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On Extended Generalized Exponential Distribution

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Abstract

In this paper, we present a generalized exponential distribution that contains four parameters. This distribution further generalizes previously established generalized exponential distributions which now serves as special cases of the new four-parameter generalized exponential distribution. The properties of the new distribution like the cumulative distribution function, the survival function, the hazard function, the moment generating function, the median, the 100p-percentile point and the mode of distribution are established. The moment function of the distribution which cannot be obtained in close form is numerically obtained and tabulated for some selected values of the parameters. A Theorem that characterized the distribution is stated and proved.

Keywords: Generalized exponential distribution; moment generating function; moments; median; percentile; mode.

1 Introduction

Exponential distribution is one of the very well known continuous probability distributions which have been used for modeling various life time data and waiting time problems. Specifically, if X is a random variable denoting the waiting time between successive occurrences of events which follow a Poisson distribution with mean λ, then X has an exponential distribution with probability density function (pdf)

$$f_X(x;\lambda) = \lambda e^{-\lambda x}, \qquad x > 0, \lambda > 0. \tag{1.1}$$

The corresponding cumulative distribution function (cdf) of the exponential distribution is

$$F_X(x;\lambda) = 1 - e^{-\lambda x}, \qquad x > 0, \lambda > 0. \tag{1.2}$$

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Recently a new distribution, named as generalized exponential (GE) distribution was introduced by Gupta and Kundu [1]. The GE distribution has the pdf

$$f_X(x;\alpha,\lambda) = \alpha\lambda(1-e^{-\lambda \mathrm{x}})^{\alpha-1}e^{-\lambda \mathrm{x}}, \qquad \mathrm{x}>0, \alpha>0, \lambda>0. \tag{1.3}$$

with cdf

$$F_X(x;\alpha,\lambda) = (1-e^{-\lambda \mathrm{x}})^{\alpha}, \qquad \mathrm{x}>0, \alpha>0, \lambda>0. \tag{1.4}$$

survival function

$$S_X(x;\alpha,\lambda) = 1-(1-e^{-\lambda \mathrm{x}})^{\alpha}, \qquad \mathrm{x}>0, \alpha>0, \lambda>0. \tag{1.5}$$

and a hazard function

$$h_X(x;\alpha,\lambda) = \frac{\alpha\lambda(1-e^{-\lambda \mathrm{x}})^{\alpha-1}e^{-\lambda \mathrm{x}}}{1-(1-e^{-\lambda \mathrm{x}})^{\alpha}}, \qquad >0, \alpha>0, \tag{1.6}$$

In that paper α is a shape parameter while λ is a scale parameter. The GE distribution with the shape parameter α and the scale parameter λ was denoted by α,λ. The GE$(1,\lambda)$ represents the exponential distribution with the scale parameter λ in equation (1.1). If the measure of location μ is introduced in equation (1.3), we have

f_X (x;μ,α,ʎ)=αʎ(1-e^(-ʎ(x-μ)))^(α-1) e^(-ʎ(x-μ)),x>0,μ>0,α>0,ʎ>0. (1.7)

It is made known in Gupta and Kundu [1] that the generalized exponential distribution can be used quite effectively in analyzing many lifetime data, particularly in place of two-parameter gamma and two-parameter Weibull distributions.

The generalized exponential distribution can have increasing and decreasing failure rates depending on the shape parameter. Gupta and Kundu [2] studied how the different estimators of the unknown parameter or parameters of the generalized exponential distribution behave for different sample sizes and for different parameter values. They compared the maximum likelihood estimator with the other estimators like method of moment estimators, estimators based on percentiles, least squares estimators, weighted least squares estimators and the estimators based on order statistics, mainly with respect to their biases and mean squared errors using extensive simulation techniques.

The main aim of this paper is to extend the generalization of the exponential distribution by introducing another generalized exponential distribution which contains four parameters as an improvement on the above generalized exponential distribution of Gupta and Kundu [1,2]. The concept of extended generalized distribution was introduced in Wu Jong-Wuu, Hung Wen-Liang and Lee Hsiu-Mei [3] for the generalized logistic distribution of George and Ojo [4]. After that, Olapade [5] obtained extended type I generalized logistic distribution for the type I generalized logistic distribution of Balakrishnan and Leung [6] and he continued in Olapade [7,8] to obtain extended type II and extended type III generalized logistic distributions respectively for the type II and type III generalized logistic distributions of Balakrishnan and Leung [6] respectively.

2. Four-Parameter Generalized Exponential Distribution

Let X be a continuous random variable, we say that the random variable X follows an extended generalized exponential distribution if its pdf is

$$f_X(x;\mu,\alpha,\beta,\lambda) = \frac{\alpha\lambda}{\beta^{\alpha}-(\beta-1)^{\alpha}}(\beta-e^{-\lambda(\mathrm{x}-\mu)})^{\alpha-1}e^{-\lambda(\mathrm{x}-\mu)},$$

$$\text{x} > 0, \mu > 0, \alpha > 1, \beta > 1\,, \lambda > 0. \tag{2.1}$$

By integrating equation (2.1) over the range of X, we could easily confirm the function to be a pdf and as the function contains four parameters, it is called a four-parameter generalized exponential distribution or extended generalized exponential (EGE) distribution where μ is a location parameter, α is a shape parameter, β is an extension parameter and, λ is a scale parameter hence, we denote the EGE with these parameters as EGE($\mu, \alpha, \beta, \lambda$). When β =1, we obtain the generalized exponential distribution of Gupta and Kundu [1] and when $\alpha = 1$, we obtain the exponential distribution with parameter λ when μ =0 which is also called negative exponential distribution in some literature. For the rest of this paper, we assume that μ =0 without loss of generality.

If X has the probability distribution function in equation (2.1), then the corresponding cdf is obtained as

$$F_X(x; \alpha, \beta, \lambda) = \frac{1}{\beta^\alpha - (\beta - 1)^\alpha}[(\beta - e^{-\lambda \text{x}})^\alpha - (\beta - 1)^\alpha], \\ \text{x} > 0, \alpha > 1, \beta > 1\,, \lambda > 0. \tag{2.2}$$

As the values of $F(x; \alpha, \beta, \lambda)$ depends on the values of α, β, λ and x, the probability that an EGE random variable X lies in an interval $(x_i\,, x_j)$ is obtained as

$$\Pr\,(x_i < X < x_j) = F_X(x_j; \alpha, \beta, \lambda) - F_X(x_i; \alpha, \beta, \lambda) = \frac{(\beta - e^{-\lambda x_j})^\alpha - (\beta - e^{-\lambda x_i})^\alpha}{\beta^\alpha - (\beta - 1)^\alpha} \tag{2.3}$$

for any real values of α, β, λ and any given interval $(x_i\,, x_j)$

If X is the lifetime of an object, then the survival function of the random variable X with EGE(α, β, λ) distribution is

$$S_X(x; \alpha, \beta, \lambda) = 1 - \frac{1}{\beta^\alpha - (\beta - 1)^\alpha}[(\beta - e^{-\lambda \text{x}})^\alpha - (\beta - 1)^\alpha]$$

$$= \frac{\beta^\alpha - (\beta - e^{-\lambda \text{x}})^\alpha}{\beta^\alpha - (\beta - 1)^\alpha}, \qquad \text{x} > 0, \alpha > 1, \beta > 1\,, \lambda > 0. \tag{2.4}$$

The hazard function of the random variable X with EGE(α, β, λ) distribution is also obtained as

$$h_X(x; \alpha, \beta, \lambda) = \frac{f_X(x; \alpha, \beta, \lambda)}{1 - F_X(x; \alpha, \beta, \lambda)} \\ = \frac{\alpha\lambda(\beta - e^{-\lambda \text{x}})^{\alpha - 1} e^{-\lambda \text{x}}}{\beta^\alpha - (\beta - e^{-\lambda \text{x}})^\alpha},$$

$$\text{x} > 0, \alpha > 1, \beta > 1\,, \lambda > 0. \tag{2.5}$$

If α=1, the hazard function becomes λ independent of x.

3. The Moment of Extended Generalized Exponential Distribution

The moment generating function of a random variable X that follows an EGE distribution is obtained as

$$M_X(t) = \int e^{tx} f_X(x;\alpha,\beta,ƙ)dx$$

$$= \frac{\alpha ƙ}{\beta^{\alpha} - (\beta-1)^{\alpha}} \int_0^{\infty} e^{tx} (\beta - e^{-ƙx})^{\alpha-1} e^{-ƙx} dx \quad (3.1)$$

$$= \frac{\alpha ƙ \beta^{\alpha-1}}{\beta^{\alpha} - (\beta-1)^{\alpha}} \int_0^{\infty} (1 - \frac{e^{-ƙx}}{\beta})^{\alpha-1} e^{-x(ƙ-t)} dx \quad (3.2)$$

Let $e^{-ƙx}/\beta = k$, then $x = -ƙ^{-1}\ln(\beta k)$ and $dx = -(ƙk)^{-1}dk$. So

$$M_X(t) = \frac{\alpha\beta^{\alpha-t/ƙ}}{\beta^{\alpha}-(\beta-1)^{\alpha}} \int_0^{\beta^{-1}} (1-k)^{\alpha-1} k^{-t/ƙ} dk. \quad (3.3)$$

By using the binomial series expansion of the argument in the integral of equation (3.3), we have

$$M_X(t) = \frac{\alpha\beta^{\alpha-\frac{t}{ƙ}}}{\beta^{\alpha} - (\beta-1)^{\alpha}} \int_0^{\beta-1} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} k^{j-\frac{t}{ƙ}} dk. \quad (3.4)$$

We interchange the summation and the integration to obtain

$$M_X(t) = \frac{\alpha\beta^{\alpha-\frac{t}{ƙ}}}{\beta^{\alpha} - (\beta-1)^{\alpha}} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} \int_0^{\beta-1} k^{j-t/ƙ} dk. \quad (3.5)$$

$$= \frac{\alpha\beta^{\alpha-t/ƙ}}{\beta^{\alpha} - (\beta-1)^{\alpha}} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} \frac{\beta^{-j}}{j+1-t/ƙ}. \quad (3.6)$$

Since the infinite series is sum able, differentiable and it has only a finite number of terms when α is an integer, we have after differentiating k times and evaluating at t=0, we obtain the k^{th} moment of the EGE distribution as

$$\mu_k = \frac{\alpha\beta^{\alpha-1}k!}{\beta^{\alpha} - (\beta-1)^{\alpha}} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} \frac{\beta^{-j}}{(j+1)^{k+1}}. \quad (3.7)$$

Since the moment generating function $M_X(t)$ is an infinite series which may be difficult to make use of, we obtained the n^{th} moment of the EGE distribution as

$$E[X^n] = \frac{\alpha ƙ}{\beta^{\alpha} - (\beta-1)^{\alpha}} \int_0^{\infty} x^n (\beta - e^{-ƙx})^{\alpha-1} e^{-ƙx} dx. \quad (3.8)$$

The tables (3-1) till (3-6) which are presented in the Appendix should be considered.

These tabulated values can be used to compute the mean, variance, skewness and kurtosis for the EGE distribution using the following relations:

$$\mu_1=\nu_1$$

$$\mu_2=\nu_2-\nu_1^2$$

$$\mu_3=\nu_3-3\nu_2\nu_1+2\nu_1^3$$

$$\mu_4=\nu_4-4\nu_3\nu_1+6\nu_2\nu_1^2-3\nu_1^4, \tag{3.9}$$

where ν_i is the i^{th} moment $E[X^i]$ and μ_1=the mean, μ_2 = the variance, skewness $\beta_1=\mu_3^2/\mu_2^3$ and the measure of kurtosis $\beta_2=\mu_4/\mu_2^2$.

4 Median of the Extended Generalized Exponential Distribution

The median of a probability density function f(x) is a point x_m on the real line which satisfies the equation

$$\int_{-\infty}^{xm} f(x)dx = 1/2$$

this implies that $F(x_m)=1/2$. For the extended generalized exponential distribution with probability distribution function in equation (2.2), $F_X(x;\alpha,\beta,\lambda)=1/2$ implies $\frac{[(\beta-e^{-\lambda x})^\alpha-(\beta-1)^\alpha]}{\beta^\alpha-(\beta-1)^\alpha}=1/2$, which implies that

$$x_{median}=-\lambda^{-1}\ln\left[\beta-\sqrt[\alpha]{\frac{\beta^\alpha-(\beta-1)^\alpha}{2}}\right]. \tag{4.1}$$

The survival function of the EGE distribution at the median is $S_{median}(x;\alpha,\beta,\lambda)=1/2$.

5 The 100p-Percentage Point of the Extended Generalized Exponential Distribution

Consider the extended generalized exponential distribution, the 100p-percentage point is obtained by equating the cumulative probability distribution function to p, where $0\le p\le 1$. That is

$$F\left(x_{(p)}\right)=p \rightarrow \frac{[(\beta-e^{-\lambda x})^\alpha-(\beta-1)^\alpha]}{\beta^\alpha-(\beta-1)^\alpha}=p.$$

Solving for $x_{(p)}$ gives

$$x_{(p)}=-\lambda^{-1}\ln\left[\beta-\sqrt[\alpha]{p\beta^\alpha-(p+1)(\beta-1)^\alpha}\right]. \tag{5.1}$$

This gives the value of the point $x_{(p\ )}$ on the real line that produce a percentage p of the distribution. We can easily test this by checking the value of $x_{(p)}$ when p=0.5 which corresponds to the median.

6 The Mode of the Extended Generalized Exponential Distribution

The mode of a probability density function is obtained by equating the derivative of the density function to zero and solve for the variable. Therefore, for the extended generalized exponential distribution

$$f_X(x;\alpha,\beta,\lambda)=\frac{\alpha\lambda}{\beta^{\alpha}-(\beta-1)^{\alpha}}(\beta-e^{-\lambda x})^{\alpha-1}e^{-\lambda x},$$

$$x>0,\alpha>1,\beta>1\ ,\lambda>0.$$

$$f'_X(x;\alpha,\beta,\lambda)=\frac{\alpha\lambda^2}{\beta^{\alpha}-(\beta-1)^{\alpha}}e^{-\lambda x}(\beta-e^{-\lambda x})^{\alpha-2}(\alpha e^{-\lambda x}-\beta). \tag{6.1}$$

By equating the derivative to zero, we have

$$e^{-\lambda x}(\beta-e^{-\lambda x})^{\alpha-2}(\alpha e^{-\lambda x}-\beta). \tag{6.2}$$

This implies

$$e^{-\lambda x}=0\ \text{ or }\ (\beta-e^{-\lambda x})=0 \text{ or }\ (\alpha e^{-\lambda x}-\beta)=0. \tag{6.3}$$

This implies that

$$x_{(mode\ )}=\ \infty\ \ or -\lambda^{-1}ln\beta\ \ or\ \ -\lambda^{-1}\ln\left(\frac{\beta}{\alpha}\right) \tag{6.4}$$

To determine the mode out of these three options, we differentiate the pdf the second time to obtain

$$f_X''(x;\alpha,\beta,\lambda)=\frac{\alpha\lambda^3}{\beta^{\alpha}-(\beta-1)^{\alpha}}[\alpha(\alpha-2)e^{-3\lambda x}(\beta-e^{-\lambda x})^{\alpha-3}-2\alpha e^{-2\lambda x}(\beta-e^{-\lambda x})^{\alpha-2}$$

$$+\beta e^{-\lambda x}(\beta-e^{-\lambda x})^{\alpha-2}-\beta(\alpha-2)e^{-2\lambda x}(\beta-e^{-\lambda x})^{\alpha-3}]. \tag{6.5}$$

When $x_{(mode\ )}=\ \infty\ \ or -\lambda^{-1}ln\beta,\ f_X''(x;\alpha,\beta,\lambda)=0,$ but when $x_{(mode\ )}=\ \ -\lambda^{-1}\ln\left(\frac{\beta}{\alpha}\right),$

$$f_X''(x;\alpha,\beta,\lambda)=\frac{\beta\lambda^3(\beta-\beta/\alpha)^{\alpha-2}(\beta-2\alpha)}{\beta^{\alpha}-(\beta-1)^{\alpha}},\quad \alpha>1,\beta>0,\ \lambda>0. \tag{6.6}$$

The only determining factor for the mode of EGE distribution to exist is that $\beta-2\alpha<0$, hence the EGE distribution will have a finite real mode whenever $\alpha>\beta/2$ and $x_{(mode\ )}=\ \ -\lambda^{-1}\ln\left(\frac{\beta}{\alpha}\right)$.
The survival function at the mode is

$$S_{mode}(x) = \frac{\beta^{\alpha}(1-(1-1/\alpha)^{\alpha})}{\beta^{\alpha}-(\beta-1)^{\alpha}}, \quad \alpha > 1, \beta > 0. \tag{6.7}$$

7 A Theorem that Characterize the Extended Generalized Exponential Distribution

Here we state and prove a Theorem that characterize this distribution.

Theorem: The random variable X follows a extended generalized exponential distribution with parameters $\alpha, \beta, ƛ$ if and only if the density function f satisfies the homogeneous differential equation

$$(\beta - e^{-ƛx})f' + ƛ(\beta - \alpha e^{-ƛx})f = 0 \tag{7.1}$$

(prime denotes differentiation).

Proof: Suppose X is a extended generalized exponential distribution random variable, then $f_X(x;\alpha,\beta,ƛ)$and $f_X'(x;\alpha,\beta,ƛ)$ are as shown in equations (2.1) and (6.1) respectively.
By substituting f(x) and f'(x) in the differential equation (7.1), the equation is satisfied.
Conversely, we assume that f satisfies equation (7.1), separate the variables and then integrate, we have

$$\int \frac{f'}{f}dx = -ƛ\beta \int \frac{dx}{\beta - e^{-ƛx}} + ƛ\alpha \int \frac{e^{-ƛx}}{\beta - e^{-ƛx}}dx. \tag{7.2}$$

$$\ln f = -ƛx + \ln(\beta - e^{-ƛx})^{\alpha-1} + \ln C. \tag{7.3}$$

Therefore,

$$f = Ce^{-ƛx}(\beta - e^{-ƛx})^{\alpha-1}, \qquad x > 0, \alpha > 1, \beta > 1, ƛ > 0. \tag{7.4}$$

Where C is a constant. The value of C that makes f a probability density function is

$$C = \alpha ƛ[\beta^{\alpha} - (\beta-1)^{\alpha}]^{-1}.$$

Possible application of the Theorem: From the homogeneous differential equation (7.1),

$$x = -\frac{1}{ƛ}\ln\left[\frac{\beta(f'+ƛf)}{f'+\alpha ƛ f}\right], \tag{7.5}$$

or equivalently,

$$x = -\frac{1}{ƛ}\ln\left[\frac{\beta(F''+ƛF)}{F'+\alpha ƛ F'}\right]. \tag{7.6}$$

Where F is the corresponding cdf of the EGE distribution. Thus the importance of this Theorem lies in the linearizing transformation (7.5) or (7.6) which could be regarded as an EGE model alternative to the Berkson's logit transform in Berkson [9] for the ordinary logistic model and Ojo [10] logit transform for generalized logistic model. Hence, equation (7.5) or (7.6) could be referred to as extended generalized exponential logit transform.

8 Conclusion

We have presented a four-parameter generalized exponential distribution and proved that it is really a probability density function. The cumulative distribution function, the survival, the hazard and the moment generating function of the distribution have been obtained and the moments function have been tabulated. The median, the 100p-percentage point and the mode of the distribution are obtained. We conclude the paper by stating and proving a Theorem that characterized the distribution.

Competing Interests

Author has declared that no competing interests exist.

References

1. Gupta RD, Kundu D. Generalized exponential distributions. Australian and New Zealand Journal of Statistics. 1999;41(2):173-188.

2. Gupta RD, Kundu D. Generalized exponential distributions. Statistical inferences. Journal of Statistical Theory and Applications. 2000;1:101-118.

3. Jong-Wuu, Hung Wen-Liang, Lee Hsiu-Mei. Some moments and limit behaviors of the generalized logistic distribution with applications. Proc. Natl. Sci. Counc. ROC. 2000;4(24)1:7-14.

4. George EO, Ojo MO. On a generalization of the logistic distribution. Annals of Statistical Mathematics. 1980;32(2A):161-169.

5. Olapade AK. On extended type I generalized logistic distribution. International Journal of Mathematics and Mathematical Sciences. 2004;57:3069-3074.

6. Balakrishnan N, Leung MY. Order statistics from the type I generalized logistic distribution. Communications in Statistics - Simulation and Computation. 1988;17(1):25-50.

7. Olapade AK. On negatively skewed extended generalized logistic distribution. Kragujevac J. Math. 2005;27:175-182.

8. Olapade AK. On symmetric extended generalized logistic distribution. J. Statist. Res. Iran. 2006;3:177-190.

9. Berkson J. Application of the logistic function to bioassay. Journal of the American Statistical Association. 1944;39:357-365.

10. Ojo MO. Some relationships between the generalized logistic and other distributions. Statistica, 1997; L7 (4): 573-579.

Appendix

Tables of moments of the extended generalized exponential distribution.

Table 3.1. ʎ = 2.0, α = 2.0

β	$E[X]$	$E[X^2]$	$E[X^3]$	$E[X^4]$
1.0000	0.7500	0.8750	1.4062	2.9062
1.5000	0.6250	0.6875	1.0781	2.2031
2.0000	0.5833	0.6250	0.9687	1.9687
2.5000	0.5625	0.5937	0.9141	1.8515
3.0000	0.5500	0.5750	0.8812	1.7812
3.5000	0.5417	0.5625	0.8594	1.7344
4.0000	0.5357	0.5536	0.8437	1.7344
4.5000	0.5312	0.5469	0.8320	1.6758
5.0000	0.5278	0.5417	0.8229	1.6562

Table 3.2. ʎ=2.0, α =2.5

β	$E[X]$	$E[X^2]$	$E[X^3]$	$E[X^4]$
1.0000	0.8402	1.0345	1.7084	3.5802
1.5000	0.6856	0.7849	1.2551	2.5898
2.0000	0.6252	0.6908	1.0868	2.2251
2.5000	0.5942	0.6429	1.0018	2.0415
3.0000	0.5754	0.6141	0.9508	1.9318
3.5000	0.5628	0.5950	0.9170	1.8587
4.0000	0.5538	0.5813	0.8928	1.8067
4.5000	0.5471	0.5711	0.8748	1.7679
5.0000	0.5418	0.5631	0.8608	1.7378

Table 3.3. ʎ=2.0, α =3.0

β	$E[X]$	$E[X^2]$	$E[X^3]$	$E[X^4]$
1.0000	0.9167	1.1805	1.9965	4.2372
1.5000	0.7436	0.8825	1.4364	2.9913
2.0000	0.6667	0.7579	1.2093	2.4937
2.5000	0.6258	0.6933	1.0928	2.2400
3.0000	0.6009	0.6542	1.0228	2.0881
3.5000	0.5841	0.6282	0.9764	1.9875
4.0000	0.5721	0.6096	0.9433	1.9160
4.5000	0.5630	0.5957	0.9186	1.8627
5.0000	0.5560	0.5849	0.8995	1.8215

Table 3.4. ƛ=3.0, α =2.0

β	$E[X]$	$E[X^2]$	$E[X^3]$	$E[X^4]$
1.0000	0.5000	0.3889	0.4167	0.5741
1.5000	0.4167	0.3056	0.3194	0.4352
2.0000	0.3889	0.2778	0.2870	0.3889
2.5000	0.3750	0.2639	0.2708	0.3657
3.0000	0.3667	0.2556	0.2611	0.3518
3.5000	0.3611	0.2500	0.2546	0.3426
4.0000	0.3571	0.2460	0.2500	0.3360
4.5000	0.3542	0.2431	0.2465	0.3310
5.0000	0.3518	0.2407	0.2438	0.3272

Table 3.5. ƛ=3.0, α =2.5

β	$E[X]$	$E[X^2]$	$E[X^3]$	$E[X^4]$
1.0000	0.5601	0.4598	0.5062	0.7072
1.5000	0.4571	0.3489	0.3719	0.5116
2.0000	0.4106	0.3070	0.3220	0.4395
2.5000	0.3961	0.2857	0.2968	0.4033
3.0000	0.3836	0.2730	0.2817	0.3816
3.5000	0.3752	0.2644	0.2717	0.3671
4.0000	0.3692	0.2583	0.2645	0.3569
4.5000	0.3647	0.2538	0.2591	0.3492
5.0000	0.3612	0.2503	0.2550	0.3433

Table 3.6. ƛ=3.0, α =3.0

β	$E[X]$	$E[X^2]$	$E[X^3]$	$E[X^4]$
1.0000	0.6111	0.5247	0.5916	0.8370
1.5000	0.4957	0.3922	0.4256	0.5909
2.0000	0.4444	0.3369	0.3583	0.4920
2.5000	0.4172	0.3081	0.3238	0.4425
3.0000	0.4006	0.2908	0.3031	0.4125
3.5000	0.3894	0.2792	0.2893	0.3926
4.0000	0.3814	0.2709	0.2795	0.3785
4.5000	0.3754	0.2648	0.2722	0.3680
5.0000	0.3707	0.2600	0.2665	0.3598

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