British Journal of Mathematics & Computer Science
4(9): 1268-1279, 2014

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A Combinatorial Approach for q-Analogue of r-Stirling Numbers

Roberto B. Corcino* and Jezer C. Fernandez
[1] *Department of Mathematics, Mindanao State University*
Marawi City, Philippines 9700, Philippines.

Original Research Article

Received: 27 November 2013
Accepted: 18 February 2014
Published: 01 March 2014

Abstract

We define a q-analogue of r-Stirling numbers of the second kind using their combinatorial interpretation in terms of set partition. Some properties are obtained including recurrence relation, explicit formula and certain symmetric formula. Moreover, a q-analogue of r-Stirling numbers of the first kind is introduced to obtain a q-analogue of the orthogonality and inverse relations of the two kinds of r-Stirling numbers.

Keywords: Stirling numbers; r-Stirling Numbers; q-binomial coefficients; q-factorial
2010 Mathematics Subject Classification: 05A10; 11B73; 11B65

1 Introduction

Several generalizations of Stirling numbers have appeared in the literature. Almost all the generalizations of Stirling numbers have been listed in [1]. One of these is the r-Stirling numbers of the first and second kind in [2] which are defined, respectively, as follows

$\begin{bmatrix} n \\ k \end{bmatrix}_r$:= number of permutations of the set $\{1, 2, \cdots, n\}$ into k nonempty disjoint cycles,

such that the numbers $1, 2, \cdots, r$ are in distinct cycles.

$\begin{Bmatrix} n \\ k \end{Bmatrix}_r$:= number of partitions of the set $\{1, 2, \cdots, n\}$ into k nonempty disjoint classes

(or blocks), such that the numbers $1, 2, \cdots, r$ are in distinct classes (or blocks).

**Corresponding author: E-mail: rcorcino@yahoo.com*

Detailed discussion on r-Stirling numbers and some related works can be found in [2, 3, 4]. Recently, the r-Stirling numbers of the second kind have been generalized further in [5] by replacing the condition

the numbers $1, 2, \cdots, r$ are in distinct classes (or blocks)

with the condition

for given subsets $R_1, \ldots, R_r$ of $\{1, 2, \ldots, n\}$ where $|R_i| = r_i$ and $R_i \cap R_j = \emptyset$, for all $i, j = 1, \ldots, r$ $i \neq j$, the elements of each subsets R_i, $i = 1, \ldots, r$ are in distinct classes (or blocks).

This generalization of r-Stirling numbers of the second kind is called the $(r_1, \ldots, r_r)$-Stirling numbers of the second kind.

On the other hand, Certain generalization of Stirling numbers has been defined in [6] by considering the normal ordering of powers $(VU)^n$ of the noncommuting variables U and V satisfying $UV = VU + hV^s$ where $h \in \mathbb{C} - \{0\}$ and $s \in \mathbb{N}_0$. More precisely,

$$(VU)^n = \sum_{k=1}^{n} \mathfrak{S}_{s;h}(n,k) V^{s(nk)+k} U^k$$

where $\mathfrak{S}_{s;h}(n,k)$ denotes their generalized Stirling numbers. In [7], the numbers $\mathfrak{S}_{s;h}(n,k)$ were expressed in terms of the unified generalization of Stirling numbers in [1]. This result was used to derive more properties for $\mathfrak{S}_{s;h}(n,k)$. Further investigation of these numbers has been done in [8] by considering the particular case $s = 2$ corresponding to the meromorphic Weyl algebra.

One of the outgrowths in generalizing Stirling numbers is the introduction of their q-analogues. The study of q-analogue has become more popular nowadays due to its application in physics and other areas in mathematics, particularly, in the study of fractals, dynamical system, quantum groups, q-deformed superalgebras, fermionic oscillator, creation-annihilation principle and Ising model. There are two main classification of q-analogues: the combinatorial q-analogues and the q-analogues extended by F.H. Jackson [9]. This present study can be classified as part of combinatorial q-analogues.

A q-analogue of a number, polynomial, theorem, identity or expression is a generalization involving a new parameter q such that when $q \to 1$, it gives back the original number, polynomial, theorem, identity or expression. For instance, a given polynomial $a_k(q)$ is a q-analogue of an integer a_k if

$$\lim_{q \to 1} a_k(q) = a_k.$$

Hence, the polynomials

$$[n]_q = \frac{q^n - 1}{q - 1}, \quad [n]_q! = \prod_{i=1}^{n} [i]_q, \quad \binom{n}{k}_q = \prod_{i=1}^{k} \frac{q^{n-i+1} - 1}{q^i - 1}$$

are the q-analogues of the integers n, $n!$, and $\binom{n}{k}$. It is important to note that a q-analogue of a number, polynomial, theorem, identity or expression is not unique. For example, a q-analogue of the classical Stirling numbers has been defined by some authors in different manner (cf [10, 11]). In 1992, a new q-analogue of Stirling numbers has been defined by Cigler in [12] using the concept of set partitions (see also [13]). This is closely related to the q-Stirling numbers defined in [14] in three different ways using generating functions. This work of Cigler motivates the present authors to define a q-analogue of r-Stirling numbers of the second kind using their combinatorial interpretation in terms of set partitions. Moreover, a q-analogue of r-Stirling numbers of the first kind is defined by means of certain generating function, which, consequently, gives the orthogonality and inverse relations of the q-analogue of both kinds of r-Stirling numbers.

2 A *q*-Analogue of *r*-Stirling Numbers of the Second Kind

The classical Stirling numbers of the second kind $S(n,k)$ were defined in [15] as the cardinality of set B of partitions of $\{0,1,2,\cdots,n-1\}$ into k nonempty disjoint subsets. Based on this definition, a q-analogue of $S(n,k)$ was defined in [12] to be the following sum

$$\sum_{\pi\in B} w(\pi), \quad w(\pi) = q^{\sum_{i\in B_0} i}$$

where B_0 is a subset in partition π which contains 0.

On the other hand, the above definition of r-Stirling numbers of the second kind $\left\{ n \atop k \right\}_r$ can be restated as follows:

$\left\{ n \atop k \right\}_r$:= number of partitions π of $\{0,1,...,n-1\}$ into k nonempty subsets $B_0, B_1, ..., B_{k-1}$ such that the first r elements are in distinct subsets.

In this section, a q-analogue of r-Stirling numbers of the second kind will be defined parallel to the work of Cigler. First, we choose B_0 so that the number $0 \in B_0$. Then, let us define the following notations:

- the weight of partition π
$$w(\pi) = q^{s(B_0)}, \;\; s(B_0) = \sum_{i\in B_0} i.$$
- the weight of each set of partitions A
$$w(A) := \sum_{\pi\in A} w(\pi)$$
- $A_{n,k,r}$:= the set of all partitions of $0,1,...,n-1$ into k nonempty parts such that the first r elements are in distinct partitions.

Now, we have the following definition:

Definition 2.1. A q-analogue $\left\{ n \atop k \right\}_{q,r}$ of r-Stirling number of the second kind is defined by

$$\left\{ n \atop k \right\}_{q,r} := w(A_{n,k,r}) \quad n,k \geq 1, \;\; n \geq k \geq r$$

where $\left\{ 0 \atop k \right\}_{q,r} := \delta_{0k}$ and $\left\{ n \atop 0 \right\}_{q,r} := \delta_{0n}, \quad n,k \geq 0$

Remark 2.1. We choose the above weight function so that, when $q = 1$,

$$\left\{ n \atop k \right\}_{1,r} = |A_{n,k,r}| = \left\{ n \atop k \right\}_r .$$

Moreover, the above weight function is a kind of variation of the weight function corresponding to the q-Stirling numbers of the second kind in [11] resulting to a new q-analogue of second kind Stirling-type numbers. One may also try to define a q-analogue of r-Stirling numbers of the second kind using the weight function in terms of non-inversion numbers.

When $n = 4, k = 3$ and $r = 2$, we have the following partitions of $\{0, 1, 2, 3\}$:

$$A_{4,3,2} = \{\{0\}\{1\}\{2,3\}\}, \{\{0,2\}\{1\}\{3\}\}, \{\{0\}\{1,2\}\{3\}\}, \{\{0,3\}\{1\}\{2\}\}, \{\{0\}\{1,3\}\{2\}\}.$$

Then

$$\left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\}_{q,2} = q^0 + q^{0+2} + q^0 + q^{0+3} + q^0 = 3 + q^2 + q^3.$$

To compute quickly the first values of the q-analogue, let us consider the following recurrence relation:

Theorem 2.1. *The number* $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{q,r}$ *satisfy the following recurrence relation*

$$\left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\}_{q,r} = \left\{ \begin{matrix} n \\ k-1 \end{matrix} \right\}_{q,r} + (k - 1 + q^n) \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{q,r}$$

where $n \geq k \geq r \geq 0$.

Proof. We write $A_{n+1,k,r} = C_1 \cup C_2 \cup C_3$ such that

- C_1 is the set of all $\pi \in A_{n+1,k,r}$ such that $\{n\}$ is one of the nonempty parts of π.
- C_2 is the set of all π such that $n \in B_i$, $i \neq 0$, and $B_i \backslash \{n\} \neq \phi$.
- C_3 is the set of all π such that $n \in B_0$.

Then we have

$$w(C_1) = \left\{ \begin{matrix} n \\ k-1 \end{matrix} \right\}_{q,r}, w(C_2) = (k-1) \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{q,r}, \text{ and } w(C_3) = q^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{q,r}.$$

□

Using this recurrence relation, we can generate the first values of the q-analogue.

The next theorem contains an explicit formula for $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{q,r}$, which is analogous to certain identity in [2]. But before that, let us consider first the following lemma.

Lemma 2.2.

$$\sum_{r \leq j_1 < j_2 < \cdots < j_i \leq n} q^{j_1 + j_2 + \cdots + j_i} = \binom{n-r+1}{i}_q q^{\frac{i(i-1+2r)}{2}}.$$

Proof. Note that from [16]

$$(a + x)(a + qx) \cdots (a + q^{n-r}x) = \sum_{i=0}^{n-r+1} \binom{n-r+1}{i}_q q^{\binom{i}{2}} x^i a^{n-r+1-i}$$

Replacing x by $q^r x$, we have

$$\begin{aligned}(a+q^r x)(a+q^{r+1}x)\cdots(a+q^n x) &= \sum_{i=0}^{n-r+1}\binom{n-r+1}{i}_q q^{\binom{i}{2}} q^{ri} x^i a^{n-r+1-i}\\ &= \sum_{i=0}^{n-r+1}\binom{n-r+1}{i}_q q^{\frac{i(i-1)}{2}} q^{ri} x^i a^{n-r+1-i}\\ &= \sum_{i=0}^{n-r+1}\binom{n-r+1}{i}_q q^{\frac{i(i-1+2r)}{2}} x^i a^{n-r+1-i}\end{aligned}$$

And comparing the coefficients of x^i at $a=1$ gives

$$\sum_{i=0}^{n-r+1}\left(\sum_{r\le j_1<j_2<\cdots<j_i\le n} q^{j_1+j_2+\cdots+j_i}\right)x^i = \sum_{i=0}^{n-r+1}\binom{n-r+1}{i}_q q^{\frac{i(i-1+2r)}{2}} x^i$$
$$\sum_{r\le j_1<j_2<\cdots<j_i\le n} q^{j_1+j_2+\cdots+j_i} = \binom{n-r+1}{i}_q q^{\frac{i(i-1+2r)}{2}}$$

□

Writing $\pi\in A_{n+1,k+1}$ in the form

$$\pi=\{0,j_1,j_2,\cdots,j_i\}/B_1/\cdots/B_k$$

where $j_l\neq 1,2,\ldots,r-1$, we get therefore

$$\begin{aligned}\left\{\begin{matrix}n+1\\k+1\end{matrix}\right\}_{q,r} &= w(A_{n+1,k+1})=\sum_{\pi\in A_{n+1,k+1}} w(\pi)\\ &= \sum_{i=0}^{n}\sum_{r\le j_1<\cdots<j_i\le n} q^{j_1+\cdots+j_i}\left\{\begin{matrix}n-i\\k\end{matrix}\right\}_{r-1}\end{aligned}$$

Thus, using Lemma 2.2, we obtain the following explicit formula.

Theorem 2.3. *The explicit formula for* $\left\{\begin{matrix}n+1\\k+1\end{matrix}\right\}_{q,r}$ *is given by*

$$\left\{\begin{matrix}n+1\\k+1\end{matrix}\right\}_{q,r}=\sum_{i=0}^{n}\binom{n-r+1}{i}_q q^{\frac{i(i-1+2r)}{2}}\left\{\begin{matrix}n-i\\k\end{matrix}\right\}_{r-1}. \tag{2.1}$$

Remark 2.2. Equation (2.1) is a q-analogue of the identity in [2], which is given by

$$\left\{\begin{matrix}n\\m\end{matrix}\right\}_r=\sum_k\binom{n-r}{k}\left\{\begin{matrix}n-p-k\\m-p\end{matrix}\right\}_{r-p} p^k,$$

when $p=1$.

Remark 2.3. The r-Stirling numbers of the second kind in [2] satisfy the following exponential generating function

$$\sum_{k\geq 0}\left\{\begin{matrix}k+r\\ m+r\end{matrix}\right\}_r\frac{z^k}{k!}=\frac{1}{m!}e^{rz}(e^z-1)^m.$$

Using the Binomial Theorem and the expansion of exponential function, this can be expressed further as

$$\sum_{k\geq 0}\left\{\begin{matrix}k+r\\ m+r\end{matrix}\right\}_r\frac{z^k}{k!}=\sum_{k\geq 0}\left\{\frac{1}{m!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(r+j)^k\right\}\frac{z^k}{k!}.$$

This implies that

$$\left\{\begin{matrix}k+r\\ m+r\end{matrix}\right\}_r=\frac{1}{m!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(r+j)^k. \tag{2.2}$$

This formula can also be obtained via (r,β)-Stirling numbers $\left\langle\begin{matrix}k\\ m\end{matrix}\right\rangle_{\beta,r}$ in [17] by taking $\beta=1$. That is,

$$\left\{\begin{matrix}k+r\\ m+r\end{matrix}\right\}_r=\frac{1}{m!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(r+j)^k=\left\langle\begin{matrix}k\\ m\end{matrix}\right\rangle_{1,r}. \tag{2.3}$$

Thus, using (2.2), the explicit formula in (2.1) can be rewritten as

$$\begin{aligned}\left\{\begin{matrix}n+1\\ k+1\end{matrix}\right\}_{q,r} &= \sum_{i=0}^{n}\sum_{j=0}^{k-r+1}\frac{(-1)^{k-r+1-j}\binom{k-r+1}{j}\binom{n-r+1}{i}_q q^{\frac{i(i-1+2r)}{2}}(r-1+j)^{n-r+1-i}}{(k-r+1)!}\\ &= \frac{1}{(k-r+1)!}\sum_{j=0}^{k-r+1}(-1)^{k-r+1-j}\binom{k-r+1}{j}(r-1+j)^{n-r+1}\times\\ &\quad\times\left\{\sum_{i=0}^{n-r+1}\binom{n-r+1}{i}_q q^{\frac{i(i-1)}{2}}\left(\frac{q^r}{r-1+j}\right)^i\right\}.\end{aligned}$$

Applying a q-identity in [15], which is given by

$$\sum_{i=0}^{n}\binom{n}{i}_q q^{\frac{i(i-1)}{2}}x^i=\prod_{i=0}^{n-1}\left(1+xq^i\right),$$

we obtain

$$\left\{\begin{matrix}n+1\\ k+1\end{matrix}\right\}_{q,r}=\frac{1}{(k-r+1)!}\sum_{j=0}^{k-r+1}(-1)^{k-r+1-j}\binom{k-r+1}{j}\prod_{i=0}^{n-r}(r-1+j+q^{r+i}). \tag{2.4}$$

This identity is a kind of q-analogue of that identity in (2.3) since, when $q=1$, (2.4) reduces immediately to (2.3).

The next theorem contains a symmetric formula for $\left\{\begin{matrix}n\\ k\end{matrix}\right\}_{q,r}$ which is analogous to the horizontal generating function of Stirling numbers of the second kind.

Theorem 2.4. *A q-analogue of r-Stirling numbers of the second kind satisties the following relation*

$$\sum_{k=0}^{n-1}\left\{\begin{matrix}n\\ k+1\end{matrix}\right\}_{q,r}(x-r+1)^{\underline{k-r+1}}=(x+q^r)(x+q^{r+1})\cdots(x+q^{n-1}).$$

Proof. From the well-known formula

$$\frac{\Delta^k}{k!}(x+r)^n|_{x=0} = \left\{ {n+r \atop k+r} \right\}_r,$$

we get

$$\begin{aligned}\left\{ {n \atop k+1} \right\}_{q,r} &= \sum_{i=0}^{n-1} \binom{n-r}{i}_q q^{\frac{i(i-1+2r)}{2}} \left\{ {n-i-1 \atop k} \right\}_{r-1} \\ &= \frac{\Delta^{k-r+1}}{(k-r+1)!} \sum_{i=0}^{n-1} \binom{n-r}{i}_q q^{\frac{i(i-1+2r)}{2}} (x+r-1)^{n-r-i}|_{x=0} \\ &= \frac{\Delta^{k-r+1}}{(k-r+1)!} \sum_{i=0}^{n-r} \binom{n-r}{i}_q q^{\frac{i(i-1+2r)}{2}} (x+r-1)^{n-i-r}|_{x=0}.\end{aligned}$$

It is known that, for a positive integer n, a real number $q \neq 1$, and an indeterminate z, we have

$$\prod_{i=1}^{n}(a+q^{i-1}z) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{k(k-1)}{2}} z^k a^{n-k}.$$

With $z = q^r$ and $a = x + r$, we obtain

$$\left\{ {n \atop k+1} \right\}_{q,r} = \frac{\Delta^{k-r+1}}{(k-r+1)!}(q^r+x+r-1)(q^{r+1}+x+r-1)\cdots(q^{n-1}+x+r-1)|_{x=0}.$$

The well-known formula for higher order difference operator yields

$$\sum_{k=0}^{n-1} \left\{ {n \atop k+1} \right\}_{q,r} (x-r+1)^{\underline{k-r+1}} = \sum_{k=0}^{n-1} \left\{ \frac{\Delta^{k-r+1}}{(k-r+1)!} \prod_{j=r}^{n-1}(q^j+x+r-1)|_{x=0} \right\} (x-r+1)^{\underline{k-r+1}}$$

$$\begin{aligned}&= \sum_{k=0}^{n-1} \frac{1}{(k-r+1)!} \left\{ \sum_{j=0}^{k-r+1} (-1)^{k-r+1-j} \binom{k-r+1}{j} \times \right. . \\ &\quad \left. . \prod_{l=r}^{n-1}(q^l+r+j-1) \right\} (x-r+1)^{\underline{k-r+1}} \\ &= \sum_{k=0}^{n-1} \frac{1}{(k-r+1)!} \left\{ \sum_{j=0}^{k-r+1} (-1)^{k-r+1-j} \binom{k-r+1}{j} \times \right. . \\ &\quad \left. . \left\{ \sum_{i=0}^{n-r} \sum_{r \le i_1 < i_2 < \cdots < i_i \le n-1} q^{i_1+i_2+\cdots+i_i} (r+j-1)^{n-r-i} \right\} \right\} (x-r+1)^{\underline{k-r+1}}\end{aligned}$$

$$\begin{aligned}&= \sum_{k=0}^{n-1} \frac{1}{(k-r+1)!} \sum_{i=0}^{n-r} \sum_{r \le i_1 < \cdots < i_i \le n-1} q^{i_1+\cdots+i_i} \times \\ &\quad \left\{ \sum_{j=0}^{k-r+1} (-1)^{k-r+1-j} \binom{k-r+1}{j} (r+j-1)^{n-r-i} \right\} (x-r+1)^{\underline{k-r+1}}.\end{aligned}$$

Using the explicit formula for (r,β)-Stirling numbers in (2.3) which also appears in [19], we have

$$\begin{aligned}\sum_{k=0}^{n-1}\left\{{n \atop k+1}\right\}_{q,r}(x-r+1)^{\underline{k-r+1}} &= \sum_{i=0}^{n-r}\sum_{r\le i_1<i_2<\cdots<i_i\le n-1} q^{i_1+i_2+\cdots+i_i}\\ &\qquad \left\{\sum_{k=0}^{n-1}\left\langle{n-r-i \atop k-r+1}\right\rangle_{1,r-1}(x-r+1)^{\underline{k-r+1}}\right\}.\end{aligned}$$

A relation in [19] implies that

$$\begin{aligned}\sum_{k=0}^{n-1}\left\{{n \atop k+1}\right\}_{q,r}(x-r+1)^{\underline{k-r+1}} &= \sum_{i=0}^{n-r}\sum_{r\le i_1<i_2<\cdots<i_i\le n-1} q^{i_1+i_2+\cdots+i_i}x^{n-r-i}\\ &= (x+q^r)(x+q^{r+1})\cdots(x+q^{n-1}).\end{aligned}$$

□

For example, when $n=4$ and $r=2$, we have

$$\begin{aligned}\sum_{k=0}^{3}\left\{{4 \atop k+1}\right\}_{q,2}(x-1)^{\underline{k-1}} &= \left\{{4 \atop 2}\right\}_{q,2}+\left\{{4 \atop 3}\right\}_{q,2}(x-1)+\left\{{4 \atop 4}\right\}_{q,2}(x-1)(x-2)\\ &= (1+q^2+q^3+q^5)+(3+q^2+q^3)(x-1)+(x-1)(x-2)\\ &= x^2+q^2x+q^3x+q^5=(x+q^2)(x+q^3).\end{aligned}$$

It is worth mentioning that certain generalization of Bell numbers, called r-Bell numbers, has been investigated in [18] resulting to several interesting properties of these numbers. These numbers were first defined in [19] as the sum of r-Stirling numbers of the second kind. It is then interesting to define a q-analogue of r-Bell numbers in terms of the above q-analogue of r-Stirling numbers of the second kind and establish some properties analogous to those obtained in [18] for r-Bell numbers.

3 A q-Analogue of r-Stirling Numbers of the First Kind

It is known that the classical Stirling numbers satisfy the following inverse relation

$$f_n=\sum_{k=0}^{n}\left[{n \atop k}\right]g_k \Longleftrightarrow g_n=\sum_{k=0}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}f_k. \tag{3.1}$$

This inverse relation can be obtained using the following generating functions

$$\begin{aligned}x^{\underline{n}} &= \sum_{k=0}^{n}\left[{n \atop k}\right]x^k\\ x^n &= \sum_{k=0}^{n}\left\{{n \atop k}\right\}x^{\underline{k}}.\end{aligned}$$

This motivates the authors to define a q-analogue of r-Stirling numbers of the first kind as follows:

Definition 3.1. A q-analogue of r-Stirling number of the first kind is defined by

$$(x-r+1)^{\underline{n-r}}=\sum_{k=0}^{n}\left[{n \atop k}\right]_{q,r}(-1)^{n-k}(x+q^r)(x+q^{r+1})\ldots(x+q^{k-1}) \tag{3.2}$$

with $r \leq k-1$. By convention, $\begin{bmatrix} n \\ k \end{bmatrix}_{q,r} = 1$ when $r = k$ and $n \geq k$, $\begin{bmatrix} n \\ 0 \end{bmatrix}_{q,r} = 1$ when $n = 0$, $\begin{bmatrix} n \\ 0 \end{bmatrix}_{q,r} = 0$ when $n > 0$ and $\begin{bmatrix} n \\ k \end{bmatrix}_{q,r} = 0$ when $n < k$ or $n, k < 0$.

Using the relation in Theorem 2.4, we have

$$\begin{aligned}(x-r+1)^{\underline{n-r}} &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q,r} (-1)^{n-k} \sum_{m=1}^{k} \begin{Bmatrix} k \\ m \end{Bmatrix}_{q,r} (x-r+1)^{\underline{m-r}} \\ &= \sum_{m=1}^{n} \left\{ \sum_{k=m}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{q,r} \begin{Bmatrix} k \\ m \end{Bmatrix}_{q,r} \right\} (x-r+1)^{\underline{m-r}}.\end{aligned}$$

Comparing the coefficients of $(x-r+1)^{\underline{n-r}}$, we obtain

$$\sum_{k=m}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{q,r} \begin{Bmatrix} k \\ m \end{Bmatrix}_{q,r} = \delta_{mn}$$

where δ_{mn} is the Kronecker delta. On the other hand, the relation in Theorem 2.4 can be written as

$$\begin{aligned}(x+q^r)\cdots(x+q^{n-1}) &= \sum_{k=1}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_{q,r} \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_{q,r} (-1)^{k-m}(x+q^r)\ldots(x+q^{m-1}) \\ &= \sum_{m=1}^{k} \left\{ \sum_{k=m}^{n} (-1)^{k-m} \begin{Bmatrix} n \\ k \end{Bmatrix}_{q,r} \begin{bmatrix} k \\ m \end{bmatrix}_{q,r} \right\} (x+q^r)\ldots(x+q^{m-1}).\end{aligned}$$

Thus, we can state formally these results in the following theorem.

Theorem 3.1. *The q-analogue of r-Stirling numbers of the first kind satisties the following orthogonality relations*

$$\begin{aligned}\sum_{k=m}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{q,r} \begin{Bmatrix} k \\ m \end{Bmatrix}_{q,r} &= \delta_{mn} \\ \sum_{k=m}^{n} (-1)^{k-m} \begin{Bmatrix} n \\ k \end{Bmatrix}_{q,r} \begin{bmatrix} k \\ m \end{bmatrix}_{q,r} &= \delta_{mn}.\end{aligned}$$

Remark 3.1. This theorem immediately implies that

$$\left((-1)^{i-j} \begin{bmatrix} i \\ j \end{bmatrix}_{q,r} \right)_{0 \leq i,j \leq n} \left(\begin{Bmatrix} i \\ j \end{Bmatrix}_{q,r} \right)^{T}_{0 \leq i,j \leq n} = I_{n+1}$$

where I_{n+1} is the identity matrix of order $n+1$. That is,

$$\left((-1)^{i-j} \begin{bmatrix} i \\ j \end{bmatrix}_{q,r} \right)^{-1}_{0 \leq i,j \leq n} = \left(\begin{Bmatrix} i \\ j \end{Bmatrix}_{q,r} \right)^{T}_{0 \leq i,j \leq n}$$

and

$$\det \left[\left((-1)^{i-j} \begin{bmatrix} i \\ j \end{bmatrix}_{q,r} \right)_{0 \leq i,j \leq n} \left(\begin{Bmatrix} i \\ j \end{Bmatrix}_{q,r} \right)^{T}_{0 \leq i,j \leq n} \right] = 1.$$

As a direct consequence of this theorem, we have the following inverse relations of q-analogue of r-Stirling numbers.

Theorem 3.2. *The q-analogue of r-Stirling numbers of the first kind satisties the following inverse relations*

$$f_n = \sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}_{q,r} g_k \iff g_n = \sum_{k=0}^{n}\begin{Bmatrix} n \\ k \end{Bmatrix}_{q,r} f_k$$

$$f_k = \sum_{n=0}^{\infty}(-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}_{q,r} g_n \iff g_k = \sum_{n=0}^{\infty}\begin{Bmatrix} n \\ k \end{Bmatrix}_{q,r} f_n.$$

For quick computation of the first values of $\begin{bmatrix} n \\ k \end{bmatrix}_{q,r}$, we need the following triangular recurrence relation.

Theorem 3.3. *The q-analogue of r-Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_{q,r}$ satisfies*

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{q,r} = \begin{bmatrix} n \\ k-1 \end{bmatrix}_{q,r} + (n-1+q^k)\begin{bmatrix} n \\ k \end{bmatrix}_{q,r}. \tag{3.3}$$

Proof. Equation (3.2) implies that

$$\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_{q,r}(-1)^{n+1-k}(x+q^r)(x+q^{r+1})\dots(x+q^{k-1}) = (x-r+1-n+r)(x-r+1)^{\underline{n-r}}$$

$$= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q,r}(-1)^{n-k}(x+q^k-q^k+1-n)(x+q^r)(x+q^{r+1})\dots(x+q^{k-1})$$

$$\begin{aligned} =\ & \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q,r}(-1)^{n-k}(x+q^r)(x+q^{r+1})\dots(x+q^{k-1})(x+q^k) \\ & +\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q,r}(-1)^{n-k}(-q^k+1-n)(x+q^r)(x+q^{r+1})\dots(x+q^{k-1}) \\ =\ & \sum_{k=0}^{n+1}\begin{bmatrix} n \\ k-1 \end{bmatrix}_{q,r}(-1)^{n+1-k}(x+q^r)(x+q^{r+1})\dots(x+q^{k-1}) \\ & +\sum_{k=0}^{n}(q^k-1+n)\begin{bmatrix} n \\ k \end{bmatrix}_{q,r}(-1)^{n+1-k}(x+q^r)(x+q^{r+1})\dots(x+q^{k-1}) \end{aligned}$$

By comparing the coefficients, we obtain the desired recurrence relation. ☐

We observe that the q-Stirling numbers $\begin{bmatrix} n \\ k \end{bmatrix}_q^*$ in [12] satisfy the relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q^* = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q^* + (n-1+q^k)\begin{bmatrix} n \\ k \end{bmatrix}_q^*$$

which is analogous to the recurrence relation in Theorem 3.3. This recurrence relation has been used to give combinatorial interpretation of $\begin{bmatrix} n \\ k \end{bmatrix}_q^*$ in terms of the weight of permutations in $\{1, 2, \ldots, n\}$ with k nonempty cycles. Hence, we can also use the recurrence relation in Theorem 3.3 to give combinatorial interpretation for $\begin{bmatrix} n \\ k \end{bmatrix}_{q,r}$ by following the same argument in constructing the combinatorial interpretation of $\begin{bmatrix} n \\ k \end{bmatrix}_q^*$.

To sketch the construction, first, we let $\mathcal{P}_n$ be the set of all permutations of $\{1, 2, \ldots, n\}$, $\mathcal{P}_{n,r}$ be the set of all permutations of $\{1, 2, \ldots, n\}$ such that elements $1, 2, \ldots, r$ are in different cycles and $w(\pi)$ be the weight of $\pi \in \mathcal{P}_n$. As defined in [12], the decomposition into nonempty cycles $C_0, C_1, \ldots, C_{k-1}$ of a permutation $\pi \in \mathcal{P}_n$ is called a *natural decomposition* if the ordering is according to decreasing largest elements of the cycles, the natural ordering. Since $max(C_0) = n$, the natural decomposition of C_0 is given by $\{n\}, C_{01}, C_{02}, \ldots, C_{0i}$. Also, in [12], for $\pi = [C_{01}|C_{02}| \ldots |C_{0i}|n]C_1|C_2| \ldots |C_{k-1} \in \mathcal{P}_n$, we define

$$w(\pi) := q^{j_1+j_2+\ldots+j_i}$$

where $j_l = m$ if C_{0l} lies between C_{m-1} and C_m in the natural ordering of cycles and $j_l = k$ if $max(C_{0l}) < max(C_{k-1})$. Then the q-analogue $\begin{bmatrix} n \\ k \end{bmatrix}_{q,r}$ of r-Stirling numbers of the first kind can be interpreted as the sum of the weights of all permutations $\pi \in \mathcal{P}_{n,r}$ such that the natural decomposition has exactly k cycles.

Acknowledgment

The authors would like to thank the referees for reviewing the manuscript.

Conflict of Interest Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

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