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Orthogonal Double Covers of $K_{n,n}$ by Infinite Classes of Disjoint Unions of Certain Complete Bipartite Spanning Subgraphs

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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ABSTRACT

An *orthogonal double cover* (ODC) of any graph H is a collection of spanning subgraphs of H such that every two of them share exactly one edge at most, and every edge of the graph H belongs to exactly two of the spanning graphs. In this paper, we are concerned with the orthogonal double covers of complete bipartite graphs by infinite classes of disjoint unions of certain complete bipartite spanning subgraphs.

Keywords: Orthogonal double cover; symmetric starter.

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1. INTRODUCTION

A generalization of notion of an ODC[1] to arbitrary underlying graphs is as follows. Let H be an arbitrary graph with n vertices and let $\mathcal{G} = \{G_0, G_1, \ldots, G_{n-1}\}$ be a collection of n

[1] *Orthogonal double cover.*

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isomorphic spanning subgraphs of H. $\mathcal{G}$ is called an ODC of H if there exists a bijective mapping $\varphi: V(H) \rightarrow \mathcal{G}$ such that:

(1) Every edge of H is contained in exactly two of the graphs $G_0, G_1, \ldots, G_{n-1}$
(2) For every choice of different vertices a, b of H,

$$|E(\varphi(a)) \cap E(\varphi(b))| = \begin{cases} 1 \text{ if } \{a,b\} \in E(H), \\ 0 \text{ otherwise.} \end{cases}$$

In this paper, we assume $H = K_{n,n}$, the complete bipartite graph with partition sets of size n each. A graph G is called a spanning subgraph of H if G contains all the vertices of H and the edges of G is a sub set of the edges of H.

Here, we use the usual notation: $K_{m,n}$ for the complete bipartite graph with partition sets of sizes m and n, P_n for the path on n vertices, C_n for the cycle on n vertices, K_n for the complete graph on n vertices, K_1 for an isolated vertex, and lG for l disjoint copies of G, $F+G$ for the disjoint union of F and G, for an illustration of disjoint union of graphs see Fig. 1.

An algebraic construction of ODCs via *symmetric starters* has been exploited to get a complete classification of ODCs of $K_{n,n}$ by G for $n \leq 9$: a few exceptions apart, for all graphs G are found by this way (see [4], Table 1).

This method has been applied in [2,4] to detect some infinite classes of graphs G for which there are ODCs of $K_{n,n}$ by G. In [5] El Shanawany et al. got some results for ODCs by using the cartesian product of two symmetric starter vectors.

Also in [6] El Shanawany et al. got some results for ODCs by using the cartesian product of three symmetric starter vectors.

All graphs here are finite, simple, and undirected. Let $\Gamma = \{\gamma_0, \ldots, \gamma_{n-1}\}$ be an (additive) abelian group of order n. The vertices of $K_{n,n}$ will be labeled by the elements of $\Gamma \times \mathbb{Z}_2$. Namely, for $(v,i) \in \Gamma \times \mathbb{Z}_2$, we will write v_i for the corresponding vertex and define $\{a_i, b_j\} \in E(K_{n,n})$ if and only if $i \neq j$, for all $a, b \in \Gamma$ and $i, j \in \mathbb{Z}_2$. If there is no chance of confusion (a,b) will be written instead of $\{a_0, b_1\}$ for the edge between the vertices a_0, b_1.

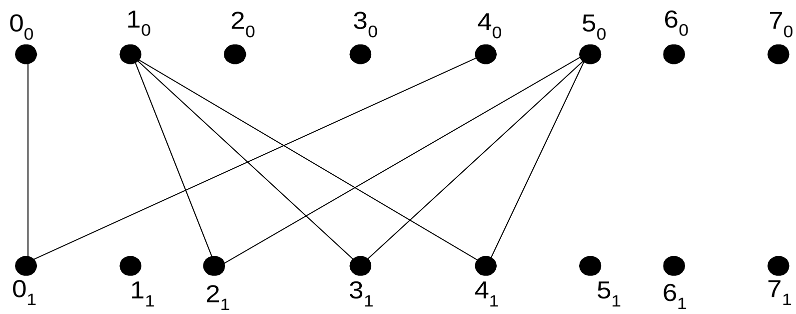

Fig. 1. Disjoint union of $K_{1,2}$ and $K_{2,3}$

Let G be a spanning subgraph of $K_{n,n}$ and let $x \in \Gamma$. Then the graph $G + x$ with $E(G+x) = \{(u+x, v+x) : (u,v) \in E(G)\}$ is called the x-translate of G. The length of an edge $e = \{(u,v) \in E(G)\}$ is defined by $d(e) = v - u$. Note that sums and differences are calculated modulo n.

G is called a *half starter* with respect to Γ if $|E(\mathrm{G})| = n$ and the lengths of all edges in G are mutually distinct; that is, $\{d(e) : e \in E(G)\} = \Gamma$.

Hereafter, a half starter will be represented by the vector $v(G) = (v_{\gamma_0}, v_{\gamma_1}, \ldots, v_{\gamma_{n-1}}) \in \Gamma^n$. Where $v_{\gamma_i} \in \Gamma$ and $(v_{\gamma_i})_0$ is the unique vertex $((v_{\gamma_i}, 0) \in \Gamma \times \{0\})$ that belongs to the unique edge of length γ_i in G.

Two half starter vectors $v(G_0)$ and $v(G_1)$ are said to be orthogonal if $\{v_\gamma(G_0) - v_\gamma(G_1) : \gamma \in \Gamma\} = \Gamma$. The subgraph G_s of $K_{n,n}$ with $E(G_s) = \{\{u_0, v_1\} : \{v_0, u_1\} \in E(G)\}$ is called the symmetric graph of G. Note that if G is a half starter, then G_s is also a half starter . A half starter G is called a symmetric starter with respect Γ if $v(G)$ and $v(G_s)$ are orthogonal. In [4], the following three results were proved.

I. If G is a half starter, then the union of all translates of G forms an edge decomposition of $K_{n,n}$. That is, $\bigcup_{x \in \Gamma} E(G+x) = E(K_{n,n})$.

II. If two half starters $v(G_0)$ and $v(G_1)$ are orthogonal, then $\mathcal{G} = \{G_{x,\,i} \,:\, (x,i) \in \Gamma \times \mathbb{Z}_2\}$ with $G_{x,\,i} = (G_i + x)$ is an ODC of $K_{n,n}$.

III. Let n be a positive integer and let G a half starter represented by $v(G)=(v_{\gamma_0},v_{\gamma_1},\ldots,v_{\gamma_{n-1}})$. Then G is symmetric starter if and only if $\{v_\gamma - v_{-\gamma}+\gamma:\gamma\in\Gamma\}=\Gamma$.

The research on this subject focused on the detection of ODCs with pages (Elements of the collection $\mathcal{G}$) isomorphic to a given graph G. For a summary of results on ODCs, see [1,2]. The other undefined terms have usual meaning and their definitions can be found in [3].

Fig. 2 shows the *ODC* of $K_{3,3}$ by $G=P_4$ (path with three edges) as an example for an illustration.

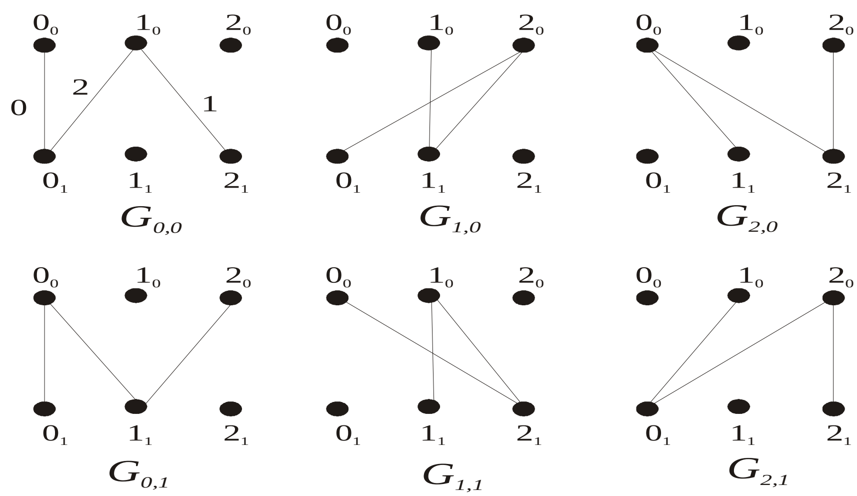

Fig. 2. ODC of $K_{3,3}$ by $G=P_4$ with $\Gamma=\mathbb{Z}_3$

Infinite classes of disjoint unions of certain complete bipartite spanning subgraphs mean at every positive integer we will get acertain result.

2. ODC OF $K_{r,r}$ BY INFINITE CLASSES OF DISJOINT UNIONS OF CERTAIN COMPLETE BIPARTITE SPANNING SUBGRAPHS

For any positive integer r, the following Theorem construct an ODC of $K_{r,r}$ by disjoint unions of trees and by disjoint unions of trees and complete bipartite graphs.

Theorem 2.1. *Let r, s, t, u, v, p, and q be positive integers. Then there is an ODC of $K_{r,r}$ by $pK_{u,s}+qK_{v,t}+\left(2r-p(u+s)-q(v+t)\right)K_1$.*

Proof.
Case 1. When r is a prime number this case is trivial, as all regular graphs of degree r have an ODC by $K_{1,r}$, see [4].

Case 2. $u = 1, v = 1, r$ is not a prime number

Subcase 2.1. $r = mn,\ p = q = 1, s + t = mn.$

for $m \geq 3, n = m - 2,$ define $v(G)$ by $v_i = 0$ if $i = 0 \bmod m, v_i = m$ otherwise. Then, for all $i \in \mathbb{Z}_{mn}$, $v_i - v_{-i} + i = i.$ Therefore the result (iii) proves that $v(G)$ is a symmetric starter vector. From the definition of $v(G),$ we get

$$E(G) = \bigcup_{\alpha=0}^{n-1} \{(0, m\alpha), (m, m\alpha + \gamma) : 1 \leq \gamma \leq m - 1\},$$

and hence G is isomorphic to $K_{1,m-1} + K_{1,m(n-1)+1} + (mn - 2)K_1.$

Subcase 2.2. $r = 2mn,\ p = q = m, s + t = 2n.$

For any positive integers n, m with $\gcd(m,3) = 1,$ define $v(G)$ by $v_i = 0$ if $i = 0 \bmod 2m, v_i = 2i$ if $0 < i < 2m, v_i = 2i - 4m$ if $2m < i < 4m, v_i = 2i - 8m$ if $4m < i < 6m,\ v_i = 2i - 12m$ if $6m < i < 8m, \ldots, v_i = 2i - (4mn - 8m)$ if $2mn - 4m < i < 2mn - 2m, v_i = 2i - (4mn - 4m)$ if $2mn - 2m < i < 2mn.$
Then,

for $i = 0 \bmod 2m, v_i - v_{-i} + i = i;$ and for $0 < i < 2m,\ v_i - v_{-i} + i = 5i - 4m;$ and for

$2m < i < 4m, v_i - v_{-i} + i = 5i - 12m;$ and for $4m < i < 6m,$

$v_i - v_{-i} + i = 5i - 20m, \ldots,$

and for $2mn - 2m < i < 2mn, v_i - v_{-i} + i = 5i - 8mn - 12m.$

Therefore the result (iii) proves that $v(G)$ is a symmetric starter vector. From the definition of $v(G),$ we get

$$E(G) = \bigcup_{\alpha=0}^{2m-1} \{(2\alpha, 2m\gamma + 3\alpha) : 0 \leq \gamma \leq n - 1\},$$

and hence G is isomorphic to $2mK_{1,n} + 2m(n-1)K_1.$

Subcase 2.3. $r = 2n,\ p = q = 1, s + t = 2n.$

For any positive integer $n,$ define $v(G)$ by $v_i = 0$ if $i = 0,\ n,\ v_i = n$ otherwise. Then, for all $i \in \mathbb{Z}_{2n},\ v_i - v_{-i} + i = i.$ Therefore the result (iii) proves that $v(G)$ is a symmetric starter vector. From the definition of $v(G),$ we get

$$E(G) = \bigcup_{\alpha=1, \alpha\neq n}^{2n-1} \{(0,0),\ (0,n),\ (n,\alpha)\},$$

and hence G is isomorphic to $K_{1,2} + K_{1,2n-2} + 2(n-1)K_1.$

Case 3. $r = 2n, u = 2, v = 1,\ p = q = 1, s + t = n + 1.$

For any positive $n,$ and $n > 1,$ define $v(G)$ by $v_i = 0$ if $i = 0, v_i = 1$ if $1 \le i \le n-1,$ $v_i = n$ if $i = n, v_i = n+1$ otherwise. Then, for $i \in \{0,\ n\},$ $v_i - v_{-i} + i = i;$ $v_i - v_{-i} + i = n + i$ otherwise . Therefore the result (iii) proves

that $v(G)$ is a symmetric starter vector. From the definition of $v(G),$ we get

$$E(G) = \bigcup_{\alpha=2}^{n} \{(0,0),\ (1,\alpha),\ (n,0),\ (n+1,\alpha)\},$$

and hence, G is isomorphic to $K_{2,n-1} + K_{1,2} + (3n-4)K_1.$ ■

For any positive integer $r,$ the following Theorem construct an ODC of $K_{r,r}$ by disjoint unions of complete bipartite graphs

Theorem 2.2. *Let* $r,\ s,\ t,$ *and* v *be positive integers. Then there is an ODC of* $K_{r,r}$ *by* $sK_{v,t} + \left(2r - s\left(v+t\right)\right)K_1.$

Proof.
Case 1. $r = 2mn, s = m, t = n, v = 2.$

For any positive integers n, m with $\gcd(m,3) = 1,$ define $v(G)$ by $v_i = 0$ if

$i = 0 \bmod 2m$, $v_i = i$ if $0 < i < 2m$, $v_i = i - 2m$ if $2m < i < 4m$, $v_i = i - 4m$ if $4m < i < 6m$, $v_i = i - 6m$ if $6m < i < 8m, \ldots, v_i = i - (2mn - 4m)$ if $2mn - 4m < i < 2mn - 2m$, $v_i = i - (2mn - 2m)$ if $2mn - 2m < i < 2mn.$
Then, for $i = 0 \bmod 2m$, $v_i - v_{-i} + i = i;$ and for $0 < i < 2m$, $v_i - v_{-i} + i = 3i - 2m;$

and for $2m < i < 4m$, $v_i - v_{-i} + i = 3i - 6m;$ and for $4m < i < 6m$,

$v_i - v_{-i} + i = 3i - 10m;$ and for $6m < i < 8m$, $v_i - v_{-i} + i = 3i - 14m, \ldots,$ and for $2mn - 4m < i < 2mn - 2m$, $v_i - v_{-i} + i = 3i - (4mn - 6m);$ and for $2mn - 2m < i < 2mn$, $v_i - v_{-i} + i = 3i - (4mn - 2m).$ Therefore the result

(iii) proves that $v(G)$ is a symmetric starter vector.

From the definition of $v(G)$,

$$E(G) = \bigcup_{\gamma=0}^{n-1} \{(\alpha + \beta, 2m\gamma + 2\beta) : \alpha \in \{0, m\}, 0 \le \beta \le m - 1\},$$

and hence G is isomorphic to $mK_{2,n} + m(3n-2)K_1.$

The following case shows also an ODC of $K_{2mn,2mn}$, but by a new symmetric starter.

Case 2. $r = 2mn, s = \frac{m}{2}, t = n, v = 4.$

For any positive integers n, m with $\gcd(m,3) = 1,$ and m is even, define $v(G)$ by

$v_i = 0$ if $i = 0 \bmod 2m$, $v_i = 3i$ if $0 < i < 2m$, $v_i = 3i - 6m$ if $2m < i < 4m$, $v_i = 3i - 12m$ if $4m < i < 6m, \ldots,$ $v_i = 3i - (6mn - 12m)$ if $2mn - 4m < i < 2mn - 2m$, $v_i = 3i - (6mn - 6m)$ if $2mn - 2m < i < 2mn.$
Then, for $i = 0 \bmod 2m$, $v_i - v_{-i} + i = i;$ and for $0 < i < 2m$, $v_i - v_{-i} + i = 7i - 6m;$ and for $2m < i < 4m$, $v_i - v_{-i} + i = 7i - 18m;$ and for $4m < i < 6m$, $v_i - v_{-i} + i = 7i - 30m, \ldots,$ and for $2mn - 2m < i < 2mn$, $v_i - v_{-i} + i = 7i - (12mn - 6m).$ Therefore the result (iii) proves that $v(G)$ is a symmetric starter vector. From the definition of $v(G)$, we get

$$E(G)=\bigcup_{\alpha=0}^{\frac{m}{2}-1}\left\{(\frac{3}{2}m\gamma+3\alpha,2m\beta+4\alpha):0\le\beta\le n-1,0\le\gamma\le 3\right\},$$

and hence G is isomorphic to $\frac{m}{2}K_{4,n}+\frac{m}{2}(7n-4)K_1$. ■

The following conjecture generalizes the above results.

Conjecture. 2.3 *Let* α_i, β_i, δ_i *and* r *be positive integers. Then there is an ODC of* $K_{r,r}$ *by* $\sum_{i=0}^{m}[\alpha_i K_{\beta_i,\delta_i}+\left(2r-\ \alpha_i\left(\beta_i+\delta_i\right)\right)K_1]$, *where* m *is the number of the disjoint complete bipartite graphs.*

3. CONCLUSION

We get an ODC of $K_{r,r}$ by infinite classes of disjoint unions of certain complete bipartite spanning subgraphs which are summarized in the following Table.

ODC of	by
$K_{mn,mn}$	$K_{1,m-1}+K_{1,m(n-1)+1}+(mn-2)K_1$
$K_{2mn,2mn}$	$2mK_{1,n}+2m(n-1)K_1$
$K_{2n,2n}$	$K_{1,2}+K_{1,2n-2}+2(n-1)K_1$
$K_{2n,2n}$	$K_{2,n-1}+K_{1,2}+(3n-4)K_1$
$K_{2mn,2mn}$	$mK_{2,n}+m(3n-2)K_1$
$K_{2mn,2mn}$	$\frac{m}{2}K_{4,n}+\frac{m}{2}(7n-4)K_1$

Comparative analysis of our work and the results given in [5].

Our work	Results given in [5]
$K_{1,m-1}+K_{1,m(n-1)+1}+(mn-2)K_1$; $m\ge 3,\ n=m-2.$	$K_{1,2m}\cup K_{1,2m(n-1)}\cup 2(mn-1)K_1$; m any positive integer, $n\ge 2.$
$2mK_{1,n}+2m(n-1)K_1.$	-
$K_{1,2}+K_{1,2n-2}+2(n-1)K_1.$	-
$K_{2,n-1}+K_{1,2}+(3n-4)K_1.$	-
$mK_{2,n}+m(3n-2)K_1$; n any positive integer, $\gcd(m,3)=1.$	$mK_{2,2n}\cup 2m(3n-1)K_1$; n any positive integer, $\gcd(m,3)=1,$ but this result gives $mK_{2,\lambda}$ where λ is an even number.
$\frac{m}{2}K_{4,n}+\frac{m}{2}(7n-4)K_1.$	-

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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