{n}{135}\right) - 2/9\tau_{2,15}(n)\\ & - 2/9\tau_{2,15}\left(\frac{n}{3}\right) - 2\tau_{2,15}\left(\frac{n}{9}\right) - 4/9\tau_{2,27}(n) - 20/9\tau_{2,27}\left(\frac{n}{5}\right)\\ & + 4/9\tau_{2,135,2}(n) + (14/117\eta - 2/39)\tau_{2,135,3}(n) + (-14/117\eta - 20/117)\tau_{2,135,4}(n)\\ & + (-22/117\eta + 2/117)\tau_{2,135,5}(n) + (22/117\eta + 8/39)\tau_{2,135,6}(n) \end{aligned}$$

$$\begin{aligned} r_{F\oplus\Lambda}(n) =& 1/6\sigma(n) + 1/3\sigma\left(\frac{n}{3}\right) - 5/6\sigma\left(\frac{n}{5}\right) + \sigma\left(\frac{n}{9}\right)\\ & - 5/3\sigma\left(\frac{n}{15}\right) + 9/2\sigma\left(\frac{n}{27}\right) - 5\sigma\left(\frac{n}{45}\right) - 45/2\sigma\left(\frac{n}{135}\right) + 17/18\tau_{2,15}(n)\\ & - 1/18\tau_{2,15}\left(\frac{n}{3}\right) + 17/2\tau_{2,15}\left(\frac{n}{9}\right) + 4/9\tau_{2,135,2}(n) + (-4/117\eta + 8/39)\tau_{2,135,3}(n)\\ & + (4/117\eta + 28/117)\tau_{2,135,4}(n) \end{aligned}$$

$$\begin{aligned} r_{F\oplus\Phi}(n) =& 2/9\sigma(n) - 8/9\sigma\left(\frac{n}{3}\right) + 10/9\sigma\left(\frac{n}{5}\right) + 8/3\sigma\left(\frac{n}{9}\right) - 40/9\sigma\left(\frac{n}{15}\right) \\ & - 6\sigma\left(\frac{n}{27}\right) + 40/3\sigma\left(\frac{n}{45}\right) - 30\sigma\left(\frac{n}{135}\right) + 2/9\tau_{2,15}(n) \\ & + 2/9\tau_{2,15}\left(\frac{n}{3}\right) + 2\tau_{2,15}\left(\frac{n}{9}\right) + 2/9\tau_{2,27}(n) + 10/9\tau_{2,27}\left(\frac{n}{5}\right) \\ & + 2/9\tau_{2,135,2}(n) + (-17/117\eta + 8/39)\tau_{2,135,3}(n) + (17/117\eta \\ & + 41/117)\tau_{2,135,4}(n) + (-1/117\eta + 32/117)\tau_{2,135,5}(n) \\ & + (1/117\eta + 11/39)\tau_{2,135,6}(n) \end{aligned}$$

$$\begin{aligned} r_{F\oplus\Psi}(n) =& 1/6\sigma(n) + 1/3\sigma\left(\frac{n}{3}\right) - 5/6\sigma\left(\frac{n}{5}\right) + \sigma\left(\frac{n}{9}\right) \\ & - 5/3\sigma\left(\frac{n}{15}\right) + 9/2\sigma\left(\frac{n}{27}\right) - 5\sigma\left(\frac{n}{45}\right) - 45/2\sigma\left(\frac{n}{135}\right) + 5/18\tau_{2,15}(n) \\ & + 23/18\tau_{2,15}\left(\frac{n}{3}\right) + 5/2\tau_{2,15}\left(\frac{n}{9}\right) + 2/3\tau_{2,45}(n) + 2\tau_{2,45}\left(\frac{n}{3}\right) + 4/9\tau_{2,135,2}(n) \\ & + (-1/117\eta + 2/39)\tau_{2,135,3}(n) + (1/117\eta + 7/117)\tau_{2,135,4}(n) + (-1/117\eta \\ & + 2/39)\tau_{2,135,5}(n) + (1/117\eta + 7/117)\tau_{2,135,6}(n) \end{aligned}$$

$$\begin{aligned} r_{\Lambda\oplus\Phi}(n) =& 1/6\sigma(n) + 1/3\sigma\left(\frac{n}{3}\right) - 5/6\sigma\left(\frac{n}{5}\right) + \sigma\left(\frac{n}{9}\right) \\ & - 5/3\sigma\left(\frac{n}{15}\right) + 9/2\sigma\left(\frac{n}{27}\right) - 5\sigma\left(\frac{n}{45}\right) - 45/2\sigma\left(\frac{n}{135}\right) + 5/18\tau_{2,15}(n) \\ & + 23/18\tau_{2,15}\left(\frac{n}{3}\right) + 5/2\tau_{2,15}\left(\frac{n}{9}\right) - 2/3\tau_{2,45}(n) - 2\tau_{2,45}\left(\frac{n}{3}\right) + 4/9\tau_{2,135,2}(n) \\ & + (-1/117\eta + 2/39)\tau_{2,135,3}(n) + (1/117\eta + 7/117)\tau_{2,135,4}(n) + (1/117\eta \\ & - 2/39)\tau_{2,135,5}(n) + (-1/117\eta - 7/117)\tau_{2,135,6}(n) \end{aligned}$$

$$\begin{aligned} r_{\Lambda\oplus\Psi}(n) =& 2/9\sigma(n) - 8/9\sigma\left(\frac{n}{3}\right) + 10/9\sigma\left(\frac{n}{5}\right) + 8/3\sigma\left(\frac{n}{9}\right) \\ & - 40/9\sigma\left(\frac{n}{15}\right) - 6\sigma\left(\frac{n}{27}\right) + 40/3\sigma\left(\frac{n}{45}\right) - 30\sigma\left(\frac{n}{135}\right) - 2/9\tau_{2,15}(n) \\ & - 2/9\tau_{2,15}\left(\frac{n}{3}\right) - 2\tau_{2,15}\left(\frac{n}{9}\right) + 2/9\tau_{2,27}(n) + 10/9\tau_{2,27}\left(\frac{n}{5}\right) - 2/9\tau_{2,135,2}(n) \\ & + (17/117\eta - 8/39)\tau_{2,135,3}(n) + (-17/117\eta - 41/117)\tau_{2,135,4}(n) + (-1/117\eta \\ & + 32/117)\tau_{2,135,5}(n) + (1/117\eta + 11/39)\tau_{2,135,6}(n) \end{aligned}$$

$$\begin{aligned} r_{\Phi\oplus\Psi}(n) =& 1/6\sigma(n) + 1/3\sigma\left(\frac{n}{3}\right) - 5/6\sigma\left(\frac{n}{5}\right) + \sigma\left(\frac{n}{9}\right) \\ & - 5/3\sigma\left(\frac{n}{15}\right) + 9/2\sigma\left(\frac{n}{27}\right) - 5\sigma\left(\frac{n}{45}\right) - 45/2\sigma\left(\frac{n}{135}\right) - 7/18\tau_{2,15}(n) \\ & - 25/18\tau_{2,15}\left(\frac{n}{3}\right) - 7/2\tau_{2,15}\left(\frac{n}{9}\right) + 4/9\tau_{2,135,2}(n) \\ & + (2/117\eta - 4/39)\tau_{2,135,3}(n) + (-2/117\eta - 14/117)\tau_{2,135,4}(n) \end{aligned}$$

Now, consider the following 21 quadratic forms of 8 variables

$$F_4 := F \oplus F \oplus F \oplus F,$$

$$\begin{aligned} & F_4, \Lambda_4, \Phi_4, \Psi_4, F_3 \oplus \Lambda, F_3 \oplus \Phi, F_3 \oplus \Psi, \\ & \Lambda_3 \oplus F, \Lambda_3 \oplus \Phi, \Lambda_3 \oplus \Psi, \Phi_3 \oplus F, \Phi_3 \oplus \Psi, \\ & \Psi_3 \oplus F, \Psi_3 \oplus \Lambda, \Psi_3 \oplus \Phi, F_2 \oplus \Lambda_2, F_2 \oplus \Phi_2 \\ & F_2 \oplus \Psi_2, \Lambda_2 \oplus \Phi_2, \Lambda_2 \oplus \Psi_2, \Phi_2 \oplus \Psi_2. \end{aligned}$$

Their theta functions are in $M_2(\Gamma_0(135))$,(see Hecke Theorem 2.1 in [26]) so again we can determine the representation numbers by the following Theorem

$$\begin{aligned}
r_{F_4}(n) =&240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right)\\
&+ 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right)\\
&+ 10\,800/52\sigma_3\left(\frac{n}{135}\right) - 64/1053\tau_{4,5}(n) - 4000/7371\tau_{4,5}\left(\frac{n}{3}\right) - 4000/819\tau_{4,5}\left(\frac{n}{9}\right)\\
&- 576/13\tau_{4,5}\left(\frac{n}{27}\right) + 64/135\tau_{4,9}(n) + 64/15\tau_{4,9}\left(\frac{n}{3}\right) + 320/27\tau_{4,9}\left(\frac{n}{5}\right)\\
&+ 320/3\tau_{4,9}\left(\frac{n}{15}\right) + 4/9\tau_{4,15,1}(n) + 4\tau_{4,15,1}\left(\frac{n}{3}\right) + 36\tau_{4,15,1}\left(\frac{n}{9}\right)\\
&+ 8/9\tau_{4,15,2}(n) + 248/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 72\tau_{4,15,1}\left(\frac{n}{9}\right) + 64/135\tau_{4,27,1}(n)\\
&+ 320/27\tau_{4,27,1}\left(\frac{n}{5}\right) + (56/459\omega + 8/17)\tau_{4,27,3}(n) + (1400/459\omega + 200/17)\tau_{4,27,3}\left(\frac{n}{5}\right)\\
&+ (-56/459\omega + 8/17)\tau_{4,27,4}(n) + (-1400/459\omega + 200/17)\tau_{4,27,4}\left(\frac{n}{5}\right) + 32/27\tau_{4,45,3}(n)\\
&+ 32/3\tau_{4,45,3}\left(\frac{n}{3}\right) + 128/405\tau_{4,45,4}(n) + 128/45\tau_{4,45,4}\left(\frac{n}{3}\right) + 112/459\tau_{4,135,2}(n)\\
&+ 1/2282985(-35848\alpha^2 + 106184\alpha + 1067376)\tau_{4,135,5}(n) + (1/2282985(35848\alpha + 70336)\beta\\
&+ 1/2282985(35848\alpha^2 - 35848\alpha + 242872))\tau_{4,135,6}(n) + (1/2282985(-35848\alpha - 70336)\beta\\
&+ 1/2282985(-70336\alpha + 313208))\tau_{4,135,7}(n)\,(1/50733(-104\alpha^2 - 4536\alpha\\
&+ 23184))\tau_{4,135,8}(n) + (1/50733(104\alpha - 4640)\beta + 1/50733(104\alpha^2 - 104\alpha\\
&+ 20792))\tau_{4,135,9}(n) + (1/50733(-104\alpha + 4640)\beta + 1/50733(4640\alpha + 16152))\tau_{4,135,10}(n)\\
&+ (-1928/35883\gamma^2 + 6200/35883\gamma + 10064/11961)\tau_{4,135,11}(n) + ((1928/35883\gamma\\
&- 3440/35883)\delta + 1928/35883\gamma^2 - 9640/35883\gamma + 14768/35883)\tau_{4,135,12}(n)\\
&+ ((-1928/35883\gamma + 3440/35883)\delta + 3440/35883\gamma - 2432/35883)\tau_{4,135,13}(n)\,,
\end{aligned}$$

$$
\begin{aligned}
r_{\Lambda_4}(n) =& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right)\\
&+ 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right)\\
&+ 2400/117\sigma_3\left(\frac{n}{45}\right) + 10800/52\sigma_3\left(\frac{n}{135}\right) - 64/1053\tau_{4,5}(n) - 4000/7371\tau_{4,5}\left(\frac{n}{3}\right)\\
&- 4000/819\tau_{4,5}\left(\frac{n}{9}\right) - 576/13\tau_{4,5}\left(\frac{n}{27}\right) + 64/135\tau_{4,9}(n) + 64/15\tau_{4,9}\left(\frac{n}{3}\right)\\
&+ 320/27\tau_{4,9}\left(\frac{n}{5}\right) + 320/3\tau_{4,9}\left(\frac{n}{15}\right) - 4/9\tau_{4,15,1}(n) - 4\tau_{4,15,1}\left(\frac{n}{3}\right)\\
&- 36\tau_{4,15,1}\left(\frac{n}{9}\right) + 8/9\tau_{4,15,2}(n) + 248/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 72\tau_{4,15,1}\left(\frac{n}{9}\right)\\
&+ 64/135\tau_{4,27,1}(n) + 320/27\tau_{4,27,1}\left(\frac{n}{5}\right) + (56/459\omega + 8/17)\tau_{4,27,3}(n) + (1400/459\omega\\
&+ 200/17)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-56/459\omega + 8/17)\tau_{4,27,4}(n) + (-1400/459\omega\\
&+ 200/17)\tau_{4,27,4}\left(\frac{n}{5}\right) - 32/27\tau_{4,45,3}(n) - 32/3\tau_{4,45,3}\left(\frac{n}{3}\right) + 128/405\tau_{4,45,4}(n)\\
&+ 128/45\tau_{4,45,4}\left(\frac{n}{3}\right) + \tau_{4,135,1}(n) + 112/459\tau_{4,135,2}(n) + 1/2282985(-35848\alpha^2\\
&+ 106184\alpha + 1067376)\tau_{4,135,5}(n) + (1/2282985(35848\alpha + 70336)\beta + 1/2282985(35848\alpha^2\\
&- 35848\alpha + 242872))\tau_{4,135,6}(n) + (1/2282985(-35848\alpha\\
&- 70336)\beta + 1/2282985(-70336\alpha\\
&+ 313208))\tau_{4,135,7}(n)\,(1/50733(104\alpha^2 + 4536\alpha - 23184))\tau_{4,135,8}(n)\\
&+ (1/50733(-104\alpha + 4640)\beta + 1/50733(-104\alpha^2 + 104\alpha - 20792))\tau_{4,135,9}(n)\\
&+ (1/50733(104\alpha - 4640)\beta + 1/50733(-4640\alpha - 16152))\tau_{4,135,10}(n)\\
&+ (1928/35883\gamma^2 - 6200/35883\gamma - 10064/11961)\tau_{4,135,11}(n)\,((-1928/35883\gamma\\
&+ 3440/35883)\delta - 1928/35883\gamma^2 + 9640/35883\gamma - 14768/35883)\tau_{4,135,12}(n)\\
&+ ((1928/35883\gamma - 3440/35883)\delta - 3440/35883\gamma + 2432/35883)\tau_{4,135,13}(n)
\end{aligned}
$$

$$
\begin{aligned}
r_{\Phi_4}(n) =& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right)\\
&+ 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right)\\
&+ 10800/52\sigma_3\left(\frac{n}{135}\right) - 64/1053\tau_{4,5}(n) - 4000/7371\tau_{4,5}\left(\frac{n}{3}\right) - 4000/819\tau_{4,5}\left(\frac{n}{9}\right)\\
&- 576/13\tau_{4,5}\left(\frac{n}{27}\right) - 32/135\tau_{4,9}(n) - 32/15\tau_{4,9}\left(\frac{n}{3}\right) - 160/27\tau_{4,9}\left(\frac{n}{5}\right)\\
&- 160/3\tau_{4,9}\left(\frac{n}{15}\right) - 4/3\tau_{4,15,1}\left(\frac{n}{3}\right) + 8/7\tau_{4,15,2}\left(\frac{n}{3}\right) + (28/459\omega + 4/17)\tau_{4,27,3}(n)\\
&+ (700/459\omega + 100/17)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-28/459\omega + 4/17)\tau_{4,27,4}(n) + (-700/459\omega\\
&+ 100/17)\tau_{4,27,4}\left(\frac{n}{5}\right) - 4/21\tau_{4,45,2}(n) - 12/7\tau_{4,45,2}\left(\frac{n}{3}\right)\\
&+ 64/189\tau_{4,45,3}(n) + 64/21\tau_{4,45,3}\left(\frac{n}{3}\right)\\
&+ 56/405\tau_{4,45,4}(n) + 56/45\tau_{4,45,4}\left(\frac{n}{3}\right) - 80/459\tau_{4,135,2}(n) + +(1/456597(7708\alpha^2\\
&+ 280\alpha - 143400))\tau_{4,135,5}(n) + (1/456597(-7708\alpha + 7988)\beta + 1/456597(-7708\alpha\hat{}2 + 7708\alpha\\
&+ 33884))\tau_{4,135,6}(n) + (1/456597(7708\alpha - 7988)\beta + 1/456597(-7988\alpha\\
&+ 41872))\tau_{4,135,7}(n) + (1/152199(1828\alpha^2 - 6416\alpha - 18984))\tau_{4,135,8}(n)\\
&+ (1/152199(-1828\alpha - 4588)\beta + 1/152199(-1828\alpha^2 + 1828\alpha + 23060))\tau_{4,135,9}(n)\\
&+ (1/152199(1828\alpha + 4588)\beta + 1/152199(4588\alpha + 18472))\tau_{4,135,10}(n)\\
&+ (-4/35883\gamma^2 + 1204/35883\gamma - 1832/11961)\tau_{4,135,11}(n) + ((4/35883\gamma + 1184/35883)\delta\\
&+ 4/35883\gamma^2 - 20/35883\gamma - 5528/35883)\tau_{4,135,12}(n)\\
&+ ((-4/35883\gamma - 1184/35883)\delta - 1184/35883\gamma + 392/35883)\tau_{4,135,13}(n)
\end{aligned}
$$

$$
\begin{aligned}
r_{\Psi_4}(n) =& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
&+ 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) \\
&+ 2400/117\sigma_3\left(\frac{n}{45}\right) + 10800/52\sigma_3\left(\frac{n}{135}\right) - 64/1053\tau_{4,5}(n) - 4000/7371\tau_{4,5}\left(\frac{n}{3}\right) \\
&- 4000/819\tau_{4,5}\left(\frac{n}{9}\right) - 576/13\tau_{4,5}\left(\frac{n}{27}\right) - 32/135\tau_{4,9}(n) - 32/15\tau_{4,9}\left(\frac{n}{3}\right) \\
&- 160/27\tau_{4,9}\left(\frac{n}{5}\right) - 160/3\tau_{4,9}\left(\frac{n}{15}\right) + 4/3\tau_{4,15,1}\left(\frac{n}{3}\right) + 8/7\tau_{4,15,2}\left(\frac{n}{3}\right) \\
&+ (28/459\omega + 4/17)\tau_{4,27,3}(n) + (700/459\omega + 100/17)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-28/459\omega \\
&+ 4/17)\tau_{4,27,4}(n) + (-700/459\omega + 100/17)\tau_{4,27,4}\left(\frac{n}{5}\right) + 4/21\tau_{4,45,2}(n) \\
&+ 12/7\tau_{4,45,2}\left(\frac{n}{3}\right) - 64/189\tau_{4,45,3}(n) - 64/21\tau_{4,45,3}\left(\frac{n}{3}\right) \\
&+ 56/405\tau_{4,45,4}(n) + 56/45\tau_{4,45,4}\left(\frac{n}{3}\right) - 80/459\tau_{4,135,2}(n) \\
&+ (1/456597(7708\alpha^2 + 280\alpha - 143400))\tau_{4,135,5}(n) + (1/456597(-7708\alpha + 7988)\beta \\
&+ 1/456597(-7708\alpha^2 + 7708\alpha + 33884))\tau_{4,135,6}(n) \\
&+ (1/456597(7708\alpha - 7988)\beta + 1/456597(-7988\alpha + 41872))\tau_{4,135,7}(n) \\
&+ (1/152199(-1828\alpha^2 + 6416\alpha + 18984))\tau_{4,135,8}(n) \\
&+ (1/152199(1828\alpha + 4588)\beta + 1/152199(1828\alpha^2 \\
&- 1828\alpha - 23060))\tau_{4,135,9}(n) + (1/152199(-1828\alpha \\
&- 4588)\beta + 1/152199(-4588\alpha - 18472))\tau_{4,135,10}(n) \\
&+ (4/35883\gamma^2 - 1204/35883\gamma + 1832/11961)\tau_{4,135,11}(n) \\
&+ ((-4/35883\gamma - 1184/35883)\delta - 4/35883\gamma^2 + 20/35883\gamma + 5528/35883)\tau_{4,135,12}(n) \\
&+ ((4/35883\gamma + 1184/35883)\delta + 1184/35883\gamma - 392/35883)\tau_{4,135,13}(n)
\end{aligned}
$$

$$\begin{aligned}
r_{F_3\oplus\Lambda_1}(n) =& 240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) - 6000/15552\sigma_3\left(\frac{n}{5}\right) \\
&+ 1200/864\sigma_3\left(\frac{n}{9}\right) + 30\,000/7776\sigma_3\left(\frac{n}{15}\right) - 720/64\sigma_3\left(\frac{n}{27}\right) - 30000/864\sigma_3\left(\frac{n}{45}\right) \\
&+ 18\,000/64\sigma_3\left(\frac{n}{135}\right) - 8/567\tau_{4,5}(n) - 164/567\tau_{4,5}\left(\frac{n}{3}\right) - 164/63\tau_{4,5}\left(\frac{n}{9}\right) \\
&- 72/7\tau_{4,5}\left(\frac{n}{27}\right) + 32/405\tau_{4,9}(n) + 88/45\tau_{4,9}\left(\frac{n}{3}\right) + 440/81\tau_{4,9}\left(\frac{n}{5}\right) \\
&+ 160/9\tau_{4,9}\left(\frac{n}{15}\right) + 19/36\tau_{4,15,1}(n) - 17/12\tau_{4,15,1}\left(\frac{n}{3}\right) + 171/4\tau_{4,15,1}\left(\frac{n}{9}\right) \\
&+ 2/7\tau_{4,15,2}(n) + 62/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 162/7\tau_{4,15,2}\left(\frac{n}{9}\right) + 4/15\tau_{4,27,1}(n) \\
&+ 20/3\tau_{4,27,1}\left(\frac{n}{5}\right) + 68/405\tau_{4,27,2}(n) - 340/81\tau_{4,27,2}\left(\frac{n}{5}\right) + (106/1377\omega \\
&+ 112/459)\tau_{4,27,3}(n) + (2650/1377\omega + 2800/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-106/1377\omega \\
&+ 112/459)\tau_{4,27,4}(n) + (-2650/1377\omega + 2000/459)\tau_{4,27,4}\left(\frac{n}{5}\right) + 4/9\tau_{4,45,3}(n) \\
&+ 4\tau_{4,45,3}\left(\frac{n}{3}\right) + 8/81\tau_{4,45,4}(n) + 8/9\tau_{4,45,4}\left(\frac{n}{3}\right) + 16/135\tau_{4,45,5}(n) \\
&- 16/15\tau_{4,45,5}\left(\frac{n}{3}\right) 116/459\tau_{4,135,2}(n) + (1/2282985(-41252\alpha^2 \\
&+ 86716\alpha + 978144))\tau_{4,135,5}(n) + (1/2282985(41252\alpha + 45464)\beta \\
&+ 1/2282985(41252\alpha^2 - 41252\alpha + 29348))\tau_{4,135,6}(n) + (1/2282985(-41252\alpha - 45464)\beta \\
&+ 1/2282985(-45464\alpha + 74812))\tau_{4,135,7}(n) + (1/760995(-15844\alpha^2 \\
&- 44088\alpha + 596128))\tau_{4,135,8}(n) + (1/760995(15844\alpha - 59932)\beta + 1/760995(15844\alpha^2 \\
&- 15844\alpha + 231716))\tau_{4,135,9}(n) + (1/760995(-15844\alpha + 59932)\beta \\
&+ 1/760995(59932\alpha + 171784))\tau_{4,135,10}(n) + (-1262/35883\gamma^2 + 2426/35883\gamma \\
&+ 3436/3987)\tau_{4,135,11}(n) + ((1262/35883\gamma - 3884/35883)\delta + 1262/35883\gamma^2 \\
&- 6310/35883\gamma + 20828/35883)\tau_{4,135,12}(n) + ((-1262/35883\gamma \\
&+ 3884/35883)\delta + 3884/35883\gamma + 1408/35883)\tau_{4,135,13}(n)
\end{aligned}$$

$$
\begin{aligned}
r_{F_3\oplus\Phi_1}(n) =& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right)\\
&+ 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right)\\
&+ 10800/52\sigma_3\left(\frac{n}{135}\right) - 64/1053\tau_{4,5}(n) - 4000/7371\tau_{4,5}\left(\frac{n}{3}\right) - 4000/819\tau_{4,5}\left(\frac{n}{9}\right)\\
&- 576/13\tau_{4,5}\left(\frac{n}{27}\right) + 8/27\tau_{4,9}(n) + 8/3\tau_{4,9}\left(\frac{n}{3}\right) + 200/27\tau_{4,9}\left(\frac{n}{5}\right)\\
&+ 200/3\tau_{4,9}\left(\frac{n}{15}\right) + 1/2\tau_{4,15,1}(n) - 5/3\tau_{4,15,1}\left(\frac{n}{3}\right) + 81/2\tau_{4,15,1}\left(\frac{n}{9}\right)\\
&+ 1/2\tau_{4,15,2}(n) + 29/7\tau_{4,15,2}\left(\frac{n}{3}\right) + 81/2\tau_{4,15,1}\left(\frac{n}{9}\right) + 16/45\tau_{4,27,1}(n)\\
&+ 80/9\tau_{4,27,1}\left(\frac{n}{5}\right) + (2/27\omega + 1/3)\tau_{4,27,3}(n) + (50/27\omega + 25/3)\tau_{4,27,3}\left(\frac{n}{5}\right)\\
&+ (-2/27\omega + 1/3)\tau_{4,27,4}(n) + (-50/27\omega + 25/3)\tau_{4,27,4}\left(\frac{n}{5}\right) + 5/18\tau_{4,45,1}(n)\\
&- 5/2\tau_{4,45,1}\left(\frac{n}{3}\right) + 1/42\tau_{4,45,2}(n) + 3/14\tau_{4,45,2}\left(\frac{n}{3}\right) + 104/189\tau_{4,45,3}(n)\\
&+ 104/21\tau_{4,45,3}\left(\frac{n}{3}\right) + 13/81\tau_{4,45,4}(n) + 13/9\tau_{4,45,4}\left(\frac{n}{3}\right) + 1/9\tau_{4,45,5}(n)\\
&- \tau_{4,45,5}\left(\frac{n}{3}\right) + 8/27\tau_{4,135,2}(n) + (1/2282985(-30316\alpha^2\\
&+ 73853)\alpha + 828012)\tau_{4,135,5}(n) + (1/2282985(30316\alpha + 43537)\beta + 1/2282985(30316\alpha^2\\
&- 30316\alpha + 130744))\tau_{4,135,6}(n) + (1/2282985(-30316\alpha - 43537)\beta + 1/2282985(-43537\alpha\\
&+ 174281))\tau_{4,135,7}(n) + (1/50733(-500\alpha^2 - 2295\alpha + 23004))\tau_{4,135,8}(n) + (1/50733(500\alpha\\
&- 2795)\beta + 1/50733(500\alpha^2 - 500\alpha + 11504))\tau_{4,135,9}(n) + (1/50733(-500\alpha + 2795)\beta\\
&+ (1/50733(2795\alpha + 8709))\tau_{4,135,10}(n) + (-665/35883\gamma\hat{}2 + 815/35883\gamma\\
&+ 6416/11961)\tau_{4,135,11}(n) + ((665/35883\gamma - 2510/35883)\delta + 665/35883\gamma^2 - 3325/35883\gamma\\
&+ 13928/35883)\tau_{4,135,12}(n) + ((-665/35883\gamma + 2510/35883)\delta\\
&+ 2510/35883\gamma + 1378/35883)\tau_{4,135,13}(n)
\end{aligned}
$$

$$
\begin{aligned}
r_{F_3\oplus\Psi_1}(n) =& 240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) - 6000/15552\sigma_3\left(\frac{n}{5}\right)\\
&+ 1200/864\sigma_3\left(\frac{n}{9}\right) + 30\,000/7776\sigma_3\left(\frac{n}{15}\right) - 720/64\sigma_3\left(\frac{n}{27}\right)\\
&- 30000/864\sigma_3\left(\frac{n}{45}\right) + 18000/64\sigma_3\left(\frac{n}{135}\right) + 64/567\tau_{4,5}(n) + 340/567\tau_{4,5}\left(\frac{n}{3}\right)\\
&+ 340/63\tau_{4,5}\left(\frac{n}{9}\right) + 576/7\tau_{4,5}\left(\frac{n}{27}\right) + 32/405\tau_{4,9}(n) + 32/9\tau_{4,9}\left(\frac{n}{3}\right)\\
&+ 800/81\tau_{4,9}\left(\frac{n}{5}\right) + 160/9\tau_{4,9}\left(\frac{n}{15}\right) + 7/36\tau_{4,15,1}(n) + 7/12\tau_{4,15,1}\left(\frac{n}{3}\right)\\
&+ 63/4\tau_{4,15,1}\left(\frac{n}{9}\right) + 17/63\tau_{4,15,2}(n) + 100/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 153/7\tau_{4,15,2}\left(\frac{n}{9}\right)\\
&+ 32/135\tau_{4,27,1}(n) + 160/27\tau_{4,27,1}\left(\frac{n}{5}\right) - 16/405\tau_{4,27,2}(n) + 80/81\tau_{4,27,2}\left(\frac{n}{5}\right)\\
&+ (124/1377\omega + 208/459)\tau_{4,27,3}(n) + (3100/1377\omega + 5200/459)\tau_{4,27,3}\left(\frac{n}{5}\right)\\
&+ (-124/1377\omega + 208/459)\tau_{4,27,4}(n) + (-3100/1377\omega + 5200/459)\tau_{4,27,4}\left(\frac{n}{5}\right)\\
&- 1/9\tau_{4,45,1}(n) + \tau_{4,45,1}\left(\frac{n}{3}\right) + 13/63\tau_{4,45,2}(n) + 13/7\tau_{4,45,2}\left(\frac{n}{3}\right)\\
&+ 4/7\tau_{4,45,3}(n) + 36/7\tau_{4,45,3}\left(\frac{n}{3}\right) + 94/405\tau_{4,45,4}(n)\\
&+ 94/45\tau_{4,45,4}\left(\frac{n}{3}\right) - 2/135\tau_{4,45,5}(n) + 2/15\tau_{4,45,5}\left(\frac{n}{3}\right)\\
&+ 128/459\tau_{4,135,2}(n) + (1/2282985(-30674\alpha^2 + 93217\alpha + 1086468))\tau_{4,135,5}(n)\\
&+ (1/2282985(30674\alpha + 62543)\beta + 1/2282985(30674\alpha^2 - 30674\alpha + 380966))\tau_{4,135,6}(n)\\
&+ (1/2282985(-30674\alpha - 62543)\beta + 1/2282985(-62543\alpha + 443509))\tau_{4,135,7}(n)\\
&+ (1/760995(-1322\alpha^2 - 49999\alpha + 338644))\tau_{4,135,8}(n) + (1/760995(1322\alpha - 51321)\beta\\
&+ 1/760995(1322\alpha^2 - 1322\alpha + 308238))\tau_{4,135,9}(n) + (1/760995(-1322\alpha + 51321)\beta\\
&+ 1/253665(17107\alpha + 85639))\tau_{4,135,10}(n) + (-518/35883\gamma^2 + 1754/35883\gamma\\
&+ 1840/3987)\tau_{4,135,11}(n) + ((518/35883\gamma - 836/35883)\delta + 518/35883\gamma^2\\
&- 2590/35883\gamma + 12416/35883)\tau_{4,135,12}(n) + ((-518/35883\gamma\\
&+ 836/35883)\delta + 836/35883\gamma + 8236/35883)\tau_{4,135,13}(n)
\end{aligned}
$$

$$
\begin{aligned}
r_{\Lambda_3\oplus F_1}(n) =& 240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) - 6000/15552\sigma_3\left(\frac{n}{5}\right)\\
&+ 1200/864\sigma_3\left(\frac{n}{9}\right) + 30000/7776\sigma_3\left(\frac{n}{15}\right) - 720/64\sigma_3\left(\frac{n}{27}\right)\\
&- 30000/864\sigma_3\left(\frac{n}{45}\right) + 18000/64\sigma_3\left(\frac{n}{135}\right) + 8/567\tau_{4,5}(n) + 164/567\tau_{4,5}\left(\frac{n}{3}\right)\\
&+ 164/63\tau_{4,5}\left(\frac{n}{9}\right) + 72/7\tau_{4,5}\left(\frac{n}{27}\right) - 88/405\tau_{4,9}(n) - 32/45\tau_{4,9}\left(\frac{n}{3}\right)\\
&- 160/81\tau_{4,9}\left(\frac{n}{5}\right) - 440/9\tau_{4,9}\left(\frac{n}{15}\right) + 19/36\tau_{4,15,1}(n) - 17/12\tau_{4,15,1}\left(\frac{n}{3}\right)\\
&+ 171/4\tau_{4,15,1}\left(\frac{n}{9}\right) - 2/7\tau_{4,15,2}(n) - 62/21\tau_{4,15,2}\left(\frac{n}{3}\right) - 162/7\tau_{4,15,2}\left(\frac{n}{9}\right)\\
&- 4/15\tau_{4,27,1}(n) - 20/3\tau_{4,27,1}\left(\frac{n}{5}\right) + 68/405\tau_{4,27,2}(n) - 340/81\tau_{4,27,2}\left(\frac{n}{5}\right)\\
&+ (-106/1377\omega - 112/459)\tau_{4,27,3}(n) + (-2650/1377\omega - 2800/459)\tau_{4,27,3}\left(\frac{n}{5}\right)\\
&+ (106/1377\omega - 112/459)\tau_{4,27,4}(n) + (2650/1377\omega - 2800/459)\tau_{4,27,4}\left(\frac{n}{5}\right)\\
&+ 4/9\tau_{4,45,3}(n) + 4\tau_{4,45,3}\left(\frac{n}{3}\right) - 8/81\tau_{4,45,4}(n) - 8/9\tau_{4,45,4}\left(\frac{n}{3}\right)\\
&+ 16/135\tau_{4,45,5}(n) - 16/15\tau_{4,45,5}\left(\frac{n}{3}\right) - 116/459\tau_{4,135,2}(n)\\
&+ (1/2282985(41252\alpha^2 - 86716\alpha - 978144))\tau_{4,135,5}(n) + (1/2282985(-41252\alpha - 45464)\beta\\
&+ 1/2282985(-41252\alpha^2 + 41252\alpha - 29348))\tau_{4,135,6}(n) + (1/2282985(41252\alpha\\
&+ 45464)\beta + 1/2282985(45464\alpha - 74812))\tau_{4,135,7}(n) + (1/760995(-15844\alpha^2 - 44088\alpha\\
&+ 596128))\tau_{4,135,8}(n) + (1/760995(15844\alpha - 59932)\beta + 1/760995(15844\alpha^2\\
&- 15844\alpha + 231716))\tau_{4,135,9}(n) + (1/760995(-15844\alpha\\
&+ 59932)\beta + 1/760995(59932\alpha + 171784))\tau_{4,135,10}(n)\\
&+ (-1262/35883\gamma^2 + 2426/35883\gamma + 3436/3987)\tau_{4,135,11}(n)\\
&((1262/35883\gamma - 3884/35883)\delta + 1262/35883\gamma^2 - 6310/35883\gamma\\
&+ 20828/35883)\tau_{4,135,12}(n) + ((-1262/35883\gamma + 3884/35883)\delta\\
&+ 3884/35883\gamma + 1408/35883)\tau_{4,135,13}(n)
\end{aligned}
$$

$$\begin{aligned}
r_{\Lambda_3\oplus\Phi_1}(n) =& 240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) - 6000/15552\sigma_3\left(\frac{n}{5}\right) \\
&+ 1200/864\sigma_3\left(\frac{n}{9}\right) + 30000/7776\sigma_3\left(\frac{n}{15}\right) - 720/64\sigma_3\left(\frac{n}{27}\right) - 30000/864\sigma_3\left(\frac{n}{45}\right) \\
&+ 18000/64\sigma_3\left(\frac{n}{135}\right) - 64/567\tau_{4,5}(n) - 340/567\tau_{4,5}\left(\frac{n}{3}\right) - 340/63\tau_{4,5}\left(\frac{n}{9}\right) \\
&- 576/7\tau_{4,5}\left(\frac{n}{27}\right) - 32/81\tau_{4,9}(n) - 32/45\tau_{4,9}\left(\frac{n}{3}\right) - 160/81\tau_{4,9}\left(\frac{n}{5}\right) \\
&- 800/9\tau_{4,9}\left(\frac{n}{15}\right) + 7/36\tau_{4,15,1}(n) + 7/12\tau_{4,15,1}\left(\frac{n}{3}\right) + 63/4\tau_{4,15,1}\left(\frac{n}{9}\right) \\
&- 17/63\tau_{4,15,2}(n) - 100/21\tau_{4,15,2}\left(\frac{n}{3}\right) - 153/7\tau_{4,15,2}\left(\frac{n}{9}\right) - 32/135\tau_{4,27,1}(n) \\
&- 160/27\tau_{4,27,1}\left(\frac{n}{5}\right) - 16/405\tau_{4,27,2}(n) + 80/81\tau_{4,27,2}\left(\frac{n}{5}\right) + (-124/1377\omega \\
&- 208/459)\tau_{4,27,3}(n) + (-3100/1377\omega - 5200/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (124/1377\omega \\
&- 208/459)\tau_{4,27,4}(n) + (3100/1377\omega - 5200/459)\tau_{4,27,4}\left(\frac{n}{5}\right) + 1/9\tau_{4,45,1}(n) \\
&- \tau_{4,45,1}\left(\frac{n}{3}\right) + 13/63\tau_{4,45,2}(n) + 13/7\tau_{4,45,2}\left(\frac{n}{3}\right) + 4/7\tau_{4,45,3}(n) \\
&+ 36/7\tau_{4,45,3}\left(\frac{n}{3}\right) - 94/405\tau_{4,45,4}(n) - 94/45\tau_{4,45,4}\left(\frac{n}{3}\right) \\
&- 2/135\tau_{4,45,5}(n) + 2/15\tau_{4,45,5}\left(\frac{n}{3}\right) - 128/459\tau_{4,135,2}(n) \\
&+ (1/2282985(30674\alpha^2 - 93217\alpha - 1086468))\tau_{4,135,5}(n) + (1/2282985(-30674\alpha - 62543)\beta \\
&+ 1/2282985(-30674\alpha^2 + 30674\alpha - 380966))\tau_{4,135,6}(n) + (1/2282985(30674\alpha + 62543)\beta \\
&+ 1/2282985(62543\alpha - 443509))\tau_{4,135,7}(n) + (1/760995(-1322\alpha^2 - 49999\alpha \\
&+ 338644))\tau_{4,135,8}(n) + (1/760995(1322\alpha - 51321)\beta + 1/760995(1322\alpha^2 \\
&- 1322\alpha + 308238))\tau_{4,135,9}(n) + (1/760995(-1322\alpha + 51321)\beta + 1/253665(17107\alpha \\
&+ 85639))\tau_{4,135,10}(n) + (-518/35883\gamma^2 + 1754/35883\gamma + 1840/3987)\tau_{4,135,11}(n) \\
&+ ((518/35883\gamma - 836/35883)\delta + 518/35883\gamma^2 - 2590/35883\gamma \\
&+ 12416/35883)\tau_{4,135,12}(n) + ((-518/35883\gamma + 836/35883)\delta + 836/35883\gamma \\
&+ 8236/35883)\tau_{4,135,13}(n)
\end{aligned}$$

$$
\begin{aligned}
r_{\Lambda_3\oplus\Psi_1}(n) =& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right)\\
&+ 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right)\\
&+ 10800/52\sigma_3\left(\frac{n}{135}\right) - 64/1053\tau_{4,5}(n) - 4000/7371\tau_{4,5}\left(\frac{n}{3}\right) - 4000/819\tau_{4,5}\left(\frac{n}{9}\right)\\
&- 576/13\tau_{4,5}\left(\frac{n}{27}\right) + 8/27\tau_{4,9}(n) + 8/3\tau_{4,9}\left(\frac{n}{3}\right) + 200/27\tau_{4,9}\left(\frac{n}{5}\right)\\
&+ 200/3\tau_{4,9}\left(\frac{n}{15}\right) - 1/2\tau_{4,15,1}(n) + 5/3\tau_{4,15,1}\left(\frac{n}{3}\right) - 81/2\tau_{4,15,1}\left(\frac{n}{9}\right)\\
&+ 1/2\tau_{4,15,2}(n) + 29/7\tau_{4,15,2}\left(\frac{n}{3}\right) + 81/2\tau_{4,15,1}\left(\frac{n}{9}\right) + 16/45\tau_{4,27,1}(n)\\
&+ 80/9\tau_{4,27,1}\left(\frac{n}{5}\right) + (2/27\omega + 1/3)\tau_{4,27,3}(n) + (50/27\omega + 25/3)\tau_{4,27,3}\left(\frac{n}{5}\right)\\
&+ (-2/27\omega + 1/3)\tau_{4,27,4}(n) + (-50/27\omega + 25/3)\tau_{4,27,4}\left(\frac{n}{5}\right) + 5/18\tau_{4,45,1}(n)\\
&- 5/2\tau_{4,45,1}\left(\frac{n}{3}\right) - 1/42\tau_{4,45,2}(n) - 3/14\tau_{4,45,2}\left(\frac{n}{3}\right) - 104/189\tau_{4,45,3}(n)\\
&- 104/21\tau_{4,45,3}\left(\frac{n}{3}\right) + 13/81\tau_{4,45,4}(n) + 13/9\tau_{4,45,4}\left(\frac{n}{3}\right) - 1/9\tau_{4,45,5}(n)\\
&+ \tau_{4,45,5}\left(\frac{n}{3}\right) + 8/27\tau_{4,135,2}(n) - 13928/35883)\tau_{4,135,12}(n)\\
&+ ((665/35883\gamma - 2510/35883)\delta + (1/2282985(-30316\alpha^2 + 73853\alpha\\
&+ 828012))\tau_{4,135,5}(n) + (1/2282985(30316\alpha + 43537)\beta + 1/2282985(30316\alpha^2 - 30316\alpha\\
&+ 130744))\tau_{4,135,6}(n) + (1/2282985(-30316\alpha - 43537)\beta\\
&+ 1/2282985(-43537\alpha + 174281))\tau_{4,135,7}(n) + (1/50733(500\alpha^2 + 2295\alpha\\
&- 23004))\tau_{4,135,8}(n) + (1/50733(-500\alpha + 2795)\beta + 1/50733(-500\alpha^2 2 + 500\alpha\\
&- 11504))\tau_{4,135,9}(n) + (1/50733(500\alpha - 2795)\beta + 1/50733(-2795\alpha - 8709))\tau_{4,135,10}(n)\\
&+ (665/35883\gamma^2 - 815/35883\gamma - 6416/11961)\tau_{4,135,11}(n) + ((-665/35883\gamma\\
&+ 2510/35883)\delta - 665/35883\gamma^2 + 3325/35883\gamma - 2510/35883\gamma - 1378/35883)\tau_{4,135,13}
\end{aligned}
$$

$$\begin{aligned}
r_{\Phi_3\oplus F_1}(n) =&240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right)\\
&+ 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right)\\
&+ 10800/52\sigma_3\left(\frac{n}{135}\right) + 488/7371\tau_{4,5}(n) + 2552/7371\tau_{4,5}\left(\frac{n}{3}\right) + 2552/819\tau_{4,5}\left(\frac{n}{9}\right)\\
&+ 4392/91\tau_{4,5}\left(\frac{n}{27}\right) - 8/135\tau_{4,9}(n) - 8/15\tau_{4,9}\left(\frac{n}{3}\right) - 40/27\tau_{4,9}\left(\frac{n}{5}\right)\\
&- 40/3\tau_{4,9}\left(\frac{n}{15}\right) + 1/18\tau_{4,15,1}(n) - 1/3\tau_{4,15,1}\left(\frac{n}{3}\right) + 9/2\tau_{4,15,1}\left(\frac{n}{9}\right)\\
&+ 19/26\tau_{4,15,2}(n) - 1/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 171/14\tau_{4,15,1}\left(\frac{n}{9}\right) + 16/135\tau_{4,27,1}(n)\\
&+ 80/27\tau_{4,27,1}\left(\frac{n}{5}\right) + (20/459\omega + 11/51)\tau_{4,27,3}(n) + (500/459\omega + 275/51)\tau_{4,27,3}\left(\frac{n}{5}\right)\\
&+ (-20/459\omega + 11/51)\tau_{4,27,4}(n) + (-500/459\omega + 275/51)\tau_{4,27,4}\left(\frac{n}{5}\right) + 1/18\tau_{4,45,1}(n)\\
&- 1/2\tau_{4,45,1}\left(\frac{n}{3}\right) + 3/14\tau_{4,45,2}(n) + 27/14\tau_{4,45,2}\left(\frac{n}{3}\right) - 16/189\tau_{4,45,3}(n)\\
&- 16/21\tau_{4,45,3}\left(\frac{n}{3}\right) + 29/405\tau_{4,45,4}(n) + 29/45\tau_{4,45,4}\left(\frac{n}{3}\right) + 1/9\tau_{4,45,5}(n)\\
&- \tau_{4,45,5}\left(\frac{n}{3}\right) + 40/459\tau_{4,135,2}(n) + (1/2282985(15278\alpha^2 + 14921\alpha - 193596))\tau_{4,135,5}(n)\\
&+ (1/2282985(-15278\alpha + 30199)\beta + 1/2282985(-15278\alpha^2 + 15278\alpha + 157798))\tau_{4,135,6}(n)\\
&+ (1/2282985(15278\alpha - 30199)\beta + 1/2282985(-30199\alpha + 187997))\tau_{4,135,7}(n)\\
&+ (1/50733(-566\alpha^2 + 897\alpha + 11700))\tau_{4,135,8}(n) + (1/50733(566\alpha + 331)\beta\\
&+ 1/50733(566\alpha^2 - 566\alpha - 1318))\tau_{4,135,9}(n) + (1/50733(-566\alpha - 331)\beta\\
&+ 1/50733(-331\alpha - 987))\tau_{4,135,10}(n) + (301/35883\gamma^2 - 2887/35883\gamma\\
&+ 2300/11961)\tau_{4,135,11}(n) + ((-301/35883\gamma - 1382/35883)\delta - 301/35883\gamma^2\\
&+ 1505/35883\gamma + 9308/35883)\tau_{4,135,12}(n) + (301/35883\gamma + 1382/35883)\delta\\
&+ 1382/35883\gamma + 2398/35883)\tau_{4,135,13}
\end{aligned}$$

$$
\begin{aligned}
r_{\Phi_3\oplus\Lambda_1}(n) =& 240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) - 6000/15552\sigma_3\left(\frac{n}{5}\right) \\
& + 1200/864\sigma_3\left(\frac{n}{9}\right) + 30000/7776\sigma_3\left(\frac{n}{15}\right) - 720/64\sigma_3\left(\frac{n}{27}\right) - 30000/864\sigma_3\left(\frac{n}{45}\right) \\
& + 18000/64\sigma_3\left(\frac{n}{135}\right) - 8/567\tau_{4,5}(n) - 164/567\tau_{4,5}\left(\frac{n}{3}\right) - 164/63\tau_{4,5}\left(\frac{n}{9}\right) \\
& - 72/7\tau_{4,5}\left(\frac{n}{27}\right) - 112/405\tau_{4,9}(n) + 16/45\tau_{4,9}\left(\frac{n}{3}\right) + 80/81\tau_{4,9}\left(\frac{n}{5}\right) \\
& - 560/9\tau_{4,9}\left(\frac{n}{15}\right) + 1/12\tau_{4,15,1}(n) - 1/12\tau_{4,15,1}\left(\frac{n}{3}\right) + 27/4\tau_{4,15,1}\left(\frac{n}{9}\right) \\
& - 1/21\tau_{4,15,2}(n) - 64/21\tau_{4,15,2}\left(\frac{n}{3}\right) - 27/7\tau_{4,15,2}\left(\frac{n}{9}\right) - 16/135\tau_{4,27,1}(n) \\
& - 80/27\tau_{4,27,1}\left(\frac{n}{5}\right) - 16/405\tau_{4,27,2}(n) + 80/81\tau_{4,27,2}\left(\frac{n}{5}\right) + (-62/1377\omega \\
& - 104/459)\tau_{4,27,3}(n) + (-1550/1377\omega - 2600/459)\tau_{4,27,3}\left(\frac{n}{5}\right) \\
& + (162/1377\omega - 104/459)\tau_{4,27,4}(n) + (1550/1377\omega \\
& - 2600/459)\tau_{4,27,4}\left(\frac{n}{5}\right) + 1/9\tau_{4,45,1}(n) - \tau_{4,45,1}\left(\frac{n}{3}\right) + 8/63\tau_{4,45,2}(n) \\
& + 8/7\tau_{4,45,2}\left(\frac{n}{3}\right) + 20/63\tau_{4,45,3}(n) + 20/7\tau_{4,45,3}\left(\frac{n}{3}\right) - 32/405\tau_{4,45,4}(n) \\
& - 32/45\tau_{4,45,4}\left(\frac{n}{3}\right) - 2/135\tau_{4,45,5}(n) + 2/15\tau_{4,45,5}\left(\frac{n}{3}\right) - 64/459\tau_{4,135,2}(n) \\
& + (1/2282985(15712\alpha^2 - 37841\alpha - 340644))\tau_{4,135,5}(n) + (1/2282985(-15712\alpha - 22129)\beta \\
& + 1/2282985(-15712\alpha^2 + 15712\alpha + 20732))\tau_{4,135,6}(n) + (1/2282985(15712\alpha \\
& + 22129)\beta + 1/2282985(22129\alpha - 1397))\tau_{4,135,7}(n) + (1/760995(2668\alpha^2 - 27739\alpha \\
& + 99604))\tau_{4,135,8}(n) + (1/760995(-2668\alpha - 25071)\beta + 1/760995(-2668\alpha^2 + 2668\alpha \\
& + 160968))\tau_{4,135,9}(n) + (1/760995(2668\alpha + 25071)\beta + 1/253665(8357\alpha \\
& + 45299))\tau_{4,135,10}(n) + (178/35883\gamma^2 - 418/35883\gamma + 4/3987)\tau_{4,135,11}(n) \\
& + (((-178/35883\gamma + 472/35883)\delta - 178/35883\gamma^2 + 890/35883\gamma + 1460/35883)\tau_{4,135,12}(n) \\
& + ((178/35883\gamma - 472/35883)\delta - 472/35883\gamma + 3820/35883)\tau_{4,135,13}
\end{aligned}
$$

$$
\begin{aligned}
r_{\Phi_3\oplus\Psi_1}(n) =& 240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) - 6000/15552\sigma_3\left(\frac{n}{5}\right) \\
&+ 1200/864\sigma_3\left(\frac{n}{9}\right) + 30\,000/7776\sigma_3\left(\frac{n}{15}\right) - 720/64\sigma_3\left(\frac{n}{27}\right) - 30000/864\sigma_3\left(\frac{n}{45}\right) \\
&+ 18\,000/64\sigma_3\left(\frac{n}{135}\right) + 4/81\tau_{4,5}(n) + 88/567\tau_{4,5}\left(\frac{n}{3}\right) + 88/63\tau_{4,5}\left(\frac{n}{9}\right) \\
&+ 36\tau_{4,5}\left(\frac{n}{27}\right) + 32/405\tau_{4,9}(n) + 16/45\tau_{4,9}\left(\frac{n}{3}\right) + 80/81\tau_{4,9}\left(\frac{n}{5}\right) \\
&+ 160/9\tau_{4,9}\left(\frac{n}{15}\right) - 1/36\tau_{4,15,1}(n) - 3/4\tau_{4,15,1}\left(\frac{n}{3}\right) - 9/4\tau_{4,15,1}\left(\frac{n}{9}\right) \\
&- 10/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 8/405\tau_{4,27,2}(n) - 40/81\tau_{4,27,2}\left(\frac{n}{5}\right) + (-38/1377\omega \\
&- 44/459)\tau_{4,27,3}(n) + (-950/1377\omega - 1100/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (38/1377\omega \\
&- 44/459)\tau_{4,27,4}(n) + (950/1377\omega - 1100/459)\tau_{4,27,4}\left(\frac{n}{5}\right) - 1/7\tau_{4,45,2}(n) \\
&- 9/7\tau_{4,45,2}\left(\frac{n}{3}\right) + 16/63\tau_{4,45,3}(n) + 16/7\tau_{4,45,3}\left(\frac{n}{3}\right) - 14/405\tau_{4,45,4}(n) \\
&- 14/45\tau_{4,45,4}\left(\frac{n}{3}\right) + 4/135\tau_{4,45,5}(n) - 4/15\tau_{4,45,5}\left(\frac{n}{3}\right) + 20/459\tau_{4,135,2}(n) \\
&+ (1/456597(-1510\alpha^2 - 4864\alpha + 36552))\tau_{4,135,5}(n) \\
&+ (1/456597(1510\alpha - 6374)\beta + 1/456597(1510\alpha^2 - 1510\alpha + 1822))\tau_{4,135,6}(n) \\
&+ (1/456597(-1510\alpha + 6374)\beta + 1/456597(6374\alpha - 4552))\tau_{4,135,7}(n) \\
&+ (1/760995(8086\alpha^2 - 38158\alpha - 81392))\tau_{4,135,8}(n) + (1/760995(\\
&- 8086\alpha - 30072)\beta + 1/760995(-8086\alpha^2 + 8086\alpha + 104586))\tau_{4,135,9}(n) \\
&+ (1/760995(8086\alpha + 30072)\beta + 1/253665(10024\alpha + 24838))\tau_{4,135,10}(n) \\
&+ (334/35883\gamma^2 + 470/35883\gamma - 988/3987)\tau_{4,135,11}(n) + ((-334/35883\gamma \\
&+ 2140/35883)\delta - 334/35883\gamma^2 + 1670/35883\gamma - 6220/35883)\tau_{4,135,12}(n) \\
&+ (334/35883\gamma - 2140/35883)\delta - 2140/35883\gamma + 4480/35883)\tau_{4,135,13}
\end{aligned}
$$

$$\begin{aligned}
r_{\Psi_3\oplus F_1}(n) =&240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) - 6000/15552\sigma_3\left(\frac{n}{5}\right)\\
&+ 1200/864\sigma_3\left(\frac{n}{9}\right) + 30\,000/7776\sigma_3\left(\frac{n}{15}\right) - 720/64\sigma_3\left(\frac{n}{27}\right) - 30000/864\sigma_3\left(\frac{n}{45}\right)\\
&+ 18\,000/64\sigma_3\left(\frac{n}{135}\right) + 8/567\tau_{4,5}(n) + 164/567\tau_{4,5}\left(\frac{n}{3}\right) + 164/63\tau_{4,5}\left(\frac{n}{9}\right)\\
&+ 72/7\tau_{4,5}\left(\frac{n}{27}\right) - 16/405\tau_{4,9}(n) + 112/45\tau_{4,9}\left(\frac{n}{3}\right) + 560/81\tau_{4,9}\left(\frac{n}{5}\right)\\
&- 80/9\tau_{4,9}\left(\frac{n}{15}\right) + 1/12\tau_{4,15,1}(n) - 1/12\tau_{4,15,1}\left(\frac{n}{3}\right) + 27/4\tau_{4,15,1}\left(\frac{n}{9}\right)\\
&+ 1/21\tau_{4,15,2}(n) + 64/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 27/7\tau_{4,15,2}\left(\frac{n}{9}\right) + 16/135\tau_{4,27,1}(n)\\
&+ 80/27\tau_{4,27,1}\left(\frac{n}{5}\right) - 16/405\tau_{4,27,2}(n) + 80/81\tau_{4,27,2}\left(\frac{n}{5}\right) + (62/1377\omega\\
&+ 104/459)\tau_{4,27,3}(n) + (1550/1377\omega + 2600/459)\tau_{4,27,3}\left(\frac{n}{5}\right)\\
&+ (-62/1377\omega + 104/459)\tau_{4,27,4}(n) + (-1550/1377\omega + 2600/459)\tau_{4,27,4}\left(\frac{n}{5}\right)\\
&- 1/9\tau_{4,45,1}(n) + \tau_{4,45,1}\left(\frac{n}{3}\right) + 8/63\tau_{4,45,2}(n) + 8/7\tau_{4,45,2}\left(\frac{n}{3}\right)\\
&+ 20/63\tau_{4,45,3}(n) + 20/7\tau_{4,45,3}\left(\frac{n}{3}\right) + 32/405\tau_{4,45,4}(n) + 32/45\tau_{4,45,4}\left(\frac{n}{3}\right)\\
&- 2/135\tau_{4,45,5}(n) + 2/15\tau_{4,45,5}\left(\frac{n}{3}\right) + 64/459\tau_{4,135,2}(n)\\
&+ (1/2282985(-15712\alpha^2 + 37841\alpha + 340644))\tau_{4,135,5}(n)\\
&+ (1/2282985(15712\alpha + 22129)\beta\\
&+ 1/2282985(15712\alpha^2 - 15712\alpha - 20732))\tau_{4,135,6}(n)\\
&+ (1/2282985(-15712\alpha - 22129)\beta + 1/2282985(-22129\alpha + 1397))\tau_{4,135,7}(n)\\
&+ (1/760995(2668\alpha^2 - 27739\alpha + 99604))\tau_{4,135,8}(n) + (1/760995(-2668\alpha\\
&- 25071)\beta + 1/760995(-2668\alpha^2 + 2668\alpha + 160968))\tau_{4,135,9}(n)\\
&+ (1/760995(2668\alpha + 25071)\beta + 1/253665(8357\alpha + 45299))\tau_{4,135,10}(n)\\
&+ (178/35883\gamma^2 - 418/35883\gamma + 4/3987)\tau_{4,135,11}(n) + ((-178/35883\gamma\\
&+ 472/35883)\delta - 178/35883\gamma^2 + 890/35883\gamma + 1460/35883)\tau_{4,135,12}(n)\\
&+ ((178/35883\gamma - 472/35883)\delta - 472/35883\gamma + 3820/35883)\tau_{4,135,13}
\end{aligned}$$

$$
\begin{aligned}
r_{\Psi_3\oplus\Lambda_1}(n) =& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
&+ 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right) \\
&+ 10800/52\sigma_3\left(\frac{n}{135}\right) + 488/7371\tau_{4,5}(n) + 2552/7371\tau_{4,5}\left(\frac{n}{3}\right) + 2552/819\tau_{4,5}\left(\frac{n}{9}\right) \\
&+ 4392/91\tau_{4,5}\left(\frac{n}{27}\right) - 8/135\tau_{4,9}(n) - 8/15\tau_{4,9}\left(\frac{n}{3}\right) - 40/27\tau_{4,9}\left(\frac{n}{5}\right) \\
&- 40/3\tau_{4,9}\left(\frac{n}{15}\right) - 1/18\tau_{4,15,1}(n) + 1/3\tau_{4,15,1}\left(\frac{n}{3}\right) - 9/2\tau_{4,15,1}\left(\frac{n}{9}\right) \\
&+ 19/26\tau_{4,15,2}(n) - 1/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 171/14\tau_{4,15,1}\left(\frac{n}{9}\right) + 16/135\tau_{4,27,1}(n) \\
&+ 80/27\tau_{4,27,1}\left(\frac{n}{5}\right) + (20/459\omega + 11/51)\tau_{4,27,3}(n) + (500/459\omega + 275/51)\tau_{4,27,3}\left(\frac{n}{5}\right) \\
&+ (-20/459\omega + 11/51)\tau_{4,27,4}(n) + (-500/459\omega + 275/51)\tau_{4,27,4}\left(\frac{n}{5}\right) + 1/18\tau_{4,45,1}(n) \\
&- 1/2\tau_{4,45,1}\left(\frac{n}{3}\right) - 3/14\tau_{4,45,2}(n) - 27/14\tau_{4,45,2}\left(\frac{n}{3}\right) + 16/189\tau_{4,45,3}(n) \\
&+ 16/21\tau_{4,45,3}\left(\frac{n}{3}\right) + 29/405\tau_{4,45,4}(n) + 29/45\tau_{4,45,4}\left(\frac{n}{3}\right) - 1/9\tau_{4,45,5}(n) \\
&+ \tau_{4,45,5}\left(\frac{n}{3}\right) + 40/459\tau_{4,135,2}(n) + (1/2282985(15278\alpha^2 + 14921\alpha \\
&- 193596))\tau_{4,135,5}(n) + (1/2282985(-15278\alpha + 30199)\beta \\
&+ 1/2282985(-15278\alpha^2 + 15278\alpha + 157798))\tau_{4,135,6}(n) \\
&+ (1/2282985(15278\alpha - 30199)\beta + 1/2282985(-30199\alpha + 187997))\tau_{4,135,7}(n) \\
&+ (1/50733(566\alpha^2 - 897\alpha - 11700))\tau_{4,135,8}(n) + (1/50733(-566\alpha - 331)\beta \\
&+ 1/50733(-566\alpha^2 + 566\alpha + 1318))\tau_{4,135,9}(n) + (1/50733(566\alpha + 331)\beta \\
&+ 1/50733(331\alpha + 987))\tau_{4,135,10}(n) + (-301/35883\gamma^2 \\
&+ 2887/35883\gamma - 2300/11961)\tau_{4,135,11}(n) + ((301/35883\gamma + 1382/35883)\delta \\
&+ 301/35883\gamma^2 - 1505/35883\gamma - 9308/35883)\tau_{4,135,12}(n) \\
&+ ((-301/35883\gamma - 1382/35883)\delta - 1382/35883\gamma - 2398/35883)\tau_{4,135,13}
\end{aligned}
$$

$$
\begin{aligned}
r_{\Psi_3\oplus\Phi_1}(n) =& 240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) - 6000/15552\sigma_3\left(\frac{n}{5}\right)\\
&+ 1200/864\sigma_3\left(\frac{n}{9}\right) + 30\,000/7776\sigma_3\left(\frac{n}{15}\right) - 720/64\sigma_3\left(\frac{n}{27}\right) - 30000/864\sigma_3\left(\frac{n}{45}\right)\\
&+ 18\,000/64\sigma_3\left(\frac{n}{135}\right) - 4/81\tau_{4,5}(n) - 88/567\tau_{4,5}\left(\frac{n}{3}\right) - 88/63\tau_{4,5}\left(\frac{n}{9}\right)\\
&- 36\tau_{4,5}\left(\frac{n}{27}\right) - 16/405\tau_{4,9}(n) - 32/45\tau_{4,9}\left(\frac{n}{3}\right) - 160/81\tau_{4,9}\left(\frac{n}{5}\right)\\
&- 80/9\tau_{4,9}\left(\frac{n}{15}\right) - 1/36\tau_{4,15,1}(n) - 3/4\tau_{4,15,1}\left(\frac{n}{3}\right) - 9/4\tau_{4,15,1}\left(\frac{n}{9}\right)\\
&+ 10/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 8/405\tau_{4,27,2}(n) - 40/81\tau_{4,27,2}\left(\frac{n}{5}\right) + (38/1377\omega + 44/459)\tau_{4,27,3}(n)\\
&+ (950/1377\omega + 1100/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-38/1377\omega + 44/459)\tau_{4,27,4}(n)\\
&+ (-950/1377\omega + 1100/459)\tau_{4,27,4}\left(\frac{n}{5}\right) - 1/7\tau_{4,45,2}(n) - 9/7\tau_{4,45,2}\left(\frac{n}{3}\right)\\
&+ 16/63\tau_{4,45,3}(n) + 16/7\tau_{4,45,3}\left(\frac{n}{3}\right) + 14/405\tau_{4,45,4}(n)\\
&+ 14/45\tau_{4,45,4}\left(\frac{n}{3}\right) + 4/135\tau_{4,45,5}(n) - 4/15\tau_{4,45,5}\left(\frac{n}{3}\right)\\
&- 20/459\tau_{4,135,2}(n) + (1/456597(1510\alpha^2 + 4864\alpha - 36552))\tau_{4,135,5}(n)\\
&+ (1/456597(-1510\alpha + 6374)\beta + 1/456597(-1510\alpha^2 + 1510\alpha - 1822))\tau_{4,135,6}(n)\\
&+ (1/456597(1510\alpha - 6374)\beta + 1/456597(-6374\alpha + 4552))\tau_{4,135,7}(n)\\
&+ (1/760995(8086\alpha^2 - 38158\alpha - 81392))\tau_{4,135,8}(n) + (1/760995(-8086\alpha - 30072)\beta\\
&+ 1/760995(-8086\alpha^2 + 8086\alpha + 104586))\tau_{4,135,9}(n)\\
&+ (1/760995(8086\alpha + 30072)\beta + 1/253665(10024\alpha + 24838))\tau_{4,135,10}(n) + (334/35883\gamma^2\\
&+ 470/35883\gamma - 988/3987)\tau_{4,135,11}(n) + ((-334/35883\gamma + 2140/35883)\delta\\
&- 334/35883\gamma^2 + 1670/35883\gamma - 6220/35883)\tau_{4,135,12}(n)\\
&+ (334/35883\gamma - 2140/35883)\delta - 2140/35883\gamma + 4480/35883)\tau_{4,135,13}
\end{aligned}
$$

$$
\begin{aligned}
r_{F_2\oplus\Lambda_2}(n) =& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
&+ 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right) \\
&+ 10800/52\sigma_3\left(\frac{n}{135}\right) + 440/2457\tau_{4,5}(n) - 1888/2457\tau_{4,5}\left(\frac{n}{3}\right) - 1888/273\tau_{4,5}\left(\frac{n}{9}\right) \\
&+ 11880/91\tau_{4,5}\left(\frac{n}{27}\right) + 32/135\tau_{4,9}(n) + 32/15\tau_{4,9}\left(\frac{n}{3}\right) + 160/27\tau_{4,9}\left(\frac{n}{5}\right) \\
&+ 160/3\tau_{4,9}\left(\frac{n}{15}\right) + 44/63\tau_{4,15,2}(n) + 24/7\tau_{4,15,2}\left(\frac{n}{3}\right) + 396/7\tau_{4,15,1}\left(\frac{n}{9}\right) \\
&+ 16/45\tau_{4,27,1}(n) + 80/9\tau_{4,27,1}\left(\frac{n}{5}\right) + (92/1377\omega + 128/459)\tau_{4,27,3}(n) + (2300/1377\omega \\
&+ 3200/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-92/1377\omega + 128/459)\tau_{4,27,4}(n) + (-2300/1377\omega \\
&+ 3200/459)\tau_{4,27,4}\left(\frac{n}{5}\right) + 4/9\tau_{4,45,1}(n) - 4\tau_{4,45,1}\left(\frac{n}{3}\right) \\
&+ 8/135\tau_{4,45,4}(n) + 8/15\tau_{4,45,4}\left(\frac{n}{3}\right) + 32/51\tau_{4,135,2}(n) + (1/760995(-5272\alpha^2 \\
&- 3304\alpha + 294144))\tau_{4,135,5}(n) + (1/760995(5272\alpha \\
&- 8576)\beta + 1/760995(5272\alpha^2 - 5272\alpha + 172888))\tau_{4,135,6}(n) \\
&+ (1/760995(-5272\alpha + 8576)\beta + 1/760995(8576\alpha + 164312))\tau_{4,135,7}(n)
\end{aligned}
$$

$$
\begin{aligned}
r_{F_2\oplus\Phi_2}(n) =& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right)\\
&+ 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right)\\
&+ 10800/52\sigma_3\left(\frac{n}{135}\right) + 176/7371\tau_{4,5}(n) + 368/7371\tau_{4,5}\left(\frac{n}{3}\right) + 368/819\tau_{4,5}\left(\frac{n}{9}\right)\\
&+ 1584/91\tau_{4,5}\left(\frac{n}{27}\right) + 16/135\tau_{4,9}(n) + 16/15\tau_{4,9}\left(\frac{n}{3}\right) + 80/27\tau_{4,9}\left(\frac{n}{5}\right)\\
&+ 80/3\tau_{4,9}\left(\frac{n}{15}\right) + 8/27\tau_{4,15,1}(n) - 20/9\tau_{4,15,1}\left(\frac{n}{3}\right) + 24\tau_{4,15,1}\left(\frac{n}{9}\right)\\
&+ 2/7\tau_{4,15,2}(n) + 4/7\tau_{4,15,2}\left(\frac{n}{3}\right) + 162/7\tau_{4,15,1}\left(\frac{n}{9}\right) + 32/135\tau_{4,27,1}(n)\\
&+ 160/27\tau_{4,27,1}\left(\frac{n}{5}\right) + (22/459\omega + 38/153)\tau_{4,27,3}(n) + (550/459\omega + 950/153)\tau_{4,27,3}\left(\frac{n}{5}\right)\\
&+ (-22/459\omega + 38/153)\tau_{4,27,4}(n) + (-550/459\omega + 950/153)\tau_{4,27,4}\left(\frac{n}{5}\right) + 2/9\tau_{4,45,1}(n)\\
&- 2\tau_{4,45,1}\left(\frac{n}{3}\right) + 8/63\tau_{4,45,2}(n) + 8/7\tau_{4,45,2}\left(\frac{n}{3}\right) + 32/189\tau_{4,45,3}(n)\\
&+ 32/21\tau_{4,45,3}\left(\frac{n}{3}\right) + 32/405\tau_{4,45,4}(n) + 32/45\tau_{4,45,4}\left(\frac{n}{3}\right) + 4/27\tau_{4,45,5}(n)\\
&- 4/3\tau_{4,45,5}\left(\frac{n}{3}\right) + 112/459\tau_{4,135,2}(n) + (1/2282985(-2074\alpha^2 + 34712\alpha\\
&+ 235128))\tau_{4,135,5}(n) + (1/2282985(2074\alpha + 32638)\beta + 1/2282985(2074\alpha^2 - 2074\alpha\\
&+ 187426))\tau_{4,135,6}(n) + (1/2282985(-2074\alpha - 32638)\beta + 1/2282985(-32638\alpha\\
&+ 220064))\tau_{4,135,7}(n) + (1/456597(-3646\alpha^2 - 6100\alpha + 138072))\tau_{4,135,8}(n)\\
&+ (1/456597(3646\alpha - 9746)\beta + 1/456597(3646\alpha^2 - 3646\alpha + 54214))\tau_{4,135,9}(n)\\
&+ (1/456597(-3646\alpha + 9746)\beta + 1/456597(9746\alpha + 44468))\tau_{4,135,10}(n)\\
&+ (26/11961\gamma^2 - 82/1329\gamma + 1276/3987)\tau_{4,135,11}(n) + ((-26/11961\gamma - 608/11961)\delta\\
&- 26/11961\gamma^2 + 130/11961\gamma + 4036/11961)\tau_{4,135,12}(n) + ((26/11961\gamma\\
&+ 608/11961)\delta + 608/11961\gamma + 332/3987)\tau_{4,135,13}.
\end{aligned}
$$

$$
\begin{aligned}
r_{F_2 \oplus \Psi_2}(n) =& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
&+ 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right) \\
&+ 10800/52\sigma_3\left(\frac{n}{135}\right) + 16/117\tau_{4,5}(n) + 1000/819\tau_{4,5}\left(\frac{n}{3}\right) + 1000/91\tau_{4,5}\left(\frac{n}{9}\right) \\
&+ 1296/13\tau_{4,5}\left(\frac{n}{27}\right) + 16/135\tau_{4,9}(n) + 16/15\tau_{4,9}\left(\frac{n}{3}\right) + 80/27\tau_{4,9}\left(\frac{n}{5}\right) \\
&+ 80/3\tau_{4,9}\left(\frac{n}{15}\right) + 4/27\tau_{4,15,1}(n) + 8/9\tau_{4,15,1}\left(\frac{n}{3}\right) + 12\tau_{4,15,1}\left(\frac{n}{9}\right) \\
&+ 8/27\tau_{4,15,2}(n) + 272/63\tau_{4,15,2}\left(\frac{n}{3}\right) + 24\tau_{4,15,1}\left(\frac{n}{9}\right) + 64/405\tau_{4,27,1}(n) \\
&+ 320/81\tau_{4,27,1}\left(\frac{n}{5}\right) + (8/153\omega + 146/459)\tau_{4,27,3}(n) + (200/153\omega \\
&+ 3650/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-8/153\omega + 146/459)\tau_{4,27,4}(n) + (-200/153\omega \\
&+ 3650/459)\tau_{4,27,4}\left(\frac{n}{5}\right) + 4/21\tau_{4,45,2}(n) + 12/7\tau_{4,45,2}\left(\frac{n}{3}\right) \\
&+ 104/189\tau_{4,45,3}(n) + 104/21\tau_{4,45,3}\left(\frac{n}{3}\right) + 8/45\tau_{4,45,4}(n) + 8/5\tau_{4,45,4}\left(\frac{n}{3}\right) \\
&+ 16/153\tau_{4,135,2}(n) + (1/760995(1938\alpha^2 + 3296\alpha + 59984))\tau_{4,135,5}(n) \\
&+ (1/760995(-1938\alpha + 5234)\beta + 1/760995(-1938\alpha^2 + 1938\alpha \\
&+ 104558))\tau_{4,135,6}(n) + (1/760995(1938\alpha - 5234)\beta \\
&+ 1/760995(-5234\alpha + 109792))\tau_{4,135,7}(n) + (1/456597(2086\alpha^2 \\
&- 16844\alpha + 74400))\tau_{4,135,8}(n) + (1/456597(-2086\alpha - 14758)\beta + 1/456597(-2086\alpha^2 \\
&+ 2086\alpha + 122378))\tau_{4,135,9}(n) + (1/456597(2086\alpha + 14758)\beta + 1/456597(14758\alpha \\
&+ 107620))\tau_{4,135,10}(n) + (-446/35883\gamma^2 + 1346/35883\gamma + 3056/11961)\tau_{4,135,11}(n) \\
&+ ((446/35883\gamma - 884/35883)\delta + 446/35883\gamma^2 - 2230/35883\gamma \\
&+ 5600/35883)\tau_{4,135,12}(n) + ((-446/35883\gamma + 884/35883)\delta + 884/35883\gamma \\
&+ 1180/35883)\tau_{4,135,13}
\end{aligned}
$$

$$
\begin{aligned}
r_{\Lambda_2\oplus\Phi_2}(n) =& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
&+ 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right) \\
&+ 10800/52\sigma_3\left(\frac{n}{135}\right) + 16/117\tau_{4,5}(n) + 1000/819\tau_{4,5}\left(\frac{n}{3}\right) + 1000/91\tau_{4,5}\left(\frac{n}{9}\right) \\
&+ 1296/13\tau_{4,5}\left(\frac{n}{27}\right) + 16/135\tau_{4,9}(n) + 16/15\tau_{4,9}\left(\frac{n}{3}\right) + 80/27\tau_{4,9}\left(\frac{n}{5}\right) \\
&+ 80/3\tau_{4,9}\left(\frac{n}{15}\right) - 4/27\tau_{4,15,1}(n) - 8/9\tau_{4,15,1}\left(\frac{n}{3}\right) - 12\tau_{4,15,1}\left(\frac{n}{9}\right) \\
&+ 8/27\tau_{4,15,2}(n) + 272/63\tau_{4,15,2}\left(\frac{n}{3}\right) + 24\tau_{4,15,1}\left(\frac{n}{9}\right) + 64/405\tau_{4,27,1}(n) \\
&+ 320/81\tau_{4,27,1}\left(\frac{n}{5}\right) + (8/153\omega + 146/459)\tau_{4,27,3}(n) + (200/153\omega \\
&+ 3650/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-8/153\omega + 146/459)\tau_{4,27,4}(n) + (-200/153\omega \\
&+ 3650/459)\tau_{4,27,4}\left(\frac{n}{5}\right) - 4/21\tau_{4,45,2}(n) - 12/7\tau_{4,45,2}\left(\frac{n}{3}\right) - 104/189\tau_{4,45,3}(n) \\
&- 104/21\tau_{4,45,3}\left(\frac{n}{3}\right) + 8/45\tau_{4,45,4}(n) + 8/5\tau_{4,45,4}\left(\frac{n}{3}\right) + 16/153\tau_{4,135,2}(n) \\
&+ ((446/35883\gamma - 884/35883)\delta - 884/35883\gamma - 1180/35883)\tau_{4,135,13} \\
&+ (1/760995(1938\alpha^2 + 3296\alpha + 59984))\tau_{4,135,5}(n) + (1/760995(-1938\alpha \\
&+ 5234)\beta + 1/760995(-1938\alpha^2 + 1938\alpha + 104558))\tau_{4,135,6}(n) + (1/760995(1938\alpha \\
&- 5234)\beta + 1/760995(-5234\alpha + 109792))\tau_{4,135,7}(n) \\
&+ (1/456597(-2086\alpha^2 + 16844\alpha + 74400))\tau_{4,135,8}(n) + (1/456597(2086\alpha + 14758)\beta \\
&+ 1/456597(2086\alpha^2 - 2086\alpha + 122378))\tau_{4,135,9}(n) \\
&+ (1/456597(-2086\alpha - 14758)\beta + 1/456597(-14758\alpha \\
&- 107620))\tau_{4,135,10}(n) + (446/35883\gamma^2 - 1346/35883\gamma - 3056/11961)\tau_{4,135,11}(n) \\
&+ ((-446/35883\gamma + 884/35883)\delta - 446/35883\gamma^2 + 2230/35883\gamma \\
&- 5600/35883)\tau_{4,135,12}(n)
\end{aligned}
$$

$$\begin{aligned} r_{\Lambda_2\oplus\Psi_2}(n) =& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\ &+ 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right) \\ &+ 10800/52\sigma_3\left(\frac{n}{135}\right) + 16/117\tau_{4,5}(n) + 1000/819\tau_{4,5}\left(\frac{n}{3}\right) + 1000/91\tau_{4,5}\left(\frac{n}{9}\right) \\ &+ 1296/13\tau_{4,5}\left(\frac{n}{27}\right) + 16/135\tau_{4,9}(n) + 16/15\tau_{4,9}\left(\frac{n}{3}\right) + 80/27\tau_{4,9}\left(\frac{n}{5}\right) \\ &+ 80/3\tau_{4,9}\left(\frac{n}{15}\right) + 4/27\tau_{4,15,1}(n) + 8/9\tau_{4,15,1}\left(\frac{n}{3}\right) + 12\tau_{4,15,1}\left(\frac{n}{9}\right) \\ &+ 8/27\tau_{4,15,2}(n) + 272/63\tau_{4,15,2}\left(\frac{n}{3}\right) + 24\tau_{4,15,1}\left(\frac{n}{9}\right) + 64/405\tau_{4,27,1}(n) \\ &+ 320/81\tau_{4,27,1}\left(\frac{n}{5}\right) + (8/153\omega + 146/459)\tau_{4,27,3}(n) + (200/153\omega \\ &+ 3650/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-8/153\omega + 146/459)\tau_{4,27,4}(n) + (-200/153\omega \\ &+ 3650/459)\tau_{4,27,4}\left(\frac{n}{5}\right) + 3650/459)\tau_{4,27,4}\left(\frac{n}{5}\right) + 4/21\tau_{4,45,2}(n) \\ &+ 12/7\tau_{4,45,2}\left(\frac{n}{3}\right) + 104/189\tau_{4,45,3}(n) + 104/21\tau_{4,45,3}\left(\frac{n}{3}\right) \\ &+ 8/45\tau_{4,45,4}(n) + 8/5\tau_{4,45,4}\left(\frac{n}{3}\right) + 16/153\tau_{4,135,2}(n) + (1/760995(1938\alpha^2 \\ &+ 3296\alpha + 59984))\tau_{4,135,5}(n) + (1/760995(-1938\alpha + 5234)\beta \\ &+ 1/760995(-1938\alpha^2 + 1938\alpha + 104558))\tau_{4,135,6}(n) + (1/760995(1938\alpha - 5234)\beta \\ &+ 1/760995(-5234\alpha + 109792))\tau_{4,135,7}(n) + (1/456597(2086\alpha^2 - 16844\alpha \\ &- 74400))\tau_{4,135,8}(n) + (1/456597(-2086\alpha - 14758)\beta + 1/456597(-2086\alpha^2 \\ &+ 2086\alpha - 122378))\tau_{4,135,9}(n) + (1/456597(2086\alpha + 14758)\beta \\ &- 1/456597(14758\alpha + 107620))\tau_{4,135,10}(n) + (-446/35883\gamma^2 + 1346/35883\gamma \\ &+ 3056/11961)\tau_{4,135,11}(n) + ((446/35883\gamma - 884/35883)\delta \\ &+ 446/35883\gamma^2 - 2230/35883\gamma + 5600/35883)\tau_{4,135,12}(n) \\ &+ ((-446/35883\gamma + 884/35883)\delta + 884/35883\gamma + 1180/35883)\tau_{4,135,13} \end{aligned}$$

4 Combination of Quadratic Forms

The representation numbers of $Q + P$,can be calculated without using quasimodular form by the method in section 3, where Q, P quadratic forms defined in section 3. Here, we would like to describe the calculation of the representation numbers of $Q + tP$, $t = 3, 9, 15, 27, 45, 135$, by combining the section 2 and section 3. Obviously

$$r_{Q+tP}(n) = r_Q(0)\, r_P\left(\frac{n}{t}\right) + r_Q(n)\, r_P(0) + \sum_{\substack{l,m\in\mathbb{N}\\ l+tm=n}} r_Q(l)\, r_P(m)$$

holds. The term $\sum_{\substack{l,m\in\mathbb{N}\\ l+tm=n}} r_Q(l)\, r_P(m)$ depends on several convolution products. Although the calculation is complicated, it can be done. Here I will only handle the following case:

$$r_{F_2+3F_2} = r_{3F_2}\left(\frac{n}{k}\right) + r_{F_2}(n) + \sum_{\substack{l,m\in\mathbb{N}\\ l+3m=n}} r_{F_2}(l)\, r_{3F_2}(m)$$, where $r_{F_2}(n)$ is given in [3.2],

$$\begin{aligned} r_{3F_2}(n) =& 2/9\sigma\left(\frac{n}{3}\right) - 8/9\sigma\left(\frac{n}{9}\right) + 10/9\sigma\left(\frac{n}{15}\right) + 8/3\sigma\left(\frac{n}{27}\right)\\ &- 40/9\sigma\left(\frac{n}{45}\right) - 6\sigma\left(\frac{n}{81}\right) + 40/3\sigma\left(\frac{n}{135}\right) - 30\sigma\left(\frac{n}{405}\right)\\ &+ 2/9\tau_{2,15}\left(\frac{n}{3}\right) + 2/9\tau_{2,15}\left(\frac{n}{9}\right) + 2\tau_{2,15}\left(\frac{n}{27}\right) + 8/9\tau_{2,27}\left(\frac{n}{3}\right)\\ &+ 40(9\tau_{2,27}\left(\frac{n}{15}\right) + 8/9\tau_{2,135,2}\left(\frac{n}{3}\right) + (16/117\eta + 20/39)\tau_{2,135,3}\left(\frac{n}{3}\right)\\ &+ (-16/117\eta + 44/117)\tau_{2,135,4}\left(\frac{n}{3}\right) + (-16/117\eta + 44/117)\tau_{2,135,5}\left(\frac{n}{3}\right)\\ &+ (16/117\eta + 20/39)\tau_{2,135,6}\left(\frac{n}{3}\right), \end{aligned}$$

$$\sum_{\substack{l,m\in\mathbb{N}\\ l+3m=n}} r_{F_2}(l)\, r_{3F_2}(m) = \sum_{\substack{l,m\in\mathbb{N}\\ l+3m=n}} (2/9\sigma(l) - 8/9\sigma\left(\tfrac{l}{3}\right) + 10/9\sigma\left(\tfrac{l}{5}\right) + 8/3\sigma\left(\tfrac{l}{9}\right)$$

$$\begin{aligned} =& - 40/9\sigma\left(\frac{l}{15}\right) - 6\sigma\left(\frac{l}{27}\right) + 40/3\sigma\left(\frac{l}{45}\right) - 30\sigma\left(\frac{l}{135}\right) + 2/9\tau_{2,15}(l)\\ &+ 2/9\tau_{2,15}\left(\frac{l}{3}\right) + 2\tau_{2,15}\left(\frac{l}{9}\right) + 8/9\tau_{2,27}(l) + 40/9\tau_{2,27}\left(\frac{l}{5}\right) + 8/9\tau_{2,135,2}(l)\\ &+ (16/117\eta + 20/39)\tau_{2,135,3}(l) + (-16/117\eta\\ &+ 44/117)\tau_{2,135,4}(l) + (-16/117\eta + 44/117)\tau_{2,135,5}(l)\\ &+ (16/117\eta + 20/39)\tau_{2,135,6}(l))(2/9\sigma(m) - 8/9\sigma\left(\frac{m}{3}\right)\\ &+ 10/9\sigma\left(\frac{m}{5}\right) + 8/3\sigma\left(\frac{m}{9}\right) - 40/9\sigma\left(\frac{m}{15}\right) + 40/3\sigma\left(\frac{m}{45}\right) - 6\sigma\left(\frac{m}{27}\right) - 30\sigma\left(\frac{m}{135}\right)\\ &+ 2/9\tau_{2,15}(m) + 2/9\tau_{2,15}\left(\frac{m}{3}\right) + 2\tau_{2,15}\left(\frac{m}{9}\right) + 8/9\tau_{2,27}(m) + 40/9\tau_{2,27}\left(\frac{m}{5}\right)\\ &+ 8/9\tau_{2,135,2}(m) + (16/117\eta + 20/39)\tau_{2,135,3}(m) + (-16/117\eta + 44/117)\tau_{2,135,4}(m)\\ &+ (-16/117\eta + 44/117)\tau_{2,135,5}(m) + (16/117\eta + 20/39)\tau_{2,135,6}(m)). \end{aligned}$$

So

$$
\begin{aligned}
r_{F_2+3F_2} =& 4/81W_3 - 16/81W_9 + 20/81W_{15} + 16/27W_{27} - 80/81W_{45} - 12/9W_{81} + 80/27W_{135} \\
& - 20/3W_{405} + 4/81W_{\sigma,3\tau_{2,15}} + 4/81W_{\sigma,9\tau_{2,15}} + 4/9W_{\sigma,27\tau_{2,15}} \\
& + 16/81W_{\sigma,3\tau_{2,27}} + 80/81W_{\sigma,15\tau_{2,27}} + 16/81W_{\sigma,3\tau_{2,135,2}} \\
& + 2/9(16/117\eta + 20/39)W_{\sigma,3\tau_{2,135,3}} \\
& + \frac{2}{9}(-16/117\eta + 44/117)W_{\sigma,3\tau_{2,135,4}} + \frac{2}{9}(-16/117\eta \\
& + 44/117)W_{\sigma,3\tau_{2,135,5}} + \frac{2}{9}(16/117\eta + 20/39)W_{\sigma,3\tau_{2,135,6}} \\
& - 4(4/81W_{3,3} - 16/81W_{3,9} + 20/81W_{3,15} + 16/27W_{3,27} - 80/81W_{3,45} - 12/9W_{3,81} \\
& + 80/27W_{3,135} - 20/3W_{3,405} + 4/81W_{3\sigma,3\tau_{2,15}} + 4/81W_{3\sigma,9\tau_{2,15}} \\
& + 4/9W_{3\sigma,27\tau_{2,15}} + 16/81W_{3\sigma,3\tau_{2,27}} + 80/81W_{3\sigma,15\tau_{2,27}} \\
& + 16/81W_{3\sigma,3\tau_{2,135,2}} + 2/9(16/117\eta + 20/39)W_{3\sigma,3\tau_{2,135,3}} \\
& + 2/9(-16/117\eta + 44/117)W_{3\sigma,3\tau_{2,135,4}} + 2/9(-16/117\eta + 44/117)W_{3\sigma,3\tau_{2,135,5}} \\
& + 2/9(16/117\eta + 20/39)W_{3\sigma,3\tau_{2,135,6}}) + 5(4/81W_{5,3} - 16/81W_{5,9} + 20/81W_{5,15} \\
& + 16/27W_{5,27} - 80/81W_{5,45} - 4/3W_{5,81} + 80/27W_{5,135} - 20/3W_{5,405} + 4/81W_{5\sigma,3\tau_{2,15}} \\
& + 4/81W_{5\sigma,9\tau_{2,15}} + 4/9W_{5\sigma,27\tau_{2,15}} + 16/81W_{5\sigma,3\tau_{2,27}} \\
& + 80/81W_{5\sigma,15\tau_{2,27}} + 16/81W_{5\sigma,3\tau_{2,135,2}} + 2/9(16/117\eta \\
& + 20/39)W_{5\sigma,3\tau_{2,135,3}} + 2/9(-16/117\eta + 44/117)W_{5\sigma,3\tau_{2,135,4}} \\
& + 2/9(-16/117\eta + 44/117)W_{5\sigma,3\tau_{2,135,5}} + 2/9(16/117\eta + 20/39)W_{5\sigma,3\tau_{2,135,6}}) \\
& + 12(4/81W_{9,3} - 16/81W_{9,9} + 20/81W_{9,15} + 16/27W_{9,27} - 80/81W_{9,45} - 4/3W_{9,81} \\
& + 80/27W_{9,135} - 20/3W_{9,405} + 4/81W_{9\sigma,3\tau_{2,15}} + 4/81W_{9\sigma,9\tau_{2,15}} \\
& + 4/9W_{9\sigma,27\tau_{2,15}} + 16/81W_{9\sigma,3\tau_{2,27}} + 80/81W_{9\sigma,15\tau_{2,27}} \\
& + 16/81W_{9\sigma,3\tau_{2,135,2}} + 2/9(\frac{16}{117}\eta + \frac{20}{39})W_{9\sigma,3\tau_{2,135,3}} \\
& + \frac{2}{9}(-16/117\eta + 44/117)W_{9\sigma,3\tau_{2,135,4}} + 2/9(-16/117\eta \\
& + 44/117)W_{9\sigma,3\tau_{2,135,5}} + 2/9(16/117\eta + 20/39)W_{9\sigma,3\tau_{2,135,6}}) \\
& - 20(4/81W_{15,3} - 16/81W_{15,9} + 20/81W_{15,15} + 16/27W_{15,27} - 80/81W_{15,45} \\
& - 4/3W_{15,81} + 80/27W_{15,135} - 20/3W_{15,405} + 4/81W_{15\sigma,3\tau_{2,15}} + 4/81W_{15\sigma,9\tau_{2,15}} \\
& + 4/9W_{15\sigma,27\tau_{2,15}} + 16/81W_{15\sigma,3\tau_{2,27}} + 80/81W_{15\sigma,15\tau_{2,27}} \\
& + 16/81W_{15\sigma,3\tau_{2,135,2}} + 2/9(16/117\eta + 20/39)W_{15\sigma,3\tau_{2,135,3}} + 2/9(-16/117\eta \\
& + 44/117)W_{15\sigma,3\tau_{2,135,4}} + 2/9(-16/117\eta + 44/117)W_{15\sigma,3\tau_{2,135,5}} \\
& + 2/9(16/117\eta + 20/39)W_{15\sigma,3\tau_{2,135,6}}) - 27(4/81W_{27,3} - 16/81W_{27,9} + 20/81W_{27,15} \\
& + 16/27W_{27,27} - 80/81W_{27,45} - 4/3W_{27,81} + 80/27W_{27,135} - 20/3W_{27,405} + 4/81W_{27\sigma,3\tau_{2,15}}
\end{aligned}
$$

$+ 4/81W_{27\sigma,9\tau_{2,15}} + 4/9W_{27\sigma,27\tau_{2,15}} + 16/81W_{27\sigma,3\tau_{2,27}}$
$+ 80/81W_{27\sigma,15\tau_{2,27}} + 16/81W_{27\sigma,3\tau_{2,135,2}} + 2/9(16/117\eta$
$+ 20/39)W_{27\sigma,3\tau_{2,135,3}} + 2/9(-16/117\eta + 44/117)W_{27\sigma,3\tau_{2,135,4}}$
$+ 2/9(-16/117\eta + 44/117)W_{27\sigma,3\tau_{2,135,5}} + 2/9(16/117\eta + 20/39)W_{27\sigma,3\tau_{2,135,6}})$
$+ 60(4/81W_{45,3} - 16/81W_{45,9} + 20/81W_{45,15} + 16/27W_{45,27} - 80/81W_{45,45} - 4/3W_{45,81}$
$+ 80/27W_{45,135} - 20/3W_{45,405} + 4/81W_{45\sigma,3\tau_{2,15}} + 4/81W_{45\sigma,9\tau_{2,15}} + 4/9W_{45\sigma,27\tau_{2,15}}$
$+ 16/81W_{45\sigma,3\tau_{2,27}} + 80/81W_{45\sigma,15\tau_{2,27}} + 16/81W_{45\sigma,3\tau_{2,135,2}}$
$+ 2/9(16/117\eta + 20/39)W_{45\sigma,3\tau_{2,135,3}} + 2/9(-16/117\eta + 44/117)W_{45\sigma,3\tau_{2,135,4}}$
$+ 2/9(-16/117\eta + 44/117)W_{45\sigma,3\tau_{2,135,5}} + 2/9(16/117\eta + 20/39)W_{45\sigma,3\tau_{2,135,6}})$
$- 135(4/81W_{135,3} - 16/81W_{135,9} + 20/81W_{135,15} + 16/27W_{135,27} - 80/81W_{135,45} - 4/3W_{135,81}$
$+ 80/27W_{135,135} - 20/3W_{135,405} + 4/81W_{135\sigma,3\tau_{2,15}} + 4/81W_{135\sigma,9\tau_{2,15}}$
$+ 4/9W_{135\sigma,27\tau_{2,15}} + 16/81W_{135\sigma,3\tau_{2,27}} + 80/81W_{135\sigma,15\tau_{2,27}}$
$+ 16/81W_{135\sigma,3\tau_{2,135,2}} + 2/9(16/117\eta + 20/39)W_{135\sigma,3\tau_{2,135,3}} + 2/9(-16/117\eta$
$+ 44/117)W_{135\sigma,3\tau_{2,135,4}} + 2/9(-16/117\eta + 44/117)W_{135\sigma,3\tau_{2,135,5}}$
$+ 2/9(16/117\eta + 20/39)W_{135\sigma,3\tau_{2,135,6}}) + (4/81W_{\sigma,3\tau_{2,15}}$
$- 16/81W_{3\sigma,3\tau_{2,15}} + 20/81W_{5\sigma,3\tau_{2,15}} + 16/27W_{9\sigma,3\tau_{2,15}}$
$- 80/81W_{15\sigma,3\tau_{2,15}} - 4/3W_{27\sigma,3\tau_{2,15}} + 80/27W_{45\sigma,3\tau_{2,15}}$
$- 20/3W_{135\sigma,3\tau_{2,15}} + 4/81W_{\tau_{2,15},3\tau_{2,15}} + 4/81W_{\tau_{2,15},9\tau_{2,15}}$
$+ 4/9W_{\tau_{2,15},27\tau_{2,15}} + 16/81W_{\tau_{2,15},3\tau_{2,27}} + 80/81W_{\tau_{2,15},15\tau_{2,27}}$
$+ 16/81W_{\tau_{2,15},3\tau_{2,135,2}} + 2/9(16/117\eta + 20/39)W_{\tau_{2,15},3\tau_{2,135,3}}$
$+ 2/9(-16/117\eta + 44/117)W_{\tau_{2,15},3\tau_{2,135,4}} + 2/9(-16/117\eta$
$+ 44/117)W_{\tau_{2,15},3\tau_{2,135,5}} + 2/9(16/117\eta + 20/39)W_{\tau_{2,15},3\tau_{2,135,6}})$
$+ (4/81W_{\sigma,27\tau_{2,15}} - 16/81W_{3\sigma,27\tau_{2,15}} + 20/81W_{5\sigma,27\tau_{2,15}}$
$+ 16/27W_{9\sigma,27\tau_{2,15}} - 80/81W_{15\sigma,27\tau_{2,15}}$
$- 4/3W_{27\sigma,27\tau_{2,15}} + 80/27W_{45\sigma,27\tau_{2,15}} - 20/3W_{135\sigma,27\tau_{2,15}}$
$+ 4/81W_{9\tau_{2,15},3\tau_{2,15}} + 4/81W_{9\tau_{2,15},9\tau_{2,15}}$
$+ 4/9W_{9\tau_{2,15},27\tau_{2,15}} + 2W_{9\tau_{2,15},3\tau_{2,27}} + 80/81W_{9\tau_{2,15},15\tau_{2,27}}$
$+ 16/81W_{9\tau_{2,15},3\tau_{2,135,2}} + 2/9(16/117\eta + 20/39)W_{9\tau_{2,15},3\tau_{2,135,3}}$
$+ 2/9(-16/117\eta + 44/117)W_{9\tau_{2,15},3\tau_{2,135,4}} + 2/9(-16/117\eta$
$+ 44/117)W_{9\tau_{2,15},3\tau_{2,135,5}} + 2/9(16/117\eta + 20/39)W_{9\tau_{2,15},3\tau_{2,135,6}})$

$+ (4/81W_{\sigma,9\tau_{2,15}} - 16/81W_{3\sigma,9\tau_{2,15}} + 20/81W_{5\sigma,9\tau_{2,15}}$
$+ 16/27W_{9\sigma,9\tau_{2,15}} - 80/81W_{15\sigma,9\tau_{2,15}} - 4/3W_{27\sigma,9\tau_{2,15}}$
$+ 80/81W_{3\tau_{2,15},15\tau_{2,27}} + 16/81W_{3\tau_{2,15},3\tau_{2,135,2}} + 2/9(16/117\eta$
$+ 20/39)W_{3\tau_{2,15},3\tau_{2,135,3}} + 2/9(-16/117\eta + 44/117)W_{3\tau_{2,15},3\tau_{2,135,4}}$
$+ 2/9(-16/117\eta + 44/117)W_{3\tau_{2,15},3\tau_{2,135,5}} + 2/9(16/117\eta$
$+ 20/39)W_{3\tau_{2,15},3\tau_{2,135,6}}) + 4(3/81W_{\sigma,3\tau_{2,27}} - 16/81W_{3\sigma,3\tau_{2,27}}$
$+ 20/81W_{5\sigma,3\tau_{2,27}} + 16/27W_{9\sigma,3\tau_{2,27}} - 80/81W_{15\sigma,3\tau_{2,27}}$
$- 4/3W_{27\sigma,3\tau_{2,27}} + 80/27W_{45\sigma,3\tau_{2,27}} - 20/9W_{135\sigma,3\tau_{2,27}}$
$+ 4/81W_{\tau_{2,27},3\tau_{2,15}} + 4/81W_{\tau_{2,27},9\tau_{2,15}} + 4/9W_{\tau_{2,27},27\tau_{2,15}}$
$+ 16/81W_{\tau_{2,27},3\tau_{2,27}} + 80/81W_{\tau_{2,27},15\tau_{2,27}}$
$+ 16/81W_{\tau_{2,27},3\tau_{2,135,2}} + 2/9(16/117\eta + 20/39)W_{\tau_{2,27},3\tau_{2,135,3}}$
$+ 2/9(-16/117\eta + 44/117)W_{\tau_{2,27},3\tau_{2,135,4}} + 2/9(-16/117\eta$
$+ 44/117)W_{\tau_{2,27},3\tau_{2,135,5}} + 2/9(16/117\eta + 20/39)W_{\tau_{2,27},3\tau_{2,135,6}})$
$+ 20(4/81W_{\sigma,3\tau 135} - 16/81W_{3\sigma,3\tau 135} + 20/81W_{5\sigma,3\tau 135}$
$+ 16/27W_{9\sigma,3\tau 135} - 80/81W_{15\sigma,3\tau 135} - 4/3W_{27\sigma,3\tau 135}$
$+ 80/27W_{45\sigma,3\tau 135} - 20/3W_{135\sigma,3\tau 135} + 4/81W_{5\tau_{2,27},3\tau_{2,15}}$
$+ 4/81W_{5\tau_{2,27},9\tau_{2,15}} + 4/9W_{5\tau_{2,27},27\tau_{2,15}} + 16/81W_{5\tau_{2,27},3\tau_{2,27}}$
$+ 80/81W_{5\tau_{2,27},15\tau_{2,27}} + 16/81W_{5\tau_{2,27},3\tau_{2,135,2}} + 2/9(16/117\eta$
$+ 20/39)W_{5\tau_{2,27},3\tau_{2,135,3}} + 2/9(-16/117\eta + 44/117)W_{5\tau_{2,27},3\tau_{2,135,4}}$
$+ 2/9(-16/117\eta + 44/117)W_{5\tau_{2,27},3\tau_{2,135,5}} + 2/9(16/117\eta$
$+ 20/39)W_{5\tau_{2,27},3\tau_{2,135,6}}) + 4(4/81W_{\sigma,3\tau_{2,135,2}}$
$- 16/81W_{3\sigma,3\tau_{2,135,2}} + 20/81W_{5\sigma,3\tau_{2,135,2}} + 16/27W_{9\sigma,3\tau_{2,135,2}}$
$- 80/81W_{15\sigma,3\tau_{2,135,2}} - 4/3W_{27\sigma,3\tau_{2,135,2}} + 80/27W_{45\sigma,3\tau_{2,135,2}}$
$- 20/3W_{135\sigma,3\tau_{2,135,2}} + 4/81W_{\tau_{2,135,2},3\tau_{2,15}}$
$+ 4/81W_{\tau_{2,135,2},9\tau_{2,15}} + 4/9W_{\tau_{2,135,2},27\tau_{2,15}}$
$+ 16/81W_{\tau_{2,135,2},3\tau_{2,27}} + 80/81W_{\tau_{2,135,2},15\tau_{2,27}}$
$+ 16/81W_{\tau_{2,135,2},3\tau_{2,135,2}} + 2/9(16/117\eta + 20/39)W_{\tau_{2,135,2},3\tau_{2,135,3}}$
$+ 2/9(-16/117\eta + 44/117)W_{\tau_{2,135,2},3\tau_{2,135,4}} + 2/9(-16/117\eta$
$+ 44/117)W_{\tau_{2,135,2},3\tau_{2,135,5}} + 2/9(16/117\eta + 20/39)W_{\tau_{2,135,2},3\tau_{2,135,6}})$
$+ (16/117\eta + 20/39)9/2(4/81W_{\sigma,3\tau_{2,135,3}} - 16/81W_{3\sigma,3\tau_{2,135,3}}$
$+ 20/81W_{5\sigma,3\tau_{2,135,3}} + 16/27W_{9\sigma,3\tau_{2,135,3}} - 80/81W_{15\sigma,3\tau_{2,135,3}}$
$- 4/3W_{27\sigma,3\tau_{2,135,3}} + 80/27W_{45\sigma,3\tau_{2,135,3}} - 20/3W_{135\sigma,3\tau_{2,135,3}}$
$+ 4/81W_{\tau_{2,135,3},3\tau_{2,15}} + 4/81W_{\tau_{2,135,3},9\tau_{2,15}}$
$+ 4/9W_{\tau_{2,135,3},27\tau_{2,15}} + 16/81W_{\tau_{2,135,3},3\tau_{2,27}}$
$+ 80/81W_{\tau_{2,135,3},15\tau_{2,27}} + 16/81W_{\tau_{2,135,3},3\tau_{2,135,2}}$
$+ 2/9(16/117\eta + 20/39)W_{\tau_{2,135,3},3\tau_{2,135,3}} + 2/9(-16/117\eta$
$+ 44/117)W_{\tau_{2,135,3},3\tau_{2,135,4}} + 2/9(-16/117\eta + 44/117)W_{\tau_{2,135,3},3\tau_{2,135,5}}$
$+ 2/9(16/117\eta + 20/39)W_{\tau_{2,135,3},3\tau_{2,135,6}}) + (-16/117\eta$

$$+ 44/117)9/2(4/81W_{\sigma,3\tau_{2,135,4}} - 16/81W_{3\sigma,3\tau_{2,135,4}}$$
$$+ 20/81W_{5\sigma,3\tau_{2,135,4}} + 16/27W_{9\sigma,3\tau_{2,135,4}}$$
$$- 80/81W_{15\sigma,3\tau_{2,135,4}} - 4/3W_{27\sigma,3\tau_{2,135,4}} + 80/27W_{45\sigma,3\tau_{2,135,4}}$$
$$- 20/3W_{135\sigma,3\tau_{2,135,4}} + 4/81W_{\tau_{2,135,4},3\tau_{2,15}}$$
$$+ 4/81W_{\tau_{2,135,4},9\tau_{2,15}} + 4/9W_{\tau_{2,135,4},27\tau_{2,15}}$$
$$+ 16/81W_{\tau_{2,135,4},3\tau_{2,27}} + 80/81W_{\tau_{2,135,4},15\tau_{2,27}}$$
$$+ 16/81W_{\tau_{2,135,4},3\tau_{2,135,2}} + 2/9(16/117\eta + 20/39)W_{\tau_{2,135,4},3\tau_{2,135,3}}$$
$$+ 2/9(-16/117\eta + 44/117)W_{\tau_{2,135,4},3\tau_{2,135,4}} + 2/9(-16/117\eta$$
$$+ 44/117)W_{\tau_{2,135,4},3\tau_{2,135,5}} + 2/9(16/117\eta + 20/39)W_{\tau_{2,135,4},3\tau_{2,135,6}})$$
$$+ (-16/117\eta + 44/117)9/2(4/81W_{\sigma,3\tau_{2,135,5}} - 16/81W_{3\sigma,3\tau_{2,135,5}}$$
$$+ 20/81W_{5\sigma,3\tau_{2,135,5}} + 16/27W_{9\sigma,3\tau_{2,135,5}} - 80/81W_{15\sigma,3\tau_{2,135,5}}$$
$$- 4/3W_{27\sigma,3\tau_{2,135,5}} + 80/27W_{45\sigma,3\tau_{2,135,5}} - 20/3W_{135\sigma,3\tau_{2,135,5}}$$
$$+ 4/81W_{\tau_{2,135,5},3\tau_{2,15}} + 4/81W_{\tau_{2,135,5},9\tau_{2,15}}$$
$$+ 4/9W_{\tau_{2,135,5},27\tau_{2,15}} + 16/81W_{\tau_{2,135,5},3\tau_{2,27}}$$
$$+ 80/81W_{\tau_{2,135,5},15\tau_{2,27}} + 16/81W_{\tau_{2,135,5},3\tau_{2,135,2}} + 2/9(16/117\eta$$
$$+ 20/39)W_{\tau_{2,135,5},3\tau_{2,135,3}} + 2/9(-16/117\eta + 44/117)W_{\tau_{2,135,5},3\tau_{2,135,4}}$$
$$+ 2/9(-16/117\eta + 44/117)W_{\tau_{2,135,5},3\tau_{2,135,5}} + 2/9(16/117\eta$$
$$+ 20/39)W_{\tau_{2,135,5},3\tau_{2,135,6}}) + (16/117\eta + 20/39)9/2(4/81W_{\sigma,3\tau_{2,135,6}}$$
$$- 16/81W_{3\sigma,3\tau_{2,135,6}} + 20/81W_{5\sigma,3\tau_{2,135,6}} + 16/27W_{9\sigma,3\tau_{2,135,6}}$$
$$- 80/81W_{15\sigma,3\tau_{2,135,6}} - 4/3W_{27\sigma,3\tau_{2,135,6}} + 80/27W_{45\sigma,3\tau_{2,135,6}}$$
$$- 20/3W_{135\sigma,3\tau_{2,135,6}} + 4/81W_{\tau_{2,135,6},3\tau_{2,15}} + 4/81W_{\tau_{2,135,6},9\tau_{2,15}}$$
$$+ 4/9W_{\tau_{2,135,6},27\tau_{2,15}} + 16/81W_{\tau_{2,135,6},3\tau_{2,27}}$$
$$+ 80/81W_{\tau_{2,135,6},15\tau_{2,27}} + 16/81W_{\tau_{2,135,6},3\tau_{2,135,2}} + 2/9(16/117\eta$$
$$+ 20/39)W_{\tau_{2,135,6},3\tau_{2,135,3}} + 2/9(-16/117\eta + 44/117)W_{\tau_{2,135,6},3\tau_{2,135,4}}$$
$$+ 2/9(-16/117\eta + 44/117)W_{\tau_{2,135,6},3\tau_{2,135,5}} + 2/9(16/117\eta$$
$$+ 20/39)W_{\tau_{2,135,6},3\tau_{2,135,6}}).$$

Proof. Here, we need to calculate essentially the following convolution products

$$W_{s\sigma',r\tau'} = \sum_{\substack{l,m\in\mathbb{N}\\ sl+rm=n}} \sigma'(l)\,\tau'(m)\,, W_{s\tau_1',r\tau_2'} = \sum_{\substack{l,m\in\mathbb{N}\\ sl+rm=n}} {\tau_1}'(l)\,\tau_2'(m)\,,$$

where

$$\sigma' = t_1\sigma; \tau', \tau_1', \tau_2' = t_2\tau_{2,N} \text{ or } t_2\tau_{2,N,i}, N|135.$$

The first one comes from the expression of the quasiform

$$E_2(t_1z)\,\Delta_N(t_2z) \text{ or } \mathrm{E}_2(t_1z)\,\Delta_{N,i}(t_2z)\ ^*$$

by the 85 elements described in section 2. The second one is easier since

$\Delta_{m_1}(t_1z)\,\Delta_{m_2}(t_2z)\,, \Delta_{m_1}(t_1z)\,\Delta_{m_{2,i}}(t_2z)$, or $\Delta_{m_{1,i}}(t_1z)\,\Delta_{m_{2,j}}(t_2z)$

are more than quasiforms , i.e., they are modular forms of weight 4 so only 60 elements contribute to it. But these can be evaluated by the same method as described in the beginning of section 2. □

5 Conclusion

1. We have calculated the following convolution sums

$$W_{135}(n), W_{3,45}(n), W_{5,27}(n), W_{9,15}(n), W_{45}(n), W_{3,15}(n), W_{5,9}(n),$$

by using quasiforms [21] in Section 2.

2. We have determined the representation numbers of the quadratic forms

$$F_2, \Lambda_2, \Phi_2, \Psi_2, F \oplus \Lambda, F \oplus \Phi, F \oplus \Psi, \Lambda \oplus \Phi, \Lambda \oplus \Psi, \Phi \oplus \Psi,$$

$$\begin{aligned} &F_4, \Lambda_4, \Phi_4, \Psi_4, F_3 \oplus \Lambda, F_3 \oplus \Phi, F_3 \oplus \Psi, \\ &\Lambda_3 \oplus F, \Lambda_3 \oplus \Phi, \Lambda_3 \oplus \Psi, \Phi_3 \oplus F, \Phi_3 \oplus \Psi, \\ &\Psi_3 \oplus F, \Psi_3 \oplus \Lambda, \Psi_3 \oplus \Phi, F_2 \oplus \Lambda_2, F_2 \oplus \Phi_2 \\ &F_2 \oplus \Psi_2, \Lambda_2 \oplus \Phi_2, \Lambda_2 \oplus \Psi_2, \Phi_2 \oplus \Psi_2, \end{aligned}$$

by means of Hecke Theorem 2.1 [26] in section 3.

3. We have discussed the determination of the representation numbers of the quadratic forms of the form

$Q + tP$, $t = 3, 9, 15, 27, 45, 135,$ where Q and P are as above, by combining the section 2 and section 3.

Competing Interests

The author declares that no competing interests exist.

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