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Formulation of a Connection for Prolongations and an Application to the Burgers-KdV Equation

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Research Article

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Abstract

The Wahlquist-Estabrook approach which has been applied to investigate the prolongation structure of many nonlinear systems is introduced. The theory which results is applied to the Burgers-KdV equation which is shown to have a nontrivial prolongation algebra. It is shown that the resulting equations can be solved to produce a very general solution. Based on the results determined for the algebra, without picking a specific representation for the algebra, a Lax pair for the equation is determined in terms of the basic generators of the algebra.

Keywords: differential system, prolongation structure, integrability, nonlinear equation, Burgers-Korteweg-de Vries

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1 Introduction

The method initiated by Wahlquist and Estabrook (Wahlquist and Estabrook, 1975; Wahlquist and Estabrook, 1976; Estabrook, 1982; Estabrook and Wahlquist, 1978) has turned out to be very effective as far as understanding and generating prolongation algebras for large classes of nonlinear differential equations. These algebras can in turn be used to write down many other quantities which are significant as far as the study of the integrability of these systems is concerned. For example, a Lax pair for the equation can usually be generated as well as a Bäcklund transformation which allows one to produce new solutions of the equation from known solutions. Many deep ideas from differential geometry and group theory have found applications in this domain (Olver, 1993).

Here, the method which has evolved from the ideas of Wahlquist and Estabrook will be reviewed and developed in a direction that has not been discussed as of yet. These results are then used to obtain a prolongation algebra for the Burgers-KdV equation from a differential ideal of two-forms. This nonlinear equation (Newell, 1985) has arisen in discussing the flow of liquids, in fluid dynamics and merits a treatment by means of this procedure. Several other equations have been studied along these lines recently (Bracken, 2007; Bracken, 2011; Bracken, 2010), and there seems to be a lot of

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interest in this way of looking at nonlinear equations (Bracken, 2010; Findley, 1996; Zhai et al., 2006, Dodd and Fordy, 1983; Leo et al., 2001)

A differential ideal of two-forms is established first for the equation and it is shown to be closed. By the Frobenius theorem, it is established that the given exterior differential ideal is integrable (Groesen and Jager, 1994). This system of forms is chosen such that solutions of the evolution equation correspond with two-dimensional transversal integral manifolds of the ideal, which can be expressed as sections in the base manifold. To carry this out, a particular fibre bundle is introduced over the base manifold and provided with a Cartan-Ehresmann connection, which is specified by an additional differential system of one-forms. When the combined exterior differential system of these forms is considered on the bundle, it is called a Cartan prolongation if it is closed. Whenever there is a transversal solution of the differential system, there should exist a transversal solution of the combined exterior differential system such that it projects down to the preceding integral manifolds. It will be seen that this approach leads directly to a Lax pair in an efficient way once the prolongation algebra is known.

2 Cartan Prolongations

Consider the space $M=\mathbb{R}^m$ with coordinates written generally in the form $(v_1,v_2,\cdots,v_m)$. Let there be defined on M a closed exterior differential system

$$\alpha_1=0,\cdots,\alpha_l=0. \tag{1}$$

Let I be the ideal generated by (1) given by $I=\{\xi=\sum_{i=1}^{l}\sigma_i\wedge\alpha_i:\sigma_i\in\Lambda_p(M),p=0,1,2\cdots\}$ with $\Lambda_p(M)$ the set of p-forms on M. Since (1) is closed, $dI\subset I$, and according to the Frobenius Theorem, then (1) is integrable.

In applications that would be considered to evolution equations $u_t=F(x,t,u,u_x,u_{xx},\cdots)$, $v_1=x$, $v_2=t$ and $v_3=u$. The system (1) is chosen in such a way that solutions $u=u(x,t)$ of an evolution equation correspond with two-dimensional transversal integral manifolds of (1). These integral manifolds can be written as sections S in M such that S is specified by

$$(x,t)\to(x,t,v_3(x,t),\cdots,v_m(x,t)),$$

and due to tansversality, $dx\wedge dt=\pi^*(dx\wedge dt)\neq 0$ where $\pi:M\to\mathbb{R}^2$ and $\pi^*\Lambda(\mathbb{R}^2)\to\Lambda(S)$.

Now, a fibre bundle $B=(\tilde{M},\tilde{p},M)$ with connection will be introduced over M such that $M\subset\tilde{M}$ and $\tilde{p}$ the projection of $\tilde{M}$ onto M, $\tilde{p}(\tilde{M})=M$. A Cartan-Ehresmann connection in the fibre bundle B is a system of one-forms $\tilde{\omega}_i$ where $i=1,\cdots,n$ in $T^*(M)$ with the property that the mapping $\tilde{p}_*$ from the vector space $H_{\tilde{m}}=\{\tilde{X}\in T_{\tilde{q}}|\,\tilde{\omega}_i(\tilde{X})=0,i=1,\cdots,n\}$ onto the tangent space T_q is a bijection for all $\tilde{q}\in\tilde{M}$. Consider the exterior differential system in $\tilde{M}$ given by

$$\tilde{\alpha}_i=\tilde{p}^*\alpha_i=0,\quad i=1,\cdots,l,\qquad \tilde{\omega}_j=0,\quad j=1,\cdots,n, \tag{2}$$

such that $\{\tilde{\omega}\}$ is a Cartan-Ehresmann connection in $\{\tilde{M},\tilde{p},M\}$. The system (2) is called a Cartan prolongation if (2) is closed and whenever S is a transversal solution of (1). Then, there should exist a transversal solution $\tilde{S}$ of (2) such that $\tilde{p}(\tilde{S})=S$. From the fact that (2) is closed, it follows that this prolongation condition may be written as

$$d\tilde{\omega}_i=\sum_{j=1}^{n}\tilde{\beta}_i^j\wedge\tilde{\omega}_j\quad mod\,\tilde{p}^*(I), \tag{3}$$

with I the ideal defined by (1).

The bundle is given by $\tilde{M}=M\times\mathbb{R}^n$ where $\mathbf{y}=(y_1,\cdots,y_n)\in\mathbb{R}^n$ are referred to as the pseudo-potential coordinates. The connection used here is given by

$$\tilde{\omega}^k=dy^k-F^k(v_1,\cdots,v_m,\mathbf{y})\,dt-G^k(v_1,\cdots,v_m,\mathbf{y})\,dx,\qquad k=1,\cdots,n. \tag{4}$$

In (4), F^k and G^k can be represented as column vectors which depend on the manifold coordinates as well as the $\mathbf{y}$. The procedure is to regard the y^k as coordinates in an extended $(m+n)$ dimensional space of variables, and to include the $\tilde{\omega}^k$ with the original set of forms.

The integrability condition requires the prolonged ideal $\{\alpha_i, \tilde{\omega}^k\}$ be closed. This means that $d\tilde{\omega}^k$ takes the form

$$d\tilde{\omega}^k = \sum_{j=1}^{l} f^{kj}\alpha_j + \sum_{i=1}^{n} \eta^{ki} \wedge \tilde{\omega}_i,$$

where f^{kj} stand for dependent functions of the bundle coordinates and η^{ki} represent a matrix of one-forms.

For a connection of the form (4), namely $\tilde{\omega}^k = dy^k - \eta^k$, the prolongation condition reduces to the form

$$-\,d\eta_i = \frac{\partial \eta_i}{\partial y_j} \wedge (dy_j - \eta_j), \quad mod\,\tilde{p}^*(I). \tag{5}$$

This result can be compressed by writing the identity

$$d\eta_i = d_M\eta_i - (\frac{\partial \eta_i}{\partial y_j}) \wedge dy_j.$$

Here d_M means differentiation with respect to just the variables of the base manifold $\{v_j\}$. The prolongation condition then becomes

$$d_M\eta_i - (\frac{\partial \eta_i}{\partial y_j}) \wedge \eta_j = 0, \quad mod\,\tilde{p}^*(I).$$

Finally, introduce the vertical valued one-form

$$\eta = \eta_i \frac{\partial}{\partial y_i},$$

along with the definitions

$$d\eta = (d_M\eta_i)\frac{\partial}{\partial y_i}, \qquad [\eta, \omega] = (\eta_j \wedge \frac{\partial \omega_i}{\partial y_j} + \omega_j \wedge \frac{\partial \eta_i}{\partial y_j})\frac{\partial}{\partial y_i}. \tag{6}$$

The prolongation condition then takes the elegant form

$$d\eta + \frac{1}{2}[\eta, \eta] = 0, \qquad mod\,\tilde{p}^*(I). \tag{7}$$

A particular example of the connection form (4) will be defined. This one is useful with respect to generation of Lax pairs. It is defined by

$$\tilde{\Omega}^k = dy^k - \eta^k = dy^k - \sum_{i=1}^{n} F^{ki}(\mathbf{v})y_i\,dt - \sum_{i=1}^{n} G^{ki}(\mathbf{v})y_i\,dx. \tag{8}$$

To work out (7), the commutator (6) is required, and after some simplification there results,

$$\begin{aligned}
[\eta, \eta] &= (G^{ji}F^{\nu j}y_i\,dx \wedge dt + F^{ji}G^{\nu j}y_i\,dt \wedge dx + F^{ji}G^{\nu j}y_i\,dt \wedge dx + G^{ji}F^{\nu j}\,y_i dx \wedge dt)\frac{\partial}{\partial y_\nu} \\
&= 2(F^{\nu j}G^{ji} - G^{\nu j}F^{ji})y_i\,\frac{\partial}{\partial y_\nu}\,dx \wedge dt \\
&= 2[F, G]^{\nu i}y_i\frac{\partial}{\partial y_\nu}\,dx \wedge dt.
\end{aligned} \tag{9}$$

With summation implied over repeated indices, the prolongation condition is

$$(\frac{\partial F^{\nu i}}{\partial v_j}\,dv_j \wedge dt + \frac{\partial G^{\nu i}}{\partial v_j}\,dv_j \wedge dx)y_i\frac{\partial}{\partial y_\nu} + [F, G]^{\nu i}\,y_i\frac{\partial}{\partial y_\nu}\,dx \wedge dt \equiv 0, \quad mod\,\tilde{p}^*(I). \tag{10}$$

If the ideal I is given in terms of two-forms $\{\alpha_j\}$, closed over I, then (10) takes the equivalent form

$$(\frac{\partial F^{\nu i}}{\partial v_j}\, dv_j \wedge dt + \frac{\partial G^{\nu i}}{\partial v_j}\, dv_j \wedge dx) + [F, G]^{\nu i}\, dx \wedge dt \equiv \lambda_j^{\nu i} \alpha^j. \tag{11}$$

It is the result in (11) that will be used to calculate the required prolongation algebra next.

3 Differential System and Associated Differential Equation

To begin the investigation, an exterior differential system which is relevant to the partial differential equation must be introduced. An exterior differential system is given which is defined over the base manifold $M = \mathbb{R}^5$, which supports the differential forms. Consequently, the remaining variables in F and G are $v_j = \{u, p, q\}$. The system of two-forms are defined to be

$$\begin{aligned} \alpha_1 &= du \wedge dt - p\, dx \wedge dt, \\ \alpha_2 &= dp \wedge dt - q\, dx \wedge dt, \\ \alpha_3 &= -du \wedge dx + up\, dx \wedge dt - \alpha q\, dx \wedge dt + \beta\, dq \wedge dt. \end{aligned} \tag{12}$$

The exterior derivatives of the α_j are calculated to be,

$$\begin{aligned} d\alpha_1 &= -dp \wedge dx \wedge dt = dx \wedge \alpha_2, \\ d\alpha_2 &= dx \wedge dq \wedge dt = \frac{1}{\beta}\, dx \wedge \alpha_3, \\ d\alpha_3 &= p\, du \wedge dx \wedge dt + u\, dp \wedge dx \wedge dt - \alpha\, dq \wedge dx \wedge dt \\ &= -du \wedge \alpha_1 - u\, dx \wedge \alpha_2 + \frac{\alpha}{\beta}\, dx \wedge \alpha_3. \end{aligned} \tag{13}$$

From (13), the ideal $I = \{\omega | \omega = \sum_{i=1}^{3} \sigma_i \wedge \alpha_i \,:\, \sigma_i \in \Lambda(M)\}$ is closed $dI \subset I$, and the system $\{\alpha_i\}$ given by (12) is integrable (Bracken, 2010). On the transversal integral manifold, it follows that differential system (12) can be sectioned to give the system of equations,

$$\begin{aligned} \alpha_1|_S &= (u_x - p)\, dx \wedge dt = 0, \\ \alpha_2|_S &= (p_x - q)\, dx \wedge dt = 0, \\ \alpha_3|_S &= -u_t\, dt \wedge dx + up\, dx \wedge dt - \alpha q\, dx \wedge dt + \beta q_x\, dx \wedge dt = 0. \end{aligned} \tag{14}$$

Therefore, the transversal integral manifolds correspond to the set of coupled equations

$$p = u_x, \qquad q = p_x = u_{xx}, \qquad u_t + up - \alpha q + \beta q_x = 0. \tag{15}$$

Upon replacing p and q in the last equation of (15), the following partial differential equation results,

$$u_t + uu_x - \alpha u_{xx} + \beta u_{xxx} = 0. \tag{16}$$

This is the Burgers-KdV equation. When the constant α is set to zero, the KdV equation results, and when β is set to zero, Burgers equation results.

4 Determining Prolongation Algebra

To generate a prolongation algebra, system (12) is substituted into the prolongation condition (10). Suppressing indices on F and G, (11) is found to take the form

$$F_t\, dt \wedge dx + F_u\, du \wedge dx + F_p\, dp \wedge dx + F_q\, dq \wedge dx$$

$$+G_x\,dx\wedge dt+G_u\,du\wedge dt+G_p\,dp\wedge dt+G_q\,dq\wedge dt+[F,G]\,dx\wedge dt$$
$$=\lambda_1(du\wedge dt-p\,dx\wedge dt)+\lambda_2(dp\wedge dt-q\,dx\wedge dt) \tag{17}$$
$$+\lambda_3(-du\wedge dx+up\,dx\wedge dt-\alpha q\,dx\wedge dt+\beta\,dq\wedge dt).$$

Equating the coefficients of the two-forms on both sides of (17), the following system of equations result,

$$F_u=-\lambda_3,\quad F_p=0,\quad F_q=0,$$
$$G_u=\lambda_1,\quad G_p=\lambda_2,\quad G_q=\beta\lambda_3,$$
$$F_t-G_x-[F,G]=\lambda_1 p+\lambda_2 q-\lambda_3 up+\alpha\lambda_3 q. \tag{18}$$

By Galilean invariance of (16) in the independent variables x and t, it suffices to suppose that F and G are independent of x and t. This introduces a considerable simplification into (18) reducing it to,

$$F=F(u),\quad G=G(u,p,q),\quad G_q=-\beta F_u,\quad pG_u+qG_p+upF_u-\alpha qF_u=-[F,G]. \tag{19}$$

It is required to find a solution to (19). In fact, it will be shown that a very general solution can be obtained. First, since F depends only on u, the third equation in (19) can be integrated to give G

$$G(u,p,q)=-\beta qF_u(u)+G'(u,p). \tag{20}$$

In (20), $G'(u,p)$ represents an integration constant which is independent of q. Substituting (20) into the fourth equation of (19), it is found that

$$-[F,-\beta qF_u+G']=-\beta pqF_{uu}+pG'_u+qG'_p+upF_u-\alpha qF_u=q(-\beta pF_{uu}+G'_p-\alpha F_u)+pG'_u+upF_u. \tag{21}$$

Equating coefficients of q on both sides of (21) yields two equations

$$\beta[F,F_u]=-\beta pF_{uu}+G'_p-\alpha F_u,\qquad -[F,G']=pG'_u+upF_u. \tag{22}$$

Solving the first equation in (22), for G'_p, we obtain

$$G'_p=\beta[F,F_u]+\beta pF_{uu}+\alpha F_u.$$

Since F depends only on u, this equation can be integrated with respect to p to obtain G',

$$G'(u,p)=\beta p[F,F_u]+\frac{1}{2}\beta p^2F_{uu}+\alpha pF_u+G''(u), \tag{23}$$

and $G''(u)$ is a final constant of integration independent of p.

Substituting $G'(u,p)$ into the second equation in (22), there results

$$\beta p[F,[F,F_u]]+\frac{1}{2}\beta p^2[F,F_{uu}]+\alpha p[F,F_u]+[F,G'']+p(\beta p[F,F_{uu}]+\frac{1}{2}\beta p^2F_{uuu}+\alpha pF_{uu}+G''_u)+upF_u=0.$$

Again, since F and G'' both just depend on u, this is a cubic polynomial in p, which we write

$$\frac{1}{2}\beta p^3F_{uuu}+p^2(\frac{1}{2}\beta[F,F_{uu}]+\beta[F,F_{uu}]+\alpha F_{uu})+p(\beta[F,[F,F_u]]+\alpha[F,F_u]+G''_u+uF_u)+[F,G'']=0. \tag{24}$$

Now, it is required to equate the coefficient of each power of p to zero in (24).

The coefficient of p^3 must vanish, which implies that $F_{uuu}=0$, hence

$$F(u)=X_1+uX_2+u^2X_3. \tag{25}$$

The X_j denote elements of a noncommutiitive algebra, such as matrices, and need not depend on any of the bundle coordinates.

The coefficient of p^2 must vanish. This yields the following equation after substituting F from (25),

$$3\beta[X_1,X_3]+2\alpha X_3+3\beta[X_2,X_3]u=0.$$

The coefficients of the u variable in this must vanish, and requiring this results in two bracket relations

$$[X_1, X_3] = -\frac{2\alpha}{3\beta} X_3 = -\gamma X_3, \qquad [X_2, X_3] = 0. \tag{26}$$

The constant $\gamma = 2\alpha/3\beta$.

The coefficient of p must vanish which implies the equation

$$\beta[F, [F, F_u]] + \alpha[F, F_u] + G''_u + uF_u = 0. \tag{27}$$

This will yield G'_u once the brackets in (27) have been specified. The bracket $[F, F_u]$ can be expressed by using (26) and defining $X_7 = [X_1, X_2]$ as follows

$$[F, F_u] = [X_1 + uX_2 + u^2X_3, X_2 + 2uX_3] = X_7 - 2\gamma uX_3. \tag{28}$$

Moreover,

$$[F, [F, F_u]] = 2\gamma^2 uX_3 + [X_1 + uX_2 + u^2X_3, X_7].$$

By applying the Jacobi identity $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$, this can be simplified by using

$$[X_3, X_7] = [X_3, [X_1, X_2]] = -[X_1, [X_2, X_3]] - [X_2, [X_3, X_1]] = -\gamma[X_2, X_3] = 0. \tag{29}$$

Introducing $X_5 = [X_1, X_7]$ and $X_6 = [X_2, X_7]$, it follows that

$$[F, [F, F_u]] = 2\gamma^2 uX_3 + X_5 + uX_6. \tag{30}$$

Substituting these results into (27), we can solve for G''_u

$$G''_u = -2\beta\gamma^2 uX_3 - \beta X_5 - \beta uX_6 - \alpha X_7 + 2\alpha\gamma uX_3 - uX_2 - 2u^2X_3.$$

The right-hand side just depends on u and constants, so this can be integrated to give G''

$$G'' = -\beta\gamma^2u^2X_3 - \beta uX_5 - \frac{1}{2}\beta u^2X_6 - \alpha uX_7 + \alpha\gamma u^2X_3 - \frac{1}{2}u^2X_2 - \frac{2}{3}u^3X_3 + X_4, \tag{31}$$

where X_4 is a final integration constant.

There remains a term in (24) that does not depend on p, namely $[F, G''] = 0$. Since both F and G'' are known from (25) and (31), a polynomial in the variable u results. The following brackets are required to study this term and can be worked out using the Jacobi identity

$$[X_2, [X_1, X_7]] = [X_1, X_6], \qquad [X_3, [X_1, X_7]] = \gamma[X_7, X_3] = 0.$$

Two more brackets can be produced from these, namely,

$$[X_2, X_5] = [X_1, X_6], \qquad [X_3, X_5] = 0. \tag{32}$$

Substituting F and G'' into $[F, G''] = 0$, a large polynomial in u results,

$$[X_1+uX_2+u^2X_3, -\beta\gamma^2u^2X_3-\beta uX_5-\frac{1}{2}\beta u^2X_6-\alpha uX_7+\alpha\gamma u^2X_3-\frac{1}{2}u^2X_2-\frac{2}{3}u^3X_3+X_4] = 0. \tag{33}$$

Now the coefficient of each power of u in (33) is equated to zero, and each such equation provides a bracket relation for the algebra. This is a long process, and we will just summarize the results after simplifying using the results above.

The coefficient of u^5 results in the bracket $[X_3, X_3] = 0$. The coefficient of u^4 gives the bracket $[X_3, X_6] = 0$. The equation which results from u^3 after some simplification is $3\beta[X_2, X_6] - 4\gamma X_3 = 0$ and the coefficient of u^2 simplifies to $X_7 = -2\alpha X_6 - 2\alpha\gamma^2X_3 + 2\beta\gamma^3X_3 - 3\beta[X_1, X_6] + 2[X_3, X_4]$. The coefficient of u produces the equation $\beta[X_1, X_5] + \alpha[X_1, X_7] - [X_2, X_4] = 0$ and finally, the term independent of u supplies one last bracket $[X_1, X_4] = 0$.

At this point, the complete algebra which has been produced by this procedure can be summarized all at once,

$$
\begin{gathered}
[X_1, X_3] = -\gamma X_3, \quad [X_2, X_3] = 0, \quad [X_3, X_7] = 0, \\
X_5 = [X_1, X_7], \quad X_6 = [X_2, X_7], \quad X_7 = [X_1, X_2], \\
[X_2, X_5] = [X_1, X_6], \qquad [X_3, X_5] = [X_3, X_6] = 0, \\
3\beta[X_2, X_6] - 4\gamma X_3 = 0, \\
X_7 = -2\alpha X_6 - 2\alpha\gamma^2 X_3 + 2\beta\gamma^3 X_3 - 3\beta[X_1, X_6] + 2[X_3, X_4], \\
\beta[X_1, X_5] + \alpha X_5 - [X_2, X_4] = 0, \qquad [X_1, X_4] = 0.
\end{gathered} \tag{34}
$$

Collecting terms and substituting $p = u_x$ and $q = u_{xx}$ from (15) into G, the following results are obtained for F and G,

$$F = X_1 + uX_2 + u^2 X_3, \tag{35}$$

$$
\begin{aligned}
G = (-\beta u_{xx} + \alpha u_x - \frac{1}{2}u^2)X_2 + (-2\beta u u_{xx} - 2\gamma\beta u u_x + \beta u_x^2 + 2\alpha u u_x - \beta\gamma^2 u^2 + \alpha\gamma u^2 - \frac{2}{3}u^3)X_3 \\
+ X_4 - \beta u X_5 - \frac{1}{2}\beta u^2 X_6 + (\beta u_x - \alpha u)X_7.
\end{aligned} \tag{36}
$$

It will now be proved that F and G can be used to build a Lax pair.

5 A Lax Pair for the Burgers-KdV EquationI

On a two-dimensional solution submanifold of the differential ideal, the forms α_i are annihilated and a Lax pair for the system can be given in terms of F and G.

Theorem 5.1. A Lax pair for equation (16) exists and is given explicitly as

$$\mathbf{y}_x = -F\mathbf{y}, \qquad \mathbf{y}_t = -G\mathbf{y}, \tag{37}$$

where F and G are given in (35) and (36) and $\mathbf{y}$ represents the set of prolongation coordinates. It is required that the X_j satisfy algebra (34) thus (37) holds irrespective of the particular representation used for the X_j.

Proof: It will be shown that the pair (37) satisfies the required zero curvature condition for n arbitrary, which is given as

$$G_x - F_t + [F, G] = 0. \tag{38}$$

To compress the presentation, let us abbreviate G as

$$G = g_2 X_2 + g_3 X_3 + X_4 + g_5 X_5 + g_6 X_6 + g_7 X_7,$$

where the g_j are defined by comparing this to (36). By straightforward calculation using (34),

$$
\begin{aligned}
FG - GF = g_2 X_7 - \gamma g_3 X_3 + g_5[X_1, X_5] + g_6[X_1, X_6] + g_7 X_5 + u[X_2, X_4] \\
+ g_5 u[X_1, X_6] - \frac{2}{3}\gamma u^3 X_3 + g_7 u X_6 + u^2[X_3, X_4].
\end{aligned}
$$

Since $g_6 + ug_5 = -\frac{3}{2}\beta u^2$, then using (34) once again, it follows that the unknown brackets $[X_2, X_4]$ and $[X_3, X_4]$ cancel out, and $[F, G]$ simplifies to the form

$$
\begin{aligned}
[F, G] &= g_2 X_7 - \gamma g_3 X_3 + u(\alpha X_5 - [X_2, X_4]) - \frac{1}{2}u^2(-X_7 - 2\alpha X_6 - 2\alpha\gamma^2 X_3 + 2\beta\gamma^3 X_3 + 2[X_3, X_4]) \\
&\quad + g_7 X_5 + u[X_2, X_4] - \frac{2}{3}\gamma u^3 X_3 + g_7 u X_6 + u^2[X_3, X_4] \\
&= (g_2 + \frac{1}{2}u^2)X_7 + (-\gamma g_3 + \alpha\gamma^2 u^2 - \beta\gamma^2 u^2 - \frac{2}{3}\gamma u^3)X_3 + (\alpha u + g_7)X_5 + (\alpha u^2 + u g_7)X_6.
\end{aligned} \tag{39}
$$

Now replacing the g_j in (39), the bracket $[F,G]$ simplifies to

$$[F,G] = (2\beta\gamma u u_{xx} + 2\gamma^2\beta u u_x - \beta\gamma u_x^2 - 2\alpha\gamma u u_x)X_3 + \beta u_x X_5 + \beta u u_x X_6 + (-\beta u_{xx} + \alpha u_x)X_7. \quad (40)$$

This is to be substituted into the zero curvature condition (38) along with G_x and F_t, which are given by

$$F_t = u_t X_2 + 2uu_t X_3,$$
$$G_x = (-\beta u_{xxx} + \alpha u_{xx} - uu_x)X_2$$
$$+(-2\beta uu_{xxx} - 2\beta u_x u_{xx} - 2\gamma\beta u_x^2 - 2\gamma\beta uu_{xx} + 2\beta u_x u_{xx} + 2\alpha u_x^2 + 2\alpha uu_{xx} - 2\beta\gamma^2 uu_x + 2\alpha\gamma uu_x - 2u^2 u_x)X_3$$
$$-\beta u_x X_5 - \beta uu_x X_6 + (\beta u_{xx} - \alpha u_x)X_7.$$

Substituting the results above along with (40) into (38), before simplifying the zero curvature condition is written

$$G_x - F_t + [F,G] = (-u_t - \beta u_{xxx} + \alpha u_{xx} - uu_x)X_2$$
$$+(-2uu_t - 2\beta uu_{xxx} - 2\beta u_x u_{xx} - 2\gamma\beta u_x^2 - 2\gamma\beta uu_{xx} + 2\beta u_x u_{xx} + 2\alpha u_x^2 + 2\alpha uu_{xx} - 2\beta\gamma^2 uu_x + 2\alpha\gamma uu_x - 2u^2 u_x)X_3$$
$$-\beta u_x X_5 - \beta uu_x X_6 + (\beta u_{xx} - \alpha u_x)X_7$$
$$+(2\beta\gamma uu_{xx} + 2\beta\gamma^2 uu_x - \beta\gamma u_x^2 - 2\alpha\gamma uu_x)X_3$$
$$+\beta u_x X_5 + \beta uu_x X_6 - (\beta u_{xx} - \alpha u_x)X_7.$$

Clearly, unknown brackets and terms in X_5, X_6 and X_7 completely cancel out. It is necessary to collect and simplify the coefficient of the X_3 term in order to formulate a conclusion. To do this, everything becomes transparent if we substitute the fact that $3\beta\gamma = 2\alpha$. The coefficient of X_3 then simplifies to,

$$-2uu_t - 2\beta u_x u_{xx} - 2\beta uu_{xxx} - \frac{4}{3}\alpha u_x^2 - \frac{4}{3}\alpha uu_{xx} + 2\beta u_x u_{xx} + 2\alpha u_x^2 + 2\alpha uu_{xx}$$
$$-2\beta\gamma^2 uu_x + 2\alpha\gamma uu_x - 2u^2 u_x + \frac{4}{3}\alpha uu_{xx} + 2\beta\gamma^2 uu_x - \frac{2}{3}\alpha u_x^2 - 2\alpha\gamma uu_x$$
$$= -2uu_t - 2\beta uu_{xxx} + 2\alpha uu_{xx} - 2u^2 u_x = -2u(u_t + \beta u_{xxx} - \alpha u_{xx} + uu_x).$$

Therefore, zero curvature condition (38) reduces to the following two terms after cancellations,

$$G_x - F_t + [F,G]$$
$$= -(u_t + \beta u_{xxx} - \alpha u_{xx} + uu_x)X_2 - 2u(u_t + \beta u_{xxx} - \alpha u_{xx} + uu_x)X_3. \quad (41)$$

Provided X_2 and X_3 are not set to zero, the result in (41) will clearly vanish if and only if u satisfies Burgers-KdV equation (16). In this case, the zero curvature condition is satisfied identically, and this completes the proof.

6 Conclusions

A closed ideal of differential forms has been found for the Burgers-KdV equation. A procedure for generating prolongations has been established and applied to this equation to obtain a prolongation algebra. The cancelation of unknown brackets in the verification of the Lax is a very interesting observation. The ideal can also be used to study the limiting case $\alpha = 0$. The algebra can in this instance be simplified to a finite form by taking $X_3 = X_4 = 0$ and $X_5 = -kX_1$, $X_6 = kX_2$ so that $[X_2, X_5] = [X_1, X_6]$ holds.There are two brackets left which involve X_7, and these two will be consistent provided that $k = -1/3\beta$. If we set $E_+ = X_1$, $X_2 = e_-$ and $X_7 = h$, algebra (34) reduces to a finite deformed $sl(2)$ algebra which can be written as

$$[e_\pm, h] = \pm\frac{1}{3\beta}e_\pm, \qquad [e_+, e_-] = h.$$

It has also been shown that (34) serves to define an algebraic Lax pair for the system, and following the strategy of Dodd and Fordy (Dodd and Fordy, 1983), it should be possible to embed it in a simple Lie algebra, but will not be done here.

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