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An Optimization Problem with Controls in the Coefficients of Parabolic Equations

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Abstract

In this paper, an optimal control parabolic problem is studied. The existence and uniqueness theorem for the solving optimal control parabolic problem is proved. Also, a theorem for the sufficient differentiability conditions of the functional and its gradient formulae has been proved.

Keywords: Optimal control parabolic problems, existence and uniqueness theorems, adjoint system, sufficient differentiability conditions, gradient formulae.

1 Introduction and Statement of the Optimal Control Problem

The optimal control of systems described by partial differential equations has received increasing attention in recent years. Many of the problems of control in air-frames design, shipbuilding industry, magneto- hydrodynamics and other engineering field are problems of control of systems with distributed parameter systems [1-4]. In [4], the existence and uniqueness theorem is proved under constrained problem with closed bounded space of E_N but here the controls $v \in L_2(\Omega)$ are

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respected to x,t. In [4] the coefficients $\lambda(u(x,t),v)$ and $B(u,v)\,u_x$ were respected to $u(x,t)$ and these coefficients also had constraints but in the present paper $\lambda(x,t,v)$ does not depend on $u(x,t)$. Also, in [4] the sufficient differentiability conditions of the functional and its gradient formulae proved using the modified functional which has given by helping the penalty function method. In this paper, an optimal control parabolic problem is studied. The existence and uniqueness theorem for the solving optimal control parabolic problem is proved. Also, a theorem for the sufficient differentiability conditions of the functional and its gradient formulae has been proved.

Let D be a bounded domain of the N-dimensional Euclidean space in E_N, let l,T are given positive numbers, $0 \le t \le T,\ \Omega = D \times (0,T],\ D = [0,l]$ .Throughout this paper, we adopt the following function spaces [5] :-

1) E_N is the N-dimensional Euclidean space with $\langle z_1, z_2 \rangle_{E_N} = \sum_{i=1}^{N} (z_1)_i\,(z_2)_i$, $\|z\|_{E_N} = \sqrt{\langle z, z \rangle_{E_N}}$.

2) $L_2(D)$ is a Banach space which consisting of all the measurable functions on D with the norm

$$\|z\|_{L_2(D)} = \left(\int_D |z|^2\, dx \right)^{\frac{1}{2}}.$$

3) $L_2(0,l)$ is a Hilbert space which consisting of all the measurable functions on $(0,l)$ with

$$\langle z_1, z_2 \rangle_{L_2(0,l)} = \int_0^l z_1(x)\ z_2(x)\ dx\ ,\quad \|z\|_{L_2(0,l)} = \sqrt{\langle z, z \rangle_{L_2(0,l)}}$$

4) $L_2(0,T)$ is a Hilbert space which consisting of all the measurable functions on $(0,T)$ with

$$\langle z_1, z_2 \rangle_{L_2(0,T)} = \int_0^T z_1(t)\ z_2(t)\ dt\ ,\quad \|z\|_{L_2(0,T)} = \sqrt{\langle z, z \rangle_{L_2(0,T)}}\ .$$

5) $L_2(\Omega)$ is a Hilbert space which consisting of all measurable functions on Ω with

$$\langle z_1, z_2 \rangle_{L_2(\Omega)} = \int_\Omega z_1(x,t)\ z_2(x,t)\, dx\, dt\ ,\ \|z\|_{L_2(\Omega)} = \sqrt{\langle z, z \rangle_{L_2(\Omega)}}\ .$$

6) $W_2^{1,0}(\Omega) = \left\{ z \in L_2(\Omega)\ and\ \dfrac{\partial z}{\partial x} \in L_2(\Omega) \right\}$ is a Hilbert space with

$$\langle z_1, z_2 \rangle_{W_2^{1,0}(\Omega)} = \int_\Omega [z_1 z_2 + \frac{\partial z_1}{\partial x}\frac{\partial z_2}{\partial x}]\, dx\, dt,\ \|z\|_{W_2^{1,0}(\Omega)} = \sqrt{\|z\|^2_{L_2(\Omega)} + \left\|\frac{\partial z}{\partial x}\right\|^2_{L_2(\Omega)}}$$

7) $V_2(\Omega)$ is a Banach space consisting of elements of the space $W_2^{1,0}(\Omega)$ with the norm

$$\| z \|_{V_2(\Omega)} = vrai \max_{0 \le t \le T} \| z \|_{L_2(D)} + \sqrt{\int_\Omega \left| \frac{\partial z}{\partial x} \right|^2 dx \ dt}$$

8) $V_2^{1,0}(\Omega)$ is a subspace of $V_2(\Omega)$, the elements of which have in sections $D_t = \{ (x,\tau) : x \in D, \tau \in t \}$ traces from $L_2(D)$ at all $t \in [0,T]$ continuously changing from $t \in [0,T]$ in the norm $L_2(D)$.

Let the controlled process be considered in Ω by the initial boundary value problem for the parabolic equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\lambda(x,t,v_0) \frac{\partial u}{\partial x} \right) = f\left(x,t,u,v_1\right), \ (x,t) \in \Omega \tag{1}$$

$$u(x,0) = \phi(x) \ , x \in D \tag{2}$$

$$\lambda(x,t,v_0) \left. \frac{\partial u}{\partial x} \right|_{x=0} = g_0(t), \quad \lambda(x,t,v_0) \left. \frac{\partial u}{\partial x} \right|_{x=l} = g_1(t) \quad 0 \le t \le T. \tag{3}$$

Here $\phi(x) \in L_2(D), g_m(t) \in L_2(0,T), m = 1,2$ are given functions, the function $f(x,t,u,v_1)$ is measurable in $(x,t) \in \Omega$ and all $(x,t) \in \Omega$ It is continuous and has continuous derivatives in u, v_1 and $\dfrac{\partial f(x,t,u,v_1)}{\partial u}, \dfrac{\partial f(x,t,u,v_1)}{\partial v_1}$ are bounded. Besides, the function $\lambda(x,t,v_0)$ is measurable in $(x,t) \in \Omega$ and all $(x,t) \in \Omega$ it is continuous and has a continuous derivative v_0 and $\dfrac{\partial \lambda(x,t,v_0)}{\partial v_0}$ is bounded.

Let $V = \{ v : v = (v_0(x,t), v_1(x,t)) : v_i \in L_2(\Omega), i = 0,1 : \| v_i \|_{L_2(\Omega)} \le \zeta_i, \zeta_i > 0 \}$ be a space of controls. We consider the following problem: minimize the functional

$$J_\alpha(v) = \beta_0 \int_0^T [u(0,t) - y_0(t)]^2 dt + \beta_1 \int_0^T [u(l,t) - y_1(t)]^2 dt + \alpha \|v - \omega\|^2_{L_2(\Omega)} \tag{4}$$

on the set V under the conditions (1)-(3), where $y_0(t), y_1(t) \in L_2(0,T)$ are given functions, $\alpha \ge 0$ and $\beta_m \ge 0, m = 0,1, \beta_0 + \beta_1 \ne 0$ are given numbers, are also given: $\omega = (\omega_0, \omega_1) \in L_2(\Omega)$.

Under the solution of problem (1)-(3) for given admissible control $v \in V$, we mean a function $u = u(x,t;v) \in V_2^{1,0}(\Omega)$ satisfying the integral identity

$$\int_\Omega \left[u \frac{\partial \eta}{\partial t} + \lambda(x,t,v_0) \frac{\partial u}{\partial x} \frac{\partial \eta}{\partial x} - f(x,t,u,v_1)\, \eta(x,t) \right] dx\, dt$$
$$= \int_0^l \phi(x)\, \eta(x,0)\, dx - \int_0^T g_0(t)\, \eta(0,t)\, dt - \int_0^T g_1(t)\, \eta(l,t)\, dt \tag{5}$$

for all $\eta = \eta(x,t) \in W_2^{1,1}(\Omega)$ that is equal to zero for $t = T$.

2 The Correctness of the Optimal Control Problem

At first we consider the correctness of the boundary value problem (1)-(4) for given $v \in V$.

Proposition 1: On the assumptions of the considering optimal control problem (1)-(4) and from [6] follows that the boundary problem for a given $v \in V$ has existence, uniqueness solution and the following estimation holds:

$$\left| \frac{\partial u(x,t;v)}{\partial x} \right| \le C_1 , \ \forall\, v \in V. \tag{6}$$

Above and everywhere below positive constants are independent of the estimated quantities and admissible controls are denoted by C_m , $m = 1,2,\ldots$.

Further we need the following theorem.

Theorem A:(a corollary of the Goebel theorem [7]).Assume that $\tilde{X}$ is a uniformly convex space, U is a closed bounded set on $\tilde{X}$, a functional $I(v)$ is lower semi continuous and bounded from below on U , and $\alpha > 0$ is a given number. Then there exists a dense subset K of the space $\tilde{X}$ such that for any $\omega \in K$ the functional $J_\alpha(v) = I(v) + \alpha \| v - \omega \|_{\tilde{X}}^2$ attains its minimal value on U at a unique element.

Optimal control problems of the coefficients of differential equations do not always have solution [8]. In this section, we will prove the existence and uniqueness of the solution of problem (1)-(4).

Theorem 1: There is an everywhere dense subset $K \subset L_2(\Omega)$ such that problem (1)-(4) has a unique solution for $\omega \in K$ and $\alpha > 0$.

Proof: The proof of this theorem is divided to three parts: in the first part we prove estimation for $\delta u = \delta u(x,t;v) \in V_2^{1,0}(\Omega)$, second part the continuity of the functional $J_0(v)$ and finally the existence and uniqueness solution of the optimal control (1)-(4) is proved.

Let $\delta v \in L_2(\Omega)$ be an increment of the control at $v \in V$ such that $v + \delta v \in V$. Let $u = u(x,t;v)$ and $\delta u = u(x,t;v+\delta v) + u(x,t;v)$. It is clear that the function $\delta u(x,t;v)$ satisfies the identity

$$\int_{\Omega} \left[-\delta u \frac{\partial \eta}{\partial t} + \lambda (x,t,v_0+\delta v_0) \frac{\partial \delta u}{\partial x} \frac{\partial \eta}{\partial x} + \delta f (x,t,u,v_1) \eta(x,t) \right] dx\, dt = 0 \quad (7)$$

We shall prove the first part that $\delta u(x,t)$ satisfies the following inequality

$$\|\delta u\|_{V_2^{1,0}(\Omega)} \le C_2 \left(\left\| \delta\lambda \frac{\partial \delta u}{\partial x} \right\|^2_{L_2(\Omega)} + \| \delta f \|^2_{L_2(\Omega)} \right)^{\frac{1}{2}}. \quad (8)$$

Let $\overline{\eta_h} = h^{-1} \int\limits_{t-h}^{t} \overline{\eta} (x,\tau)\, d\tau$, $0 < h < T$ where $\overline{\eta}$ is an arbitrary element of $V_2^{1,0}(\Omega_{t_1})$ vanishing for $t > t_1\, (t_1 \le T-h)$ and $\Omega_{t_1} = D \times (0,t_1]$. We replace $\eta(x,t)$ in (7) by $\overline{\eta_h}(x,t)$ and using the result in [6, p. 116-118], we obtain

$$\frac{1}{2} \int_D \left(\delta u(x,t_1)\right)^2 dx + \int_{\Omega_{t_1}} \left[\overline{\lambda} \left(\frac{\partial \delta u}{\partial x}\right)^2 + \delta\lambda \frac{\partial u}{\partial x} \frac{\partial \delta u}{\partial x} + \delta f\, \delta u \right] dx\, dt = 0, \quad (9)$$

$\forall\, t_1 \in [0,T]$, $\overline{\lambda} = \lambda(x,t,v_0+\delta v_0)$.

Hence at the made assumption $|\lambda(x,t,v_0)| < v_0$, $v_0 > 0$ is positive number and applying Cauchy-Bunyakovsky Inequality, we obtain

$$\begin{aligned} &\frac{1}{2} \int_D (\delta u(x,t_1))^2 dx + \gamma_0 \int_{\Omega_{t_1}} \left(\frac{\partial \delta u}{\partial x}\right)^2 dxdt \\ &\le \left\{ \int_{\Omega_{t_1}} \left(\delta\lambda \frac{\partial u}{\partial x}\right)^2 dxdt \right\}^{\frac{1}{2}} \left\{ \int_{\Omega_{t_1}} \left(\frac{\partial \delta u}{\partial x}\right)^2 dxdt \right\}^{\frac{1}{2}} + \left\{ \int_{\Omega_{t_1}} (\delta f)^2 dxdt \right\}^{\frac{1}{2}} \left\{ \int_{\Omega_{t_1}} (\delta u)^2 dxdt \right\}^{\frac{1}{2}} \end{aligned} \quad (10)$$

Take $\varepsilon_1 = \varepsilon_2 = \gamma_0^{-1}$ and apply the Cauchy inequality with $\varepsilon \left(|a\, b| \le \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2 \right)$ and summands on the right hand side of (10); multiplying both sides by two we obtain

$$\begin{aligned} &\| \delta u(x,t_1) \|^2_{L_2(D)} + 2\gamma_0 \left\| \frac{\partial \delta u}{\partial x} \right\|^2_{L_2(\Omega_{t_1})} \\ &\le \frac{2}{\gamma_0} \| \delta u \|^2_{L_2(\Omega_{t_1})} + \frac{2}{\gamma_0} \left(\left\| \delta\lambda \frac{\partial u}{\partial x} \right\|^2_{L_2(\Omega_{t_1})} + \| \delta f \|^2_{L_2(\Omega_{t_1})} \right) \end{aligned} \quad (11)$$

Now we set

$$y(t_1) = \| \delta u(x,t_1) \|^2_{L_2(D)} \ , \quad \Im = \left\| \delta\lambda \ \frac{\partial u}{\partial x} \right\|^2_{L_2(\Omega_{t_1})} + \| \delta f \|^2_{L_2(\Omega_{t_1})}$$

Then the inequality (11) yields the two following inequalities

$$y(t_1) \le C_3 \int_0^{t_1} y(t)\, dt + \frac{2\Im}{\gamma_0} \tag{12}$$

$$\left\| \frac{\partial u}{\partial x} \right\|^2_{L_2(\Omega_{t_1})} \le \frac{2\ C_3}{\gamma_0} \| \delta u \|^2_{L_2(\Omega_{t_1})} + \frac{4\Im}{(\gamma_0)^2} \quad . \tag{13}$$

From the known estimate [6, p. 116-118] it follows that the following three inequalities

$$y(t_1) \le C_4\ \Im\ ,\ \ 0 \le \max_t \le t_1 \left(\| \delta u \|^2_{L_2(D)} \right) \le C_4\ \Im^{\frac{1}{2}}\ ,\ \left\| \frac{\partial u}{\partial x} \right\|^2_{L_2(\Omega_{t_1})} \le C_5\ \Im^{\frac{1}{2}} \tag{14}$$

If we combine the estimates for δu and $\dfrac{\partial u}{\partial x}$, then we obtain

$$\| \delta u \|_{V_2^{1,0}(\Omega_{t_1})} = 0 \le \max_t \le t_1 \left(\| \delta u \|^2_{L_2(D)} \right) + \left\| \frac{\partial u}{\partial x} \right\|^2_{L_2(\Omega_{t_1})} \le C_7\ \Im^{\frac{1}{2}} \tag{15}$$

then the validity of estimation (8) follows.

We prove the second part, under the above assumptions the right side of estimation (8) converges to zero as $\| \delta v \|_{L_2(\Omega)} \to 0$ therefore $\| \delta u \|_{V_2^{1,0}(\Omega)} \to 0$ as $\| \delta v \|_{L_2(\Omega)} \to 0$. Hence from the trace theorem [9] we get

$$\| \delta u(0,t) \|_{L_2(0,T)} \to 0\ ,\ \| \delta u(l,t) \|_{L_2(0,T)} \to 0 \ \ as\ \ \| \delta v \|_{L_2(\Omega)} \to 0 \tag{16}$$

Now we consider the functional of the form

$$J_0(v) = \beta_0 \int_0^T [\ u(0,t) - y_0(t)\ ]^2\, dt + \beta_1 \int_0^T [\ u(l,t) - y_1(t)\ ]^2\, dt \tag{17}$$

Let $\delta v = (\delta v_0, \delta v_1)$ be an increment of control on an element $v \in V$ such that $v + \delta v \in V$. For the increment of $J_0(v)$ we have

$$\begin{aligned} \delta J_0(v) &= 2\beta_0 \int_0^T [\ u(0,t) - y_0(t)\ ]\ \delta u(0,t)\, dt \\ &\quad + 2\beta_1 \int_0^T [\ u(l,t) - y_1(t)\ ]\, \delta u(l,t)\, dt \\ &\quad + \ \beta_0 \int_0^T [\ \delta u(0,t)\ ]^2\ \ dt + \beta_1 \int_0^T [\ \delta u(l,t)\ ]^2\, dt \end{aligned} \tag{18}$$

Applying Cauchy-Bunyakovsky Inequality and the estimation (16), we get the continuity of the functional (17).

Now we prove the last part as follows: hence we get the continuity of the functional $J_0(v)$ on any element $v \in V$, i.e., on the set V. The latter set is a closed bounded convex subset of the uniformly convex space $L_2(\Omega)$ [10]. Then due to Theorem A and the boundedness and continuity of the functional $J_0(v)$ on the set V, there exists an everywhere dense subset K of the space $L_2(\Omega)$ such that $\forall\ \omega \in K$ with $\alpha > 0$ problem (1)–(4) has a unique solution. This completes the proof of Theorem 1.

3 Sufficient differentiability conditions of the functional (4)

Let the following conditions be fulfilled:

Condition A1: The functions $\lambda(x,t,v_0)$, $f(x,t,u,v_1)$ satisfy the Lipschitz condition for $v \in V$.

Condition A2: The first derivatives of the functions $\lambda(x,t,v_0)$, $f(x,t,u,v_1)$ with respect to v_0, v_1 are continuous functions in their domain of definition.

Condition A3: The operators $\dfrac{\partial \lambda(x,t,v_0)}{\partial v_0}$, $\dfrac{\partial f(x,t,u,v_1)}{\partial u}$ and $\dfrac{\partial f(x,t,u,v_1)}{\partial v_1}$ are bounded in $L_2(\Omega)$.

For finding the adjoint system for the problem (1)-(3), we define the Lagrangian function [11] as follows

$$\begin{aligned} L(x,t,v,u,\Theta) = \beta_0 \int_0^T \left[u(0,t) - y_0(t)\right]^2 dt + \beta_1 \int_0^T [\ u(l,t) - y_1(t)]^2\, dt \\ + \alpha\ \|v - w\|^2_{L_2(\Omega)} \\ + \int_\Omega \Theta(x,t) \left[\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\lambda(x,t,v_0) \frac{\partial u}{\partial x}\right) - f(x,t,u,v_1) \right] dx\ dt \end{aligned} \tag{19}$$

Then the problem (1)-(3) we introduce the adjoint state $\Theta(x,t) = \Theta(x,t;v) \in V_2^{1,0}(\Omega)$ as a solution of the problem

$$\frac{\partial \Theta}{\partial t} + \frac{\partial}{\partial x}\left(\lambda(x,t,v_0)\ \frac{\partial \Theta}{\partial x}\right) = -\ \frac{\partial f\left(x,t,u,v_1\right)}{\partial u}\ \Theta(x,t), \quad (x,t) \in \Omega \tag{20}$$

$$\Theta(x,T) = 0\ \ , x \in D \tag{21}$$

$$\lambda(x,t,v_0)\,\frac{\partial\Theta}{\partial x}\bigg|_{x=0} = 2\,\beta_0\,\left[\,u(0,t)-y_0(t)\,\right]$$
$$\lambda(x,t,v_0)\,\frac{\partial\Theta}{\partial x}\bigg|_{x=l} = -2\,\beta_1\,\left[\,u(l,t)-y_1(t)\,\right] \quad 0\le t\le T \tag{22}$$

As a solution of the problem (20)-(22) for the given $v\in V$, we take the function $\Psi=\Psi\;(x,t;v)$ from $W_2^{1,1}(\Omega)$ satisfying the integral identity

$$\int_{\Omega}[\Theta\frac{\partial\Psi}{\partial t}+\lambda(x,t,v_0)\,\frac{\partial\Theta}{\partial x}\frac{\partial\Psi}{\partial x}-\frac{\partial f\left(x,t,u,v_1\right)}{\partial u}\;\Theta(x,t)\;\Psi(x,t)]\;dxdt$$
$$=2\;\beta_0\int_0^T\left[\,u(0,t)-y_0(t)\right]\Theta(0,t)\,dt-2\;\beta_1\int_0^T\left[\,u(l,t)-y_1(t)\right]\Theta(l,t)\,dt \tag{23}$$

for all $\Psi(x,t)=\Psi(x,t;v)\in W_2^{1,0}(\Omega)$ that is equal to zero for $t=0$.

Proposition 2: On the basis of adopted assumptions and the results of [12] follows that for every $v\in V$ the solution of the adjoint problem (21)-(22) is existed, unique and $\left|\frac{\partial\Psi(x,t)}{\partial x}\right|\le C_8$ almost at all $(x,t)\in\Omega$, $\forall\; v\in V$.

The sufficient differentiability conditions of function (4) and its gradient formulae will be obtained by defining the Hamiltonian function $\Pi\,(x,u,\Theta,v)$ as [13]:

$$\Pi(x,t,\Theta,v)=-[\;\lambda(x,t,v_0)\;\frac{\partial u}{\partial x}\frac{\partial\Theta}{\partial x}-f(x,t,u,v_2)\,\Theta\,(x,t;v)+\alpha\sum_{m=0}^{1}(v_m-w_m)^2] \tag{24}$$

Theorem 2: Let the above assumptions A1-A2 be satisfied. Then the functional $J_\alpha(v)$ is continuously differentiable by Fréchetin V , and its gradient satisfies the equality

$$\frac{\partial J_\alpha(v)}{\partial\;v}=-\frac{\partial\,\Pi(x,u,\Theta,v)}{\partial\;v}\equiv(-\frac{\partial\,\Pi(x,u,\Theta,v)}{\partial\;v_0},-\frac{\partial\,\Pi(x,u,\Theta,v)}{\partial\;v_1}) \tag{25}$$

where

$$\frac{\partial\,\Pi(x,u,\Theta,v)}{\partial\;v_0}=-[\frac{\partial\,\lambda\,(x,t,v_0)}{\partial\;v_0}\;\frac{\partial\,u}{\partial\;x}\;\frac{\partial\,\Theta}{\partial\;x}+\;2\,\alpha\;\;(v_0-w_0)\,]$$
$$\frac{\partial\,\Pi(x,u,\Theta,v)}{\partial\;v_1}=-[\frac{\partial\,f\;(x,t,u,v_1)}{\partial\;v_1}\;\Theta\,(x,t)+\;2\,\alpha\;\;(v_1-w_1)\,]$$

Proof: Let $\delta v = (\delta v_0, \delta v_1) \in V$ be an arbitrary increment of the control v such that $v + \delta v \in V$. Using the formula of Lagrange's finite increments one can obtain that the function $\delta u = u(x,t;v+\delta v) - u(x,t;v)$ is a solution from the class $V_2^{1,0}(\Omega)$ of the following boundary value problem:

$$\begin{aligned} &\frac{\partial \delta u}{\partial t} - \frac{\partial}{\partial x}\left(\lambda(x,t,v_0+\delta v_0)\,\frac{\partial \delta u}{\partial x}\right) - \frac{\partial f(\zeta\theta_1)}{\partial u}\delta u \\ &= \frac{\partial}{\partial x}\left([\lambda(x,t,v_0+\delta v_0) - \lambda(x,t,v_0)]\frac{\partial u}{\partial x}\right) + \frac{\partial f(\zeta\theta_2)}{\partial v_1}\delta v_1\,,\ (x,t)\in\Omega \end{aligned} \tag{26}$$

$$\delta u\,(x,0) = 0 \ \ , x \in D \tag{27}$$

$$\left.\begin{aligned} &\left.\left(\delta\lambda\ \frac{\partial u}{\partial x} + \lambda(x,t,v_0+\delta v_0)\ \frac{\partial \delta u}{\partial x}\right)\right|_{x=0} = 0 \\ &\left.\left(\delta\lambda\ \frac{\partial u}{\partial x} + \lambda(x,t,v_0+\delta v_0)\ \frac{\partial \delta u}{\partial x}\right)\right|_{x=l} = 0 \end{aligned}\right\}\ ,\ \ 0 \le t \le T \tag{28}$$

where $\zeta\,\theta_1 = (x,t,u+\theta_1\,\delta u, v_1+\delta v_1)$, $\zeta\,\theta_2 = (x,t,u,v_1+\theta_2\,\delta v_1), \theta_1\,,\ \theta_2 \in [0,1]$ and $\delta\lambda = \lambda(x,t;v_0+\delta v_0) - \lambda(x,t;v_0)$ are some numbers.

Using the conditions A1-A2 and estimating the right hand side of (8) we establish that

$$\|\delta u\|_{V_2^{1,0}(\Omega)} \le C_9\ \|\delta v\|_{L_2(\Omega)} \tag{29}$$

and from the trace theorem and (29) we obtain

$$\|\delta u(0,t)\|_{L_2(\Omega)}^2 + \|\delta u(l,t)\|_{L_2(\Omega)}^2 \le C_{10}\ \|\delta v\|_{L_2(\Omega)}\ . \tag{30}$$

Now, using again the Lagrange formula for the increment of the functional (4) we obtain the formula

$$\begin{aligned} \delta\ J_\alpha(v)\ &= J_\alpha(v+\delta v) - J_\alpha(v) \\ &= 2\beta_0\ \int_0^T [\ u(0,t) - y_0(t)\ ]\ \ \delta u(0,t)dt \\ &+ 2\ \beta_1\ \int_0^T [\ u(l,t) - y_1(t)\ ]\ \delta u(l,t)\ dt \\ &+ 2\alpha \int_\Omega [v - \omega]\ \delta v\ \ dx\ dt \end{aligned} \tag{31}$$

where

$$R_1(\delta v) = \beta_0 \int_0^T [\, \delta u(0,t)]^2 \, dt + \beta_1 \int_0^T [\, \delta u(l,t)]^2 \, dt + \alpha \int_\Omega [\delta v]^2 dx\, dt . \quad (32)$$

If in (7), we put $\eta = \Theta(x,t;v)$ and $\Psi = \delta u(x,t;v)$ in (23) and subtract the obtained the results, we obtain

$$\begin{aligned} &2\beta_0 \int_0^T [\, u(0,t) - y_0(t)] \,\, \delta u(0,t)dt + 2\,\beta_1 \int_0^T [\, u(l,t) - y_1(t)] \,\, \delta u(l,t)\, dt \\ &= \int_\Omega \left(\lambda(x,t,v_0 + \delta v_0) \frac{\partial u}{\partial x} \frac{\partial \Theta}{\partial x} - f(x,t,u,v_1 + \delta v_1) \,\, \Theta(x,t) \right) dx\, dt + R_2(\delta v) \end{aligned} \quad (33)$$

where

$$\begin{aligned} R_1(\delta v) = &\int_\Omega [\, \lambda(x,t,v_0 + \delta v_0)\, \frac{\partial \delta u}{\partial x} \frac{\partial \Theta}{\partial x}]\, dx\, dt \\ &+ \int_\Omega [\, \frac{\partial f(\zeta \,\, \theta_1)}{\partial u} - \frac{\partial f(x,t,u,v_1)}{\partial u}]\, \delta u(x,t) \,\, \Theta(x,t) \,\, dx \,\, dt \end{aligned} \quad (34)$$

$$\begin{aligned} \lambda(x,t,v_0 + \delta\, v_0) &= \left\langle \frac{\partial\, \lambda(x,t,v_0)}{\partial\, v_0} ,\, \delta v_0 \right\rangle + O\left(\left\| \delta v_0 \right\|_{L_2(\Omega)} \right) \\ f(x,t,u,v_1 + \delta\, v_1) &= \left\langle \frac{\partial\, f(x,t,u,v_1)}{\partial\, v_1} ,\, \delta v_1 \right\rangle + O\left(\left\| \delta v_1 \right\|_{L_2(\Omega)} \right) \end{aligned} \quad (35)$$

Taking and from (35) and (31), we obtain

$$\delta\, J_\alpha(v) = \int_\Omega \left\langle \frac{\partial \lambda}{\partial v_0} \frac{\partial \Theta}{\partial x} - \frac{\partial f}{\partial v_1} \,\, \Theta(x,t) \, + \, 2 \,\, \alpha \,\, (v - \omega), \delta v \right\rangle_{L_2(\Omega)} + R_3(\delta v) \quad (36)$$

and from the formulae of $R_3(\delta v) = R_1(\delta v) \, + \, R_2(\delta v) + \, O\,(\left\| \delta v \right\|_{L_2(\Omega)})$ we obtain

$$\left| R_3(\delta v) \right| \le C_{11} \, \left\| \delta v \right\|^2_{L_2(\Omega)} \quad (37)$$

Hence, in the right hand side of the expression for the Hamilton-Pontryagin function, we obtain

$$\delta\, J_\alpha(v) = \int_\Omega \left\langle - \frac{\partial\, \Pi(x,t,u,\Theta,v)}{\partial\, v} \,, \delta v \right\rangle_{L_2(\Omega)} + O\left(\left\| \delta v \right\|_{L_2(\Omega)} \right) \quad (38)$$

And this proves the Fréchet differentiability of the functional (4) and also gives its gradient formulae. This completes the proof of the theorem.

4 Conclusion

In [4], the existence and uniqueness theorem is proved under constrained problem with control space E_N but here the space of controls $v \in L_2(\Omega)$ are respected to x,t. Also, in [4] the coefficient of higher-order derivatives was $\lambda(u(x,t),v)$, but in the present paper it is $\lambda(x,t,v)$. In this paper, an optimal control parabolic problem is studied. The existence and uniqueness theorem for the solving optimal control parabolic problem is proved. Also, a theorem for the sufficient differentiability conditions of the functional and its gradient formulae has been proved. A parabolic optimal boundary control problem is accepted for publishing in International Journal of Computational Engineering Research. Therefore, the numerical solution of the considering optimal control problem will be reported later.

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Competing Interests

Authors have declared that no competing interests exist.

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