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Local Convexity Shape-Preserving Surface Data Visualization by Spline Function

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Research Article

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Abstract

The main purpose of this paper is the visualization of convex surface data to present a smooth, visually pleasing and interactive convexity preserving surfaces. The rational cubic function with three free parameters is extended to rational bi-cubic partially blended function to preserve the shape of convex surface data. The function involves twelve free parameters in each rectangular patch. Data dependent constraints are derived for four of these parameters to preserve the shape of convex surface data while other eight are left free to user for the refinement of convexity preserving surface of data. Moreover, the scheme under discussion is C^1, flexible, simple, local and economical as compared to existing schemes

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1 Introduction

The study of curves and surfaces is a key element in computer aided geometric design (CAGD) that has been around for quite some time. The methods of CAGD have arisen from the need of efficient computer representation of practical curves and surfaces used in engineering design. Spline interpolation is a powerful tool in Computer Graphics, CAGD and Engineering as well. Therefore, in these fields, it is often desirable to generate a convexity preserving interpolating curve and surface according to the given convex data. The aspiration of this paper is to preserve the hereditary attribute that is the convexity of data.

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Convexity is a substantial shape characteristic of the data. The significance of the convexity preserving interpolation problems in industry cannot be denied. A number of examples can be quoted in this regard, like the modelling of cars in automobile industry, aero-plane and ship design. A crumpled surface is an unwanted characteristic. Human aesthetic sense demands convexity preserving nice and smooth surfaces without wiggles.

Designing well shaped smooth surfaces also arise in manufacturing the TV-screens. In order to accomplish with the demands of the customer, as flat as possible TV-screens are most appreciated. In the surface designing sense we can say that the screens must preserve the convexity.

Butt (1991) discussed some other areas to demonstrate the importance of convexity of convex data, like, designing telecommunication systems, nonlinear programming arising in engineering problems, approximation of functions, optimal control and parameter estimation.

The problem of convexity preserving interpolation has been considered by many authors. Asaturyan (1990) developed a global scheme for the convex surfaces. In the scheme, each rectangular grid was divided into nine sub rectangles. The scheme is not local. Any change made in x direction of single rectangle edge causes a change throughout the grid of all remaining edges of sub rectangle. Asaturyan et al. (2001) constructed a six degree piecewise polynomial interpolant for the space curves to satisfy the shape-preserving properties for collinear and coplanar data.

Brodlie and Butt (1991) developed a piecewise rational cubic function to preserve the shape of convex data. The authors inserted extra knots in the interval where the interpolation loses the convexity of convex data that is the drawback of this scheme. Carnicer et al. (1996) analyzed the convexity preserving properties of rational Bézier and non-uniform rational B-spline curves from a geometric point of view and characterized totally positive systems of functions in terms of geometric convexity preserving properties of the rational curves.

Clements (1992) developed a C^2 parametric rational cubic interpolant with tension parameter to preserve the convexity. Sufficient conditions were derived to preserve the convexity of the function on strictly left/right winding polygonal line segments. Costantini and Fontanella (1990) preserved the convexity of data by semi global method. The scheme has some research gaps like the degree of rectangular patches in the interpolant were too large, the resulting surfaces were not visually pleasing and smooth.

Dodd et al. (1983) presented a method to preserve the convexity of the surface along the rectangle grid lines but in the interior of the grids, the convexity of the surface was not preserved by quadratic splines. They produced undesirable flat spots due to vanishing of second order mixed partial derivatives on the boundary of the rectangles. Sufficient conditions for a tensor-product Bézier surface to be convex were derived by Floater (1994). The convexity condition was generalized to C^1 tensor-product B-spline surfaces. These sufficient conditions were in the form of inequalities which involved control points. The schemes (Costantini and Fontanella, 1990; Devore and Yan 1986; Dodd, 1983; Floater, 1994) failed to preserve the convexity of convex data when data with derivatives was given. The rational spline was represented in terms of first derivative values at the knots and provided an alternative to the spline-under-tension to preserve the shape of monotone and convex data by Gregory (1986).

Hussain et al. (2011) developed a surface C^0 interpolation for convex and positive data. The authors used a rational bi-quadratic spline function with eight shape parameters to preserve the convexity of convex surface data. Hussain and Maria (2008) developed a rational bi-cubic function with four free parameters to preserve the shape of convex surface data. Data dependent sufficient conditions were derived on the free parameters to preserve the convexity of convex data. Hussain et al. (2008) used a rational bi-quartic partially blended function with eight shape parameters to visualize the convex surface data. The authors derived data dependent conditions on parameters to preserve the shape of convex surface data.

Hussain and Maria (2006) discussed the problem of visualization of convex surface data. A piecewise rational bi-cubic function with two free parameters was used to preserve the shape of convex surface data. Maria and Hussain (2008) proposed a local convexity preserving scheme for 3D convex data arranged over rectangular grid. Constraints on free parameters in the description of rational bi-cubic partially blended patches with eight free parameters were derived to preserve the shape of convex 3D data. The schemes (Hussain et al., 2011; Hussain and Maria, 2008; Hussain et al., 2008; Hussain and Maria, 2006; Maria and Hussain, 2008)) are local but unfortunately did not provide the liberty to the user for the refinement of surfaces as desired.

McAllister and Roulier (1981), Passow and Roulier (1977) and Roulier (1987) considered the problem of interpolating monotonic and convex data in the sense of monotonicity and convexity preserving. They used a piecewise polynomial Bernstein- Bézier function and introduce additional knots into their schemes. Such scheme for quadratic spline interpolation was described by McAllister and Roulier (1981) and this idea was more developed by Schumaker (1983) who used piecewise quadratic polynomial which was very economical but the method generally inserts an extra knot in each interval to interpolate.

The rational Bernstein-Bézier cubic interpolation, cubic and bi-cubic Hermite schemes are discussed in comprehensive form Farin (1996), Hoscheck and Lasser (1993). The rational cubic function can be extended to rational bi-cubic function (for tensor product patches) and rational bi-cubic partially blended function (for coon patches).The former is hard to compute and implement whereas the latter is easy to work out and execute. Moreover it is computationally efficient and time saving due to less number of conditions applied on shape parameters, no need of extra knots in the interpolant and does not require any modification in data.

In this paper, we extend the rational cubic function with three free parameters to rational bi-cubic partially blended function. There are twelve free parameters in each rectangular patch of rational bi-cubic partially blended function. Data dependent sufficient constraints are developed for four free parameters to visualize the shape of convex surface data while the other eight are left free to user's choice to refine the shape of convex data. The proposed scheme has a number of attributes over the existing schemes.

- In (Hussain et al. (2011), Schumaker, (1983)), the smoothness of surface interpolation is C^0 while in this paper the surface interpolant attained C^1 .
- The developed scheme has been demonstrated through different numerical examples and observed that the scheme is not only local, computationally economical, easy to compute, time saving but also visually pleasant as compared to existing schemes (Hussain et al. (2011), Hussain and Maria (2008), Hussain et al. (2008), Hussain and Maria (2006), Maria and Hussain (2008)). In contrast to the schemes (Asaturyan (1990),

Asaturyan et al. (2001), Carnicer et al. (1996), Clements (1992), Costantini and Fontanella (1990)) are global.

- Surfaces can be made more visually pleasing and smooth (still preserving the convexity), as desired by the designer, by merely adjusting some of the shape parameters in the description of the rational bi-cubic partially blended interpolant. It has more freedom for the user and without effecting the data. In contrast to the schemes Hussain et al. (2011), Hussain and Maria (2008), Hussain et al. (2008), Hussain and Maria (2006), Maria and Hussain (2008) were not flexible to designer to refine the convexity surfaces.
- Data dependent sufficient constraints on shape parameters are attained which guarantee to preserve the convex surface. The constraints on shape parameters are not dependent to each other.
- No additional points (knots) are inserted in the interpolant. In contrast, piecewise polynomial Bernstein- Bézier function methods of McAllister and Roulier (1981), Passow and Roulier (1977) and Roulier (1987), the quadratic spline methods of Schumaker, (1983) and the cubic interpolation method of Brodlie and Butt (1991) and Butt (1991) require the introduction of additional knots when interpolant loses the required shape of data.
- The schemes Costantini and Fontanella (1990), Devore and Yan (1986), Dodd (1983), Floater (1994) are unsuccessful to preserve the convexity of convex data when data with derivatives is given. In contrast no data with derivative constraints are imposed, making the proposed scheme more flexible.

The remaining part of paper is organized as: A review of rational cubic function is given in section 2. The rational cubic function is extended to rational bi-cubic partially blended function is given in section 3. Derivatives approximation method is discussed in section 4. The problem of shape-preserving convex surface is discussed in section 5. Some numerical examples for convex surface data to support usefulness of the scheme are discussed in sections 6. Finally, the conclusion of this work is given in section 7.

2 Review of Rational Cubic Spline Function

Let $\{(x_i, f_i) : i = 0,1,2,...,n\}$ be the given set of data points such as $x_0 < x_1 < x_2 < ... < x_n$. The rational cubic function with three free parameters see Abbas et al. (2012), in each subinterval $I_i = [x_i, x_{i+1}], i = 0,1,2,....,n-1$ is defined as:

$$S_i(x) = \frac{p_i(\theta)}{q_i(\theta)}, \tag{1}$$

with,

$$p_i(\theta) = u_i f_i (1-\theta)^3 + (w_i f_i + u_i h_i d_i)\theta(1-\theta)^2 + (w_i f_{i+1} - v_i h_i d_{i+1})\theta^2 (1-\theta) + v_i f_{i+1}\theta^3,$$
$$q_i(\theta) = u_i (1-\theta)^3 + w_i\theta(1-\theta) + v_i\theta^3,$$

where $\theta = (x - x_i)/h_i$, $h_i = x_{i+1} - x_i$, and u_i, v_i, w_i are the positive free parameters and d_i denotes the derivative values at knots. It is worth noting that when we use the values of these free parameters as $u_i = 1, v_i = 1$ *and* $w_i = 3$ then the C^1 piecewise rational cubic function (1) reduces to standard cubic Hermite spline.

The piecewise rational cubic function has the following interpolatory conditions,

$$\begin{cases} S_i(x_i) = f_i & S_i(x_{i+1}) = f_{i+1} \\ S_i'(x_i) = d_i & S_i'(x_{i+1}) = d_{i+1} \end{cases} \tag{2}$$

where $S_i'(x)$ denotes the derivative with respect to $'x'$.

Abbas et al. (2012) developed the following result for the convexity shape-preserving of 2D convex data.

Theorem [Abbas et al. (2012)]

The C^1 piecewise rational cubic function (1) preserves the convexity of convex data, if in each subinterval $I_i = [x_i, x_{i+1}], i = 0,1,2,....,n$, the free parameters satisfy the following sufficient conditions,

$$w_i > \max\left\{0, \frac{d_{i+1}v_i}{(d_{i+1}-\Delta_i)}, \frac{d_{i+1}v_i}{(\Delta_i - d_i)}, \frac{2u_iv_i(d_{i+1}-\Delta_i)}{(d_{i+1}v_i - \Delta_i u_i)}, \frac{2u_iv_i(\Delta_i - d_i)}{(\Delta_i v_i - d_i u_i)}, \frac{u_iv_i(d_{i+1}-d_i)}{\Delta_i(u_i+v_i)}\right\},\ u_i, v_i > 0$$

The above constraints are rearranged as:

$$w_i = l_i + \max\left\{0, \frac{d_{i+1}v_i}{(d_{i+1}-\Delta_i)}, \frac{d_{i+1}v_i}{(\Delta_i - d_i)}, \frac{2u_iv_i(d_{i+1}-\Delta_i)}{(d_{i+1}v_i - \Delta_i u_i)}, \frac{2u_iv_i(\Delta_i - d_i)}{(\Delta_i v_i - d_i u_i)}, \frac{u_iv_i(d_{i+1}-d_i)}{\Delta_i(u_i+v_i)}\right\}, l_i \geq 0, u_i, v_i > 0$$

where $\Delta_i = (f_{i+1} - f_i)/h_i$.

3 Rational Bi-Cubic Partially Blended Spline Function

We extend a C^1 piecewise rational cubic function (1) to rational bi-cubic partially blended function $S(x, y)$ over the rectangular Domain $\Omega = [a,b]\times[c,d]$. The partition of arbitrary intervals $[a,b]$ and $[c,d]$ is defined as $\pi : a = x_0 < x_1 < x_2 < ... < x_n = b,\ \hat{\pi} : c = y_0 < y_1 < y_2 < ... < y_m = d$ respectively. The rational bi-cubic partially blended function over each rectangular patch $[x_i, x_{i+1}]\times[y_j, y_{j+1}], i = 0,1,2,...,n-1; j = 0,1,2,...,m-1$ is defined as:

$$S(x, y) = -A(\theta)FB^T(\varphi) \tag{3}$$

where,

$$F = \begin{pmatrix} 0 & S(x, y_j) & S(x, y_{j+1}) \\ S(x_i, y) & S(x_i, y_j) & S(x_i, y_{j+1}) \\ S(x_{i+1}, y) & S(x_{i+1}, y_j) & S(x_{i+1}, y_{j+1}) \end{pmatrix} \tag{4}$$

$$\begin{aligned} A(\theta) &= \left(a_0(\theta)\ \ a_1(\theta)\ \ a_2(\theta)\right) \\ B(\varphi) &= \left(b_0(\varphi)\ \ b_1(\varphi)\ \ b_2(\varphi)\right) \end{aligned} \tag{5}$$

with,

$$\begin{cases} a_0(\theta) = -1, a_1(\theta) = (1-\theta)^2(1+2\theta), a_2(\theta) = \theta^2(3-2\theta) \\ b_0(\varphi) = -1, b_1(\varphi) = (1-\varphi)^2(1+2\varphi), b_2(\varphi) = \varphi^2(3-2\varphi). \end{cases}$$

In the above expression $\theta = (x - x_i)/h_i$, $\varphi = (y - y_j)/\hat{h}_j$, with $h_i = x_{i+1} - x_i, \hat{h}_j = y_{j+1} - y_j$, the rational cubic functions $S(x, y_j)$, $S(x, y_{j+1})$, $S(x_i, y)$ and $S(x_{i+1}, y)$ are defined as:

$$S(x, y_j) = \frac{\sum_{i=0}^{3} (1-\theta)^{3-i} \theta^i L_i}{q_1(\theta)}, \tag{6}$$

where,

$L_0 = u_{i,j} F_{i,j}$,

$L_1 = w_{i,j} F_{i,j} + u_{i,j} h_i F_{i,j}^x$,

$L_2 = w_{i,j} F_{i+1,j} - v_{i,j} h_i F_{i+1,j}^x$,

$L_3 = v_{i,j} F_{i+1,j}$

$q_1(\theta) = u_{i,j}(1-\theta)^3 + w_{i,j}\theta(1-\theta) + v_{i,j}\theta^3$.

$$S(x, y_{j+1}) = \frac{\sum_{i=0}^{3} (1-\theta)^{3-i} \theta^i M_i}{q_2(\theta)}, \tag{7}$$

such that,

$M_0 = u_{i,j+1} F_{i,j+1}$,

$M_1 = w_{i,j+1} F_{i,j+1} + u_{i,j+1} h_i F_{i,j+1}^x$,

$M_2 = w_{i,j+1} F_{i+1,j+1} - v_{i,j+1} h_i F_{i+1,j+1}^x$,

$M_3 = v_{i,j+1} F_{i+1,j+1}$

$q_2(\theta) = u_{i,j+1}(1-\theta)^3 + w_{i,j+1}\theta(1-\theta) + v_{i,j+1}\theta^3$.

$$S(x_i, y) = \frac{\sum_{i=0}^{3} (1-\varphi)^{3-i} \varphi^i N_i}{q_3(\varphi)}, \tag{8}$$

with,

$N_0 = \hat{u}_{i,j} F_{i,j}$,

$N_1 = \hat{w}_{i,j} F_{i,j} + \hat{u}_{i,j} \hat{h}_j F_{i,j}^y$,

$N_2 = \hat{w}_{i,j} F_{i,j+1} - \hat{v}_{i,j} \hat{h}_j F_{i,j+1}^y$,

$N_3 = \hat{v}_{i,j} F_{i,j+1}$

$q_3(\theta) = \hat{u}_{i,j}(1-\varphi)^3 + \hat{w}_{i,j}\varphi(1-\varphi) + \hat{v}_{i,j}\varphi^3$.

$$S(x_{i+1}, y) = \frac{\sum_{i=0}^{3} (1-\varphi)^{3-i} \varphi^i O_i}{q_4(\varphi)}, \tag{9}$$

with,

$O_0 = \hat{u}_{i+1,j} F_{i+1,j}$,

$O_1 = \hat{w}_{i+1,j} F_{i+1,j} + \hat{u}_{i+1,j} \hat{h}_j F_{i+1,j}^y$,

$$O_2 = \hat{w}_{i+1,j} F_{i+1,j+1} - \hat{v}_{i+1,j} \hat{h}_j F^y_{i+1,j+1},$$
$$O_3 = \hat{v}_{i+1,j} F_{i+1,j+1}$$
$$q_4(\theta) = \hat{u}_{i+1,j}(1-\varphi)^3 + \hat{w}_{i+1,j}\varphi(1-\varphi) + \hat{v}_{i+1,j}\varphi^3.$$

4 Determination of Derivatives

Mostly, the derivative $F^x_{i,j}, F^y_{i,j}$ and $F^{xy}_{i,j}$ at the knots are not given. These derivatives must be derived either at the given data set $\{(x_i, y_j, F_{i,j}) : i = 0,1,2,...,n-1;\ j = 0,1,2,...,m-1\}$ or by some other means. In this paper, these values are calculated by following Arithmetic mean method for 3D data in such a way that the smoothness of the interpolant (3) is maintained. Let us denote $F^x_{i,j}$ and $F^y_{i,j}$ as the first order derivatives with respect to x and y, respectively, at the data point $F_{i,j}$. Similarly, let the mixed derivatives be denoted by $F^{xy}_{i,j}$.

4.1 Arithmetic Mean Method for 3D Data

$$F^x_{0,j} = \Delta_{0,j} + \frac{(\Delta_{0,j} - \Delta_{1,j})h_0}{(h_0 + h_1)}, F^x_{n,j} = \Delta_{n-1,j} + \frac{(\Delta_{n-1,j} - \Delta_{n-2,j})h_{n-1}}{(h_{n-1} + h_{n-2})},$$
$$F^x_{i,j} = 0.5(\Delta_{i,j} + \Delta_{i-1,j}) \quad, i = 1,2,3,...,n-1;\ j = 0,1,2,...,m$$
$$F^y_{i,0} = \hat{\Delta}_{i,0} + \frac{(\hat{\Delta}_{i,0} - \hat{\Delta}_{i,1})\hat{h}_0}{(\hat{h}_0 + \hat{h}_1)}, F^y_{i,m} = \hat{\Delta}_{i,m-1} + \frac{(\hat{\Delta}_{i,m-1} - \hat{\Delta}_{i,m-2})\hat{h}_{m-1}}{(\hat{h}_{m-1} + \hat{h}_{m-2})}$$
$$F^y_{i,j} = 0.5(\hat{\Delta}_{i,j} + \hat{\Delta}_{i,j-1}), i = 0,1,2,...,n;\ j = 1,2,3,...,m-1$$
$$F^{xy}_{i,j} = \frac{1}{2}\left\{\frac{F^x_{i,j+1} - F^x_{i,j-1}}{\hat{h}_{j-1} + \hat{h}_j} + \frac{F^y_{i+1,j} - F^y_{i-1,j}}{h_{i-1} + h_i}\right\}, i = 1,2,...,n-1;\ \ j = 1,2,...,m-1,$$

where $\Delta_{i,j} = \dfrac{F_{i+1,j} - F_{i,j}}{h_i}$, $\hat{\Delta}_{i,j} = \dfrac{F_{i,j+1} - F_{i,j}}{\hat{h}_j}$.

5 Convexity Shape-Preserving Interpolation

The rational bi-cubic partially blended function (3) does not guarantee to preserve the shape of convex surface data. So, it is required to assign suitable constraints on the free parameters by some mathematical treatment to preserve the convexity of convex data.

Theorem 5.1

The rational bi-cubic partially blended function (3) preserves the convexity of 3D convex data, if in each rectangular patch $[x_i, x_{i+1}] \times [y_j, y_{j+1}], i = 0,1,2,...,n-1; j = 0,1,2,...,m-1$; the free parameters satisfy the following sufficient conditions

$u_{i,j}, u_{i,j+1}, \hat{u}_{i,j}, \hat{u}_{i+1,j} > 0$, $v_{i,j}, v_{i,j+1}, \hat{v}_{i,j}, \hat{v}_{i+1,j} > 0$

$w_{i,j} > \max\left\{0, D_k, 1 \le k \le 5, k \in Z^+\right\}$, $w_{i,j+1} > \max\left\{0, D_k, 6 \le k \le 10, k \in Z^+\right\}$,

$\hat{w}_{i,j} > \max\left\{0, D_k, 11 \le k \le 15, k \in Z^+\right\}, \hat{w}_{i+1,j} > \max\left\{0, D_k, 16 \le k \le 20, k \in Z^+\right\}$,

where,

$$D_1 = \frac{F^x_{i+1,j} v_{i,j}}{\left(F^x_{i+1,j} - \Delta_{i,j}\right)}, D_2 = \frac{F^x_{i+1,j} v_{i,j}}{\left(\Delta_{i,j} - F^x_{i,j}\right)}, D_3 = \frac{2u_{i,j} v_{i,j}\left(F^x_{i+1,j} - \Delta_{i,j}\right)}{\left(F^x_{i+1,j} v_{i,j} - \Delta_{i,j} u_{i,j}\right)}, D_4 = \frac{2u_{i,j} v_{i,j}\left(\Delta_{i,j} - F^x_{i,j}\right)}{\left(\Delta_{i,j} v_{i,j} - F^x_{i,j} u_{i,j}\right)},$$

$$D_5 = \frac{u_{i,j} v_{i,j}\left(F^x_{i+1,j} - F^x_{i,j}\right)}{\Delta_{i,j}\left(u_{i,j} + v_{i,j}\right)}, D_6 = \frac{F^x_{i+1,j+1} v_{i,j+1}}{\left(F^x_{i+1,j+1} - \Delta_{i,j+1}\right)}, D_7 = \frac{F^x_{i+1,j+1} v_{i,j+1}}{\left(\Delta_{i,j+1} - F^x_{i,j+1}\right)}, D_8 = \frac{2u_{i,j+1} v_{i,j+1}\left(F^x_{i+1,j+1} - \Delta_{i,j+1}\right)}{\left(F^x_{i+1,j+1} v_{i,j+1} - \Delta_{i,j+1} u_{i,j+1}\right)},$$

$$D_9 = \frac{2u_{i,j+1} v_{i,j+1}\left(\Delta_{i,j+1} - F^x_{i,j+1}\right)}{\left(\Delta_{i,j+1} v_{i,j+1} - F^x_{i,j+1} u_{i,j+1}\right)}, D_{10} = \frac{u_{i,j+1} v_{i,j+1}\left(F^x_{i+1,j+1} - F^x_{i,j+1}\right)}{\Delta_{i,j+1}\left(u_{i,j+1} + v_{i,j+1}\right)}, D_{11} = \frac{F^y_{i,j+1} \hat{v}_{i,j}}{\left(F^y_{i,j+1} - \hat{\Delta}_{i,j}\right)}, D_{12} = \frac{F^y_{i,j+1} \hat{v}_{i,j}}{\left(\hat{\Delta}_{i,j} - F^y_{i,j}\right)},$$

$$D_{13} = \frac{2\hat{u}_{i,j} \hat{v}_{i,j}\left(F^y_{i,j+1} - \hat{\Delta}_{i,j}\right)}{\left(F^y_{i,j+1} \hat{v}_{i,j} - \hat{\Delta}_{i,j} \hat{u}_{i,j}\right)}, D_{14} = \frac{2\hat{u}_{i,j} \hat{v}_{i,j}\left(\hat{\Delta}_{i,j} - F^y_{i,j}\right)}{\left(\hat{\Delta}_{i,j} \hat{v}_{i,j} - F^y_{i,j} \hat{u}_{i,j}\right)}, D_{15} = \frac{\hat{u}_{i,j} \hat{v}_{i,j}\left(F^y_{i,j+1} - F^y_{i,j}\right)}{\hat{\Delta}_{i,j}\left(\hat{u}_{i,j} + \hat{v}_{i,j}\right)}, D_{16} = \frac{F^y_{i+1,j+1} \hat{v}_{i+1,j}}{\left(F^y_{i+1,j+1} - \hat{\Delta}_{i+1,j}\right)},$$

$$D_{17} = \frac{F^y_{i+1,j+1} \hat{v}_{i+1,j}}{\left(\hat{\Delta}_{i+1,j} - F^y_{i+1,j}\right)}, D_{18} = \frac{2\hat{u}_{i+1,j} \hat{v}_{i+1,j}\left(F^y_{i+1,j+1} - \hat{\Delta}_{i+1,j}\right)}{\left(F^y_{i+1,j+1} \hat{v}_{i+1,j} - \hat{\Delta}_{i+1,j} \hat{u}_{i+1,j}\right)}, D_{19} = \frac{2\hat{u}_{i+1,j} \hat{v}_{i+1,j}\left(\hat{\Delta}_{i+1,j} - F^y_{i+1,j}\right)}{\left(\hat{\Delta}_{i+1,j} \hat{v}_{i+1,j} - F^y_{i+1,j} \hat{u}_{i+1,j}\right)}, D_{20} = \frac{\hat{u}_{i+1,j} \hat{v}_{i+1,j}\left(F^y_{i+1,j+1} - F^y_{i+1,j}\right)}{\hat{\Delta}_{i+1,j}\left(\hat{u}_{i+1,j} + \hat{v}_{i+1,j}\right)}$$

The above results are rearranged as:

$$w_{i,j} = \alpha_{i,j} + \max\left\{0, D_k, 1 \le k \le 5, k \in Z^+\right\},\ w_{i,j+1} = \beta_{i,j} + \max\left\{0, D_k, 6 \le k \le 10, k \in Z^+\right\},\ \alpha_{i,j} > 0, \beta_{i,j} > 0$$

$$\hat{w}_{i,j} = \gamma_{i,j} + \max\left\{0, D_k, 11 \le k \le 15, k \in Z^+\right\}, \hat{w}_{i+1,j} = \delta_{i,j} + \max\left\{0, D_k, 16 \le k \le 20, k \in Z^+\right\}, \gamma_{i,j} > 0, \delta_{i,j} > 0$$

Proof:

Let $\left\{(x_i, y_j, F_{i,j}) : i = 0,1,2,...,n-1; j = 0,1,2,...,m-1\right\}$ be a convex surface data arranged over a rectangular patch $\left[x_i, x_{i+1}\right] \times \left[y_j, y_{j+1}\right], i = 0,1,2,...,n-1; j = 0,1,2,...,m-1$ such that,

$$\begin{cases} F^x_{i,j} < F^x_{i+1,j},\quad F^y_{i,j} < F^y_{i,j+1},\ \Delta_{i,j} < \Delta_{i+1,j},\ \hat{\Delta}_{i,j} < \hat{\Delta}_{i,j+1} \\ \Delta_{i,j} < F^x_{i,j} < \Delta_{i+1,j},\qquad \hat{\Delta}_{i,j} < F^y_{i,j} < \hat{\Delta}_{i,j+1} \qquad \forall i, j \end{cases} \tag{10}$$

and the free parameters are

$$\begin{cases} u_{i,j} > 0, v_{i,j} > 0, u_{i,j+1} > 0, v_{i,j+1} > 0 \\ \hat{u}_{i,j} > 0, \hat{v}_{i,j} > 0, \hat{u}_{i+1,j} > 0, \hat{v}_{i+1,j} > 0 \end{cases} \tag{11}$$

Casciola and Romani (2002) developed the result as: "The bi-cubic partially blended rational surface patch inherits all the properties of network of boundary curves". According to this fact the surface patch (3) preserves the convexity if the four boundary curves $S(x, y_j), S(x, y_{j+1}), S(x_i, y)$ and $S(x_{i+1}, y)$ are defined in equations (6)-(9) are convex.

The boundary curve $S(x, y_j)$ is convex if $S^{(2)}(x, y_j) > 0$ *i.e.*,

$$S^{(2)}(x, y_j) = \frac{\sum_{k=1}^{8}(1-\theta)^{8-k}\theta^{k-1}P_k}{h_i(q_1(\theta))^3} > 0, \tag{12}$$

with,

$$P_1 = 2v_{i,j}^2\left(w_{i,j}\left(F_{i+1,j}^x - \Delta_{i,j}\right) + F_{i,j}^x u_{i,j} - F_{i+1,j}^x v_{i,j}\right),$$

$$P_2 = 4P_1 + 6v_{i,j}^2\left(F_{i+1,j}^x v_{i,j} - \Delta_{i,j}u_{i,j}\right),$$

$$P_3 = \left\{\left(P_2 - P_1\right) + 6v_{i,j}\left\{w_{i,j}\left(F_{i+1,j}^x v_{i,j} - \Delta_{i,j}u_{i,j}\right) - 2u_{i,j}v_{i,j}\left(F_{i+1,j}^x - \Delta_{i,j}\right)\right\}\right\},$$

$$P_4 = \left\{\left(P_3 + P_1 - P_2\right) + 2w_{i,j}\left\{w_{i,j}\left(\Delta_{i,j}\left(u_{i,j} + v_{i,j}\right)\right) - u_{i,j}v_{i,j}\left(F_{i+1,j}^x - F_{i,j}^x\right)\right\}\right\} + 14u_{i,j}v_{i,j}\left(F_{i+1,j}^x v_{i,j} - F_{i,j}^x u_{i,j}\right),$$

$$P_5 = \left\{\left(P_6 + P_8 - P_7\right) + 2w_{i,j}\left\{w_{i,j}\left(\Delta_{i,j}\left(u_{i,j} + v_{i,j}\right)\right) - u_{i,j}v_{i,j}\left(F_{i+1,j}^x - F_{i,j}^x\right)\right\}\right\} + 14u_{i,j}v_{i,j}\left(F_{i+1,j}^x v_{i,j} - F_{i,j}^x u_{i,j}\right),$$

$$P_6 = \left\{\left(P_7 - P_8\right) + 6u_{i,j}\left\{w_{i,j}\left(\Delta_{i,j}v_{i,j} - F_{i,j}^x u_{i,j}\right) - 2u_{i,j}v_{i,j}\left(\Delta_{i,j} - F_{i,j}^x\right)\right\}\right\},$$

$$P_7 = 4P_8 + 6u_{i,j}^2\left(\Delta_{i,j}v_{i,j} - F_{i,j}^x u_{ij}\right),$$

$$P_8 = 2u_{i,j}^2\left(w_{i,j}\left(\Delta_{i,j} - F_{i,j}^x\right) + F_{i,j}^x u_{i,j} - F_{i+1,j}^x v_{i,j}\right)$$

The rational cubic function $S^{(2)}(x, y_j) > 0$, if $\sum_{k=1}^{8}(1-\theta)^{8-k}\theta^{k-1}P_k > 0$ and $q_1(\theta) > 0$.

We have $q_1(\theta) > 0$ if equation (11) is satisfied and $\sum_{k=1}^{8}(1-\theta)^{8-k}\theta^{k-1}P_k > 0$ if $P_k > 0 \quad k = 1, 2, ..., 8$.

Further $P_k > 0 \quad \forall k$ if

$$w_{i,j} > \max\left\{\begin{array}{l} 0, \dfrac{F_{i+1,j}^x v_{i,j}}{\left(F_{i+1,j}^x - \Delta_{i,j}\right)}, \dfrac{F_{i+1,j}^x v_{i,j}}{\left(\Delta_{i,j} - F_{i,j}^x\right)}, \dfrac{2u_{i,j}v_{i,j}\left(F_{i+1,j}^x - \Delta_{i,j}\right)}{\left(F_{i+1,j}^x v_{i,j} - \Delta_{i,j}u_{i,j}\right)}, \dfrac{2u_{i,j}v_{i,j}\left(\Delta_{i,j} - F_{i,j}^x\right)}{\left(\Delta_{i,j}v_{i,j} - F_{i,j}^x u_{i,j}\right)}, \\ \dfrac{u_{i,j}v_{i,j}\left(F_{i+1,j}^x - F_{i,j}^x\right)}{\Delta_{i,j}\left(u_{i,j} + v_{i,j}\right)} \end{array}\right\} \tag{13}$$

Similarly, the boundary curve $S(x, y_{j+1})$ is convex if $S^{(2)}(x, y_{j+1}) > 0$ *i.e.*,

$$S^{(2)}(x, y_{j+1}) = \frac{\sum_{k=1}^{8}(1-\theta)^{8-k}\theta^{k-1}Q_k}{h_i(q_2(\theta))^3} > 0, \tag{14}$$

with,

$Q_1 = 2v_{i,j+1}^2\left(w_{i,j+1}\left(F_{i+1,j+1}^x - \Delta_{i,j+1}\right) + F_{i,j+1}^x u_{i,j+1} - F_{i+1,j+1}^x v_{i,j+1}\right),$

$Q_2 = 4Q_1 + 6v_{i,j+1}^2\left(F_{i+1,j+1}^x v_{i,j+1} - \Delta_{i,j+1}u_{i,j+1}\right),$

$Q_3 = \left\{\left(Q_2 - Q_1\right) + 6v_{i,j+1}\left\{w_{i,j+1}\left(F_{i+1,j+1}^x v_{i,j+1} - \Delta_{i,j+1}u_{i,j+1}\right) - 2u_{i,j+1}v_{i,j+1}\left(F_{i+1,j+1}^x - \Delta_{i,j+1}\right)\right\}\right\},$

$Q_4 = \left\{\left(Q_3 + Q_1 - Q_2\right) + 2w_{i,j+1}\left\{\begin{matrix} w_{i,j+1}\left(\Delta_{i,j+1}\left(u_{i,j+1} + v_{i,j+1}\right)\right) \\ -u_{i,j+1}v_{i,j+1}\left(F_{i+1,j+1}^x - F_{i,j+1}^x\right)\end{matrix}\right\} + 14u_{i,j+1}v_{i,j+1}\left(F_{i+1,j+1}^x v_{i,j+1} - F_{i,j+1}^x u_{i,j+1}\right)\right.,$

$Q_5 = \left\{\left(Q_6 + Q_8 - Q_7\right) + 2w_{i,j+1}\left\{\begin{matrix} w_{i,j+1}\left(\Delta_{i,j+1}\left(u_{i,j+1} + v_{i,j+1}\right)\right) \\ -u_{i,j+1}v_{i,j+1}\left(F_{i+1,j+1}^x - F_{i,j+1}^x\right)\end{matrix}\right\} + 14u_{i,j+1}v_{i,j+1}\left(F_{i+1,j+1}^x v_{i,j+1} - F_{i,j+1}^x u_{i,j+1}\right)\right.,$

$Q_6 = \left\{\left(Q_7 - Q_8\right) + 6u_{i,j+1}\left\{w_{i,j+1}\left(\Delta_{i,j+1}v_{i,j+1} - F_{i,j+1}^x u_{i,j+1}\right) - 2u_{i,j+1}v_{i,j+1}\left(\Delta_{i,j+1} - F_{i,j+1}^x\right)\right\}\right\},$

$Q_7 = 4Q_8 + 6u_{i,j+1}^2\left(\Delta_{i,j+1}v_{i,j+1} - F_{i,j+1}^x u_{i\,j+1}\right),$

$Q_8 = 2u_{i,j+1}^2\left(w_{i,j+1}\left(\Delta_{i,j+1} - F_{i,j+1}^x\right) + F_{i,j+1}^x u_{i,j+1} - F_{i+1,j+1}^x v_{i,j+1}\right)$

The rational cubic function $S^{(2)}(x, y_{j+1}) > 0$, if $\sum_{k=1}^{8}(1-\theta)^{8-k}\theta^{k-1}Q_k > 0$ and $q_1(\theta) > 0$.

We have $q_1(\theta) > 0$ if equation (11) is satisfied and $\sum_{k=1}^{8}(1-\theta)^{8-k}\theta^{k-1}Q_k > 0$ if $Q_k > 0 \quad k = 1,2,...,8$.

Further $Q_k > 0 \quad \forall k$ if,

$$w_{i,j+1} > \max\left\{\begin{matrix} 0, \dfrac{F_{i+1,j+1}^x v_{i,j+1}}{\left(F_{i+1,j+1}^x - \Delta_{i,j+1}\right)}, \dfrac{F_{i+1,j+1}^x v_{i,j+1}}{\left(\Delta_{i,j+1} - F_{i,j+1}^x\right)}, \dfrac{2u_{i,j+1}v_{i,j+1}\left(F_{i+1,j+1}^x - \Delta_{i,j+1}\right)}{\left(F_{i+1,j+1}^x v_{i,j+1} - \Delta_{i,j+1}u_{i,j+1}\right)}, \dfrac{2u_{i,j+1}v_{i,j+1}\left(\Delta_{i,j+1} - F_{i,j+1}^x\right)}{\left(\Delta_{i,j+1}v_{i,j+1} - F_{i,j+1}^x u_{i,j+1}\right)}, \\ \dfrac{u_{i,j+1}v_{i,j+1}\left(F_{i+1,j+1}^x - F_{i,j+}^x\right)}{\Delta_{i,j+1}\left(u_{i,j+1} + v_{i,j+1}\right)} \end{matrix}\right\} \tag{15}$$

Similarly, the boundary curve $S(x_i, y)$ is convex if $S^{(2)}(x_i, y) > 0$ *i.e.*,

$$S^{(2)}(x_i, y) = \frac{\sum_{k=1}^{8}(1-\varphi)^{8-k}\varphi^{k-1}R_k}{\hat{h}_j(q_3(\varphi))^3} > 0, \tag{16}$$

with,

$R_1 = 2\hat{v}_{i,j}^2\left(\hat{w}_{i,j}\left(F_{i,j+1}^y - \hat{\Delta}_{i,j}\right) + F_{i,j}^y\hat{u}_{i,j} - F_{i,j+1}^y\hat{v}_{i,j}\right),$

$R_2 = 4R_1 + 6\hat{v}_{i,j}^2\left(F_{i,j+1}^y\hat{v}_{i,j} - \hat{\Delta}_{i,j}\hat{u}_{i,j}\right),$

$R_3 = \left\{\left(R_2 - R_1\right) + 6\hat{v}_{i,j}\left\{\hat{w}_{i,j}\left(F_{i,j+1}^y\hat{v}_{i,j} - \hat{\Delta}_{i,j}\hat{u}_{i,j}\right) - 2\hat{u}_{i,j}\hat{v}_{i,j}\left(F_{i,j+1}^y - \hat{\Delta}_{i,j}\right)\right\}\right\},$

$$R_4 = \left\{ \left(R_3 + R_1 - R_2\right) + 2\hat{w}_{i,j} \left\{ \begin{array}{l} \hat{w}_{i,j}\left(\hat{\Delta}_{i,j}\left(\hat{u}_{i,j} + \hat{v}_{i,j}\right)\right) \\ -\hat{u}_{i,j}\hat{v}_{i,j}\left(F^y_{i,j+1} - F^y_{i,j}\right) \end{array} \right\} + 14\hat{u}_{i,j}\hat{v}_{i,j}\left(F^y_{i,j+1}\hat{v}_{i,j} - F^y_{i,j}\hat{u}_{i,j}\right) \right.,$$

$$R_5 = \left\{ \left(R_6 + R_8 - R_7\right) + 2\hat{w}_{i,j} \left\{ \begin{array}{l} \hat{w}_{i,j}\left(\hat{\Delta}_{i,j}\left(\hat{u}_{i,j} + \hat{v}_{i,j}\right)\right) \\ -\hat{u}_{i,j}\hat{v}_{i,j}\left(F^y_{i,j+1} - F^y_{i,j}\right) \end{array} \right\} + 14\hat{u}_{i,j}\hat{v}_{i,j}\left(F^y_{i,j+1}\hat{v}_{i,j} - F^y_{i,j}\hat{u}_{i,j}\right) \right.,$$

$$R_6 = \left\{\left(R_7 - R_8\right) + 6\hat{u}_{i,j}\left\{\hat{w}_{i,j}\left(\Delta_{i,j}\hat{v}_{i,j} - F^y_{i,j}\hat{u}_{i,j}\right) - 2\hat{u}_{i,j}\hat{v}_{i,j}\left(\hat{\Delta}_{i,j} - F^y_{i,j}\right)\right\}\right\},$$

$$R_7 = 4R_8 + 6\hat{u}^2_{i,j}\left(\hat{\Delta}_{i,j}\hat{v}_{i,j} - F^y_{i,j}\hat{u}_{i\,j}\right),$$

$$R_8 = 2u^2_{i,j}\left(\hat{w}_{i,j}\left(\hat{\Delta}_{i,j} - F^y_{i,j}\right) + F^y_{i,j}\hat{u}_{i,j} - F^y_{i,j+1}\hat{v}_{i,j}\right)$$

The rational cubic function $S^{(2)}(x_i, y) > 0$, if $\sum_{k=1}^{8}(1-\varphi)^{8-k}\varphi^{k-1}R_k > 0$ and $q_1(\theta) > 0$.

We have $q_1(\theta) > 0$ if equation (11) is satisfied and $\sum_{k=1}^{8}(1-\varphi)^{8-k}\varphi^{k-1}R_k > 0$ if $R_k > 0 \quad k = 1,2,...,8$. Further $R_k > 0 \quad \forall k$ if,

$$\hat{w}_{i,j} > \max\left\{ \begin{array}{l} 0, \dfrac{F^y_{i,j+1}\hat{v}_{i,j}}{\left(F^y_{i,j+1} - \hat{\Delta}_{i,j}\right)}, \dfrac{F^y_{i,j+1}\hat{v}_{i,j}}{\left(\hat{\Delta}_{i,j} - F^y_{i,j}\right)}, \dfrac{2\hat{u}_{i,j}\hat{v}_{i,j}\left(F^y_{i,j+1} - \hat{\Delta}_{i,j}\right)}{\left(F^y_{i,j+1}\hat{v}_{i,j} - \hat{\Delta}_{i,j}\hat{u}_{i,j}\right)}, \dfrac{2\hat{u}_{i,j}\hat{v}_{i,j}\left(\hat{\Delta}_{i,j} - F^y_{i,j}\right)}{\left(\hat{\Delta}_{i,j}\hat{v}_{i,j} - F^y_{i,j}\hat{u}_{i,j}\right)}, \\ \dfrac{\hat{u}_{i,j}\hat{v}_{i,j}\left(F^y_{i,j+1} - F^y_{i,j}\right)}{\hat{\Delta}_{i,j}\left(\hat{u}_{i,j} + \hat{v}_{i,j}\right)} \end{array} \right\} \tag{17}$$

Similarly, the boundary rational curve $S(x_{i+1}, y)$ is convex if $S^{(2)}(x_{i+1}, y) > 0$ *i.e.*,

$$S^{(2)}(x_{i+1}, y) = \frac{\sum_{k=1}^{8}(1-\varphi)^{8-k}\varphi^{k-1}S_k}{\hat{h}_j(q_4(\varphi))^3} > 0, \tag{18}$$

with,

$$S_1 = 2\hat{v}^2_{i+1,j}\left(\hat{w}_{i+1,j}\left(F^y_{i+1,j+1} - \hat{\Delta}_{i+1,j}\right) + F^y_{i+1,j}\hat{u}_{i+1,j} - F^y_{i+1,j+1}\hat{v}_{i+1,j}\right),$$

$$S_2 = 4S_1 + 6\hat{v}^2_{i+1,j}\left(F^y_{i+1,j+1}\hat{v}_{i+1,j} - \hat{\Delta}_{i+1,j}\hat{u}_{i+1,j}\right),$$

$$S_3 = \left\{\left(S_2 - S_1\right) + 6\hat{v}_{i+1,j}\left\{\hat{w}_{i+1,j}\left(F^y_{i+1,j+1}\hat{v}_{i+1,j} - \hat{\Delta}_{i+1,j}\hat{u}_{i+1,j}\right) - 2\hat{u}_{i+1,j}\hat{v}_{i+1,j}\left(F^y_{+1i,j+1} - \hat{\Delta}_{i+1,j}\right)\right\}\right\},$$

$$S_4 = \left\{ \left(S_3 + S_1 - S_2\right) + 2\hat{w}_{i+1,j} \left\{ \begin{array}{l} \hat{w}_{i+1,j}\left(\hat{\Delta}_{i+1,j}\left(\hat{u}_{i+1,j} + \hat{v}_{i+1,j}\right)\right) \\ -\hat{u}_{i+1,j}\hat{v}_{i+1,j}\left(F^y_{i+1,j+1} - F^y_{i+1,j}\right) \end{array} \right\} + 14\hat{u}_{i+1,j}\hat{v}_{i+1,j}\left(F^y_{i+1,j+1}\hat{v}_{i+1,j} - F^y_{i+1,j}\hat{u}_{i,j}\right) \right.,$$

$$S_5 = \left\{ \left(S_6 + S_8 - S_7\right) + 2\hat{w}_{i+1,j} \left\{ \begin{array}{l} \hat{w}_{i+1,j}\left(\hat{\Delta}_{i+1,j}\left(\hat{u}_{i+1,j} + \hat{v}_{i+1,j}\right)\right) \\ -\hat{u}_{i+1,j}\hat{v}_{i+1,j}\left(F^y_{i+1,j+1} - F^y_{i+1,j}\right) \end{array} \right\} + 14\hat{u}_{i+1,j}\hat{v}_{i+1,j}\left(F^y_{i+1,j+1}\hat{v}_{i+1,j} - F^y_{i+1,j}\hat{u}_{i,j}\right) \right.,$$

$$S_6 = \left\{ \left(S_7 - S_8 \right) + 6\hat{u}_{i+1,j} \left\{ \hat{w}_{i+1,j} \left(\hat{\Delta}_{i+1,j} \hat{v}_{i+1,j} - F^y_{i+1,j} \hat{u}_{i+1,j} \right) - 2\hat{u}_{i+1,j} \hat{v}_{i+1,j} \left(\hat{\Delta}_{i+1,j} - F^y_{i+1,j} \right) \right\} \right\},$$

$$S_7 = 4 S_8 + 6\hat{u}^2_{i+1,j} \left(\hat{\Delta}_{i+1,j} \hat{v}_{i+1,j} - F^y_{i+1,j} \hat{u}_{i+1j} \right),$$

$$S_8 = 2u^2_{i+1,j} \left(\hat{w}_{i+1,j} \left(\hat{\Delta}_{i+1,j} - F^y_{i+1,j} \right) + F^y_{i+1,j} \hat{u}_{i+1,j} - F^y_{i+1,j+1} \hat{v}_{i+1,j} \right)$$

The rational cubic function $S^{(2)}(x_{i+1}, y) > 0$, if $\sum_{k=1}^{8} (1-\varphi)^{8-k} \varphi^{k-1} S_k > 0$ and $q_1(\theta) > 0$.

We have $q_1(\theta) > 0$ if equation (11) is satisfied and $\sum_{k=1}^{8} (1-\varphi)^{8-k} \varphi^{k-1} S_k > 0$ if $S_k > 0 \quad k = 1, 2, ..., 8$.

Further $S_k > 0 \quad \forall k$ if,

$$\hat{w}_{i+1,j} > \max \left\{ \begin{array}{l} 0, \dfrac{F^y_{i+1,j+1} \hat{v}_{i+1,j}}{\left(F^y_{i+1,j+1} - \hat{\Delta}_{i+1,j} \right)}, \dfrac{F^y_{i+1,j+1} \hat{v}_{i+1,j}}{\left(\hat{\Delta}_{i+1,j} - F^y_{i+1,j} \right)}, \dfrac{2\hat{u}_{i+1,j} \hat{v}_{i+1,j} \left(F^y_{i+1,j+1} - \hat{\Delta}_{i+1,j} \right)}{\left(F^y_{i+1,j+1} \hat{v}_{i+1,j} - \hat{\Delta}_{i+1,j} \hat{u}_{i+1,j} \right)}, \dfrac{2\hat{u}_{i+1,j} \hat{v}_{i+1,j} \left(\hat{\Delta}_{i+1,j} - F^y_{i+1,j} \right)}{\left(\hat{\Delta}_{i+1,j} \hat{v}_{i+1,j} - F^y_{i+1,j} \hat{u}_{i+1,j} \right)}, \\ \dfrac{\hat{u}_{i+1,j} \hat{v}_{i+1,j} \left(F^y_{i+1,j+1} - F^y_{i+1,j} \right)}{\hat{\Delta}_{i+1,j} \left(\hat{u}_{i+1,j} + \hat{v}_{i+1,j} \right)} \end{array} \right\} \tag{19}$$

6 Numerical Examples

Example 6.1

The convex surface data set taken in Table 1 is obtained from

$$F_1(x, y) = x^{14} + y^2 (y^{12} + 1), \quad x, y \in [0, 1.5]$$

The surface in Fig.1 is generated by bi-cubic Hermite spline that does not preserve the convexity of convex surface data. Fig.2 (a) and Fig.2 (b) represent different view of Fig.1. It is to note that these figures do not preserve the shape of data. To overcome this flaw, Fig.3 is produced by the scheme developed in section 5 with the values of free parameters $u_{i,j} = 0.1, \hat{u}_{i,j} = 0.1, v_{i,j} = 0.1, \hat{v}_{i,j} = 0.1$ to preserve the shape of convex surface data. Fig.4 (a) and Fig.4 (b) are representing the xz- and yz-view of Fig.3. It is clearly shown that these figures not only preserve the shape of convex data but also visually pleasant.

Table 1. Convex surface data

y/x	**0**	**0.3**	**0.6**	**0.9**	**1.2**	**1.5**
0	0.0000	0.0900	0.36078	1.0388	14.2790	294.1800
0.3	0.0000	0.0900	0.36078	1.0388	14.2790	294.1800
0.6	0.0000	0.0907	0.3615	1.0396	14.2800	294.1800
0.9	0.2287	0.3187	0.5895	1.2675	14.5080	294.4100
1.2	12.8390	12.9290	13.2000	13.8780	27.1180	307.0200
1.5	291.9300	292.0200	292.2900	292.9700	306.2100	586.1100

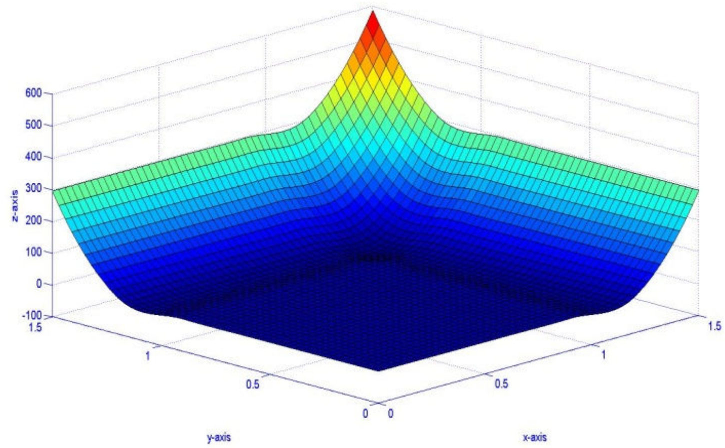
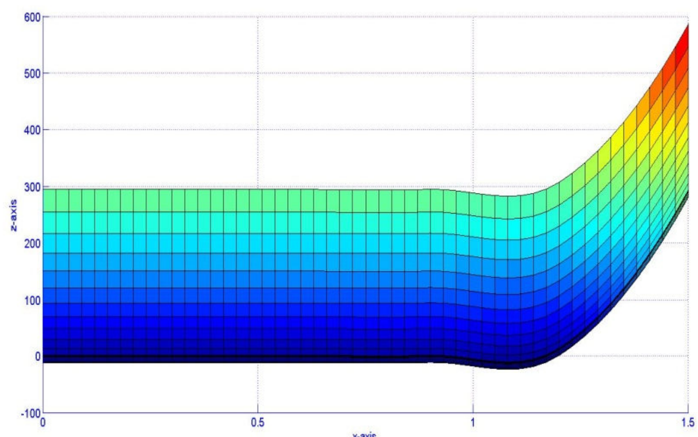


Fig. 1. Bi-cubic Hermite spline with $u_{i,j} = \hat{u}_{i,j} = 1,\ v_{i,j} = \hat{v}_{i,j} = 1,\ w_{i,j} = \hat{w}_{i,j} = 3$ (without Theorem 5.1)

a)



b)

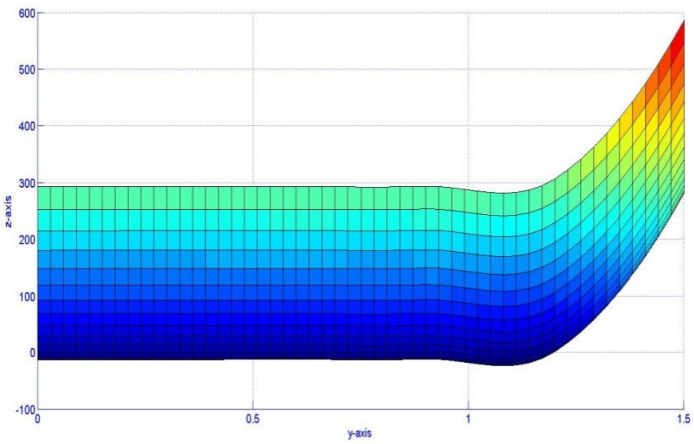


Fig. 2. Bi-cubic Hermite spline; a) xz-view of Fig. 1 b) yz-view of Fig. 1

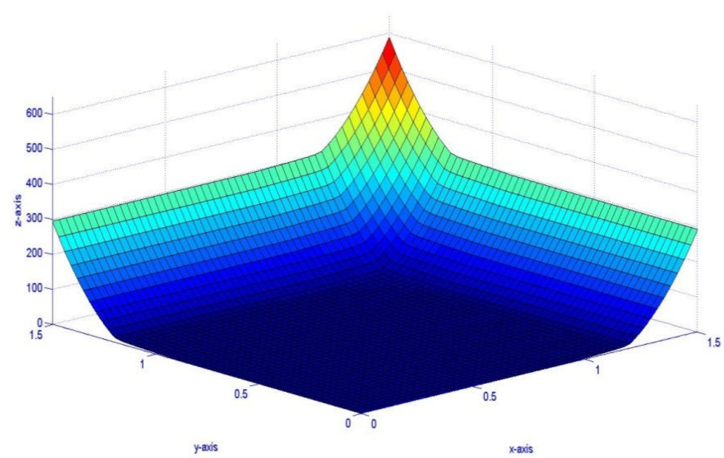
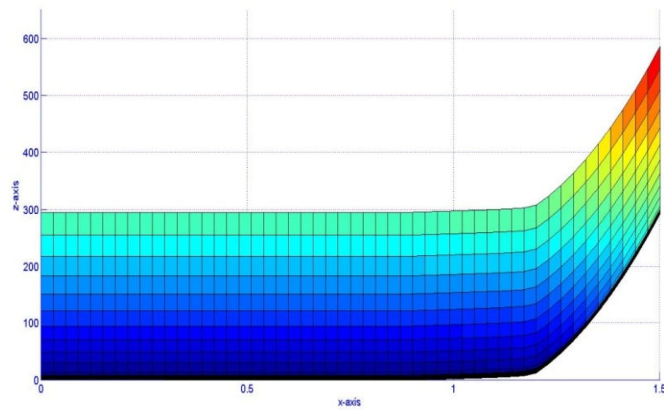


Fig. 3. Convexity preserving rational bi-cubic partially blended surface

a)



b)

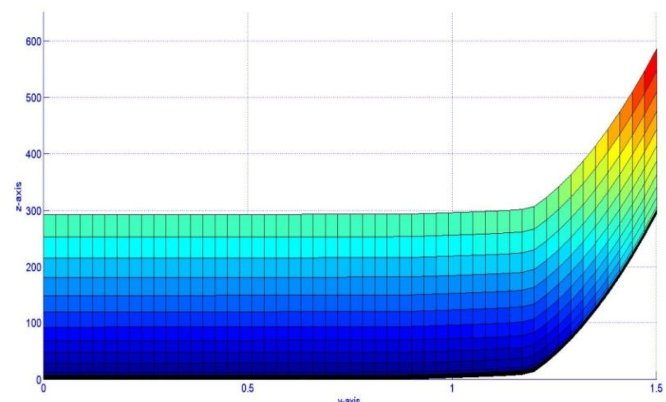


Fig. 4. Convexity preserving rational bi-cubic partially blended surface; a) xz-view of Fig. 3 b) yz-view of Fig. 3

Example 6.2

The data set taken in Table 2 is generated by following function,

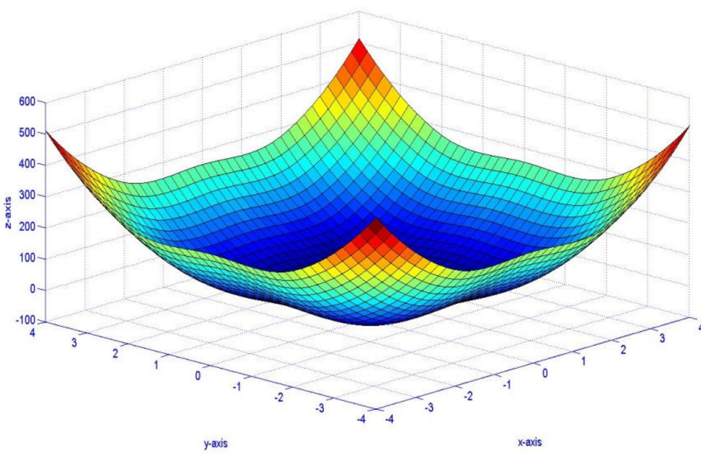
$$F_2(x, y) = x^4 + y^4, \quad -4 \le x, y \le 4.$$

The Fig.5 (a), Fig.5 (b) and Fig.5 (c) give three different views of the surface generated by bi-cubic Hermite spline. It is easy to see that these figures do not preserve the convexity of data. Fig.6 (a) is generated by convexity preserving rational bi-cubic partially blended function with the values of free parameters $u_{i,j} = 0.1, \hat{u}_{i,j} = 0.1, v_{i,j} = 0.1, \hat{v}_{i,j} = 0.1$ to preserve the shape of convex 3D data. Fig.6 (b) and Fig.6 (c) represent the xz- and yz-view of Fig.6 (a). It is to note that these figures are smooth and depict the convexity.

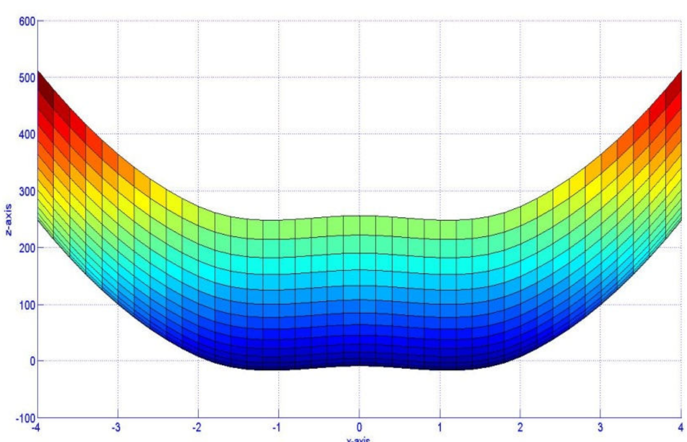
Table 2. Convex 3D data

y/x	**-4**	**-2**	**0**	**2**	**4**
-4	512	272	256	272	512
-2	272	32	16	32	272
0	256	16	0	16	256
2	272	32	16	32	272
4	512	272	256	272	512

a)



b)



c)

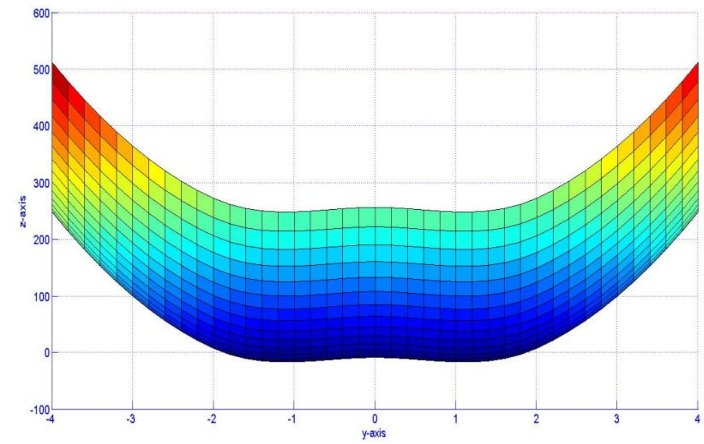
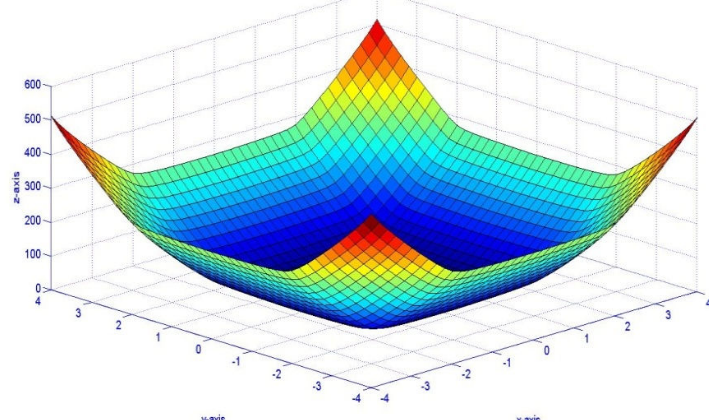
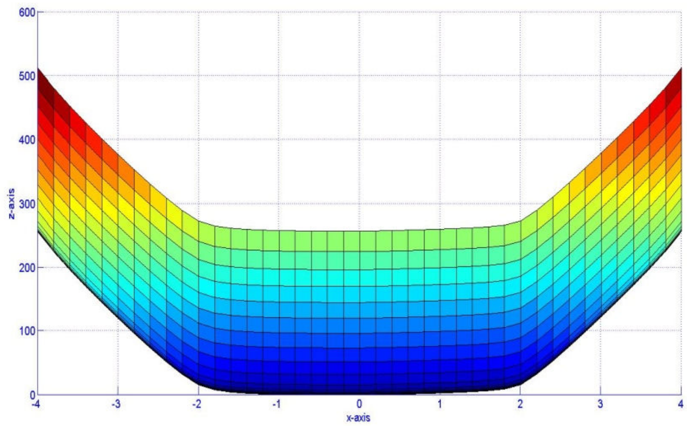


Fig. 5. a) Bi-cubic Hermite scheme with $u_{i,j} = \hat{u}_{i,j} = 1,\ v_{i,j} = \hat{v}_{i,j} = 1,\ w_{i,j} = \hat{w}_{i,j} = 3$ (without Theorem 5.1); b) xz-view; c) yz-view

a)



b)



c)

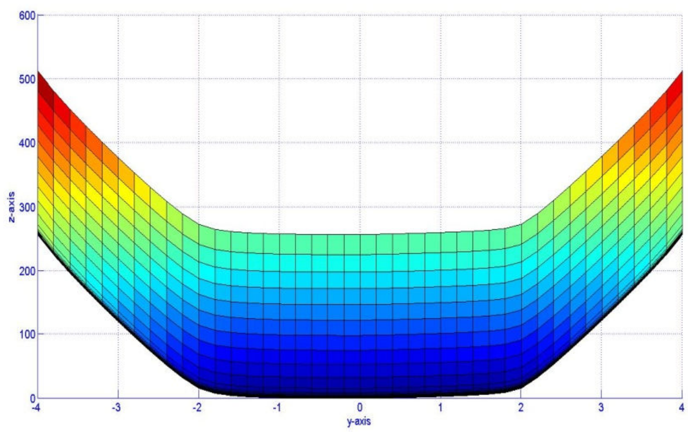


Fig. 6. a) Convexity preserving rational bi-cubic partially blended surface; b) xz-view; c) yz-view

Table 3. Convex surface data

y/x	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**
0.1	9950.1	621.98	122.94	38.967	16.02	7.777
0.2	9800.9	612.74	121.19	38.48	15.877	7.7583
0.3	9553.7	597.37	118.23	37.604	15.572	7.658
0.4	9211	576.03	114.08	36.347	15.105	7.4754
0.5	8776.3	548.93	108.78	34.719	14.48	7.2103
0.6	8253.9	516.33	102.39	32.735	13.701	6.8635

Example 6.3

The data set taken in Table 3 is generated by following function,

$$F_3(x,y) = \frac{(xy^4+1)\ \mathrm{Cos}\ x}{y^4}, \qquad x, y \in [0.1, 0.6]$$

Fig.7 is generated by bi-cubic Hermite spline. It is easy to see that Fig.7 does not preserve the convexity of convex surface data. Fig.8 represents yz-view of Fig.7. To remove this defect, Fig.9 is generated by convexity preserving rational bi-cubic partially blended function with the values of free parameters $u_{i,j} = 0.5, \hat{u}_{i,j} = 0.5, v_{i,j} = 0.5, \hat{v}_{i,j} = 0.5$ to preserve the shape of convex 3D data. The yz-view of Fig.9 can be seen in Fig. 10.

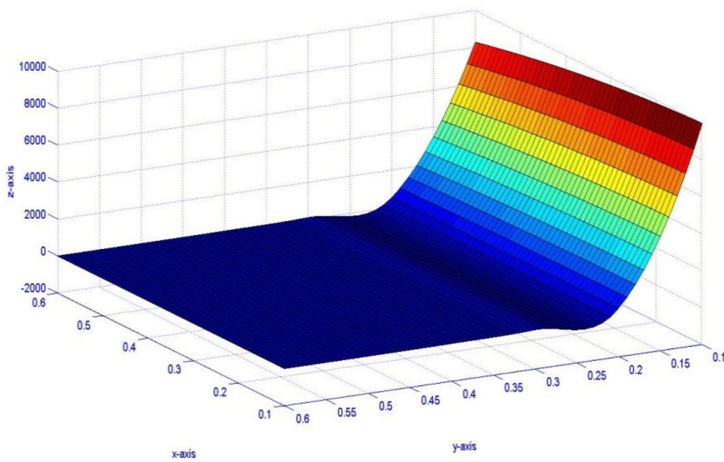


Fig. 7. Bi-cubic Hermite spline with $u_{i,j} = \hat{u}_{i,j} = 1,\ v_{i,j} = \hat{v}_{i,j} = 1,\ w_{i,j} = \hat{w}_{i,j} = 3$ **(without Theorem 5.1)**

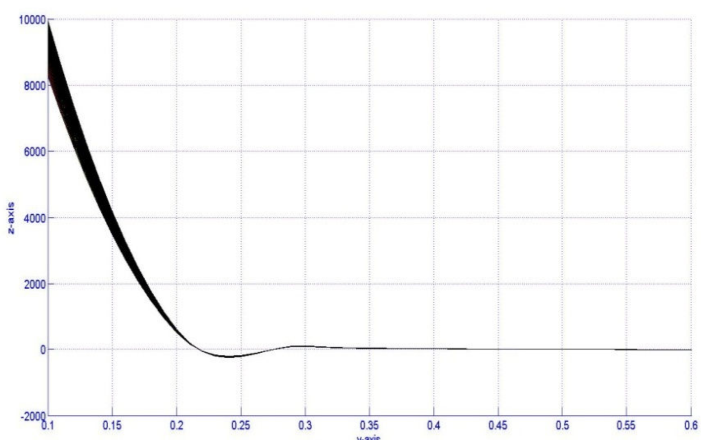


Fig. 8. yz-view of Bi-cubic Hermite spline

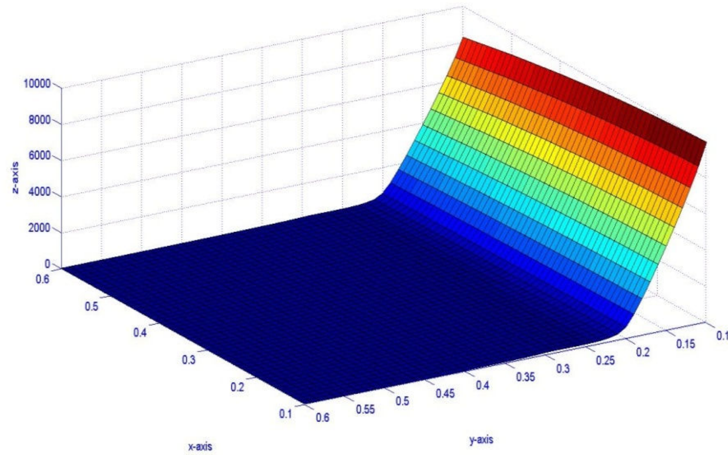


Fig. 9. Convexity preserving rational bi-cubic partially blended surface

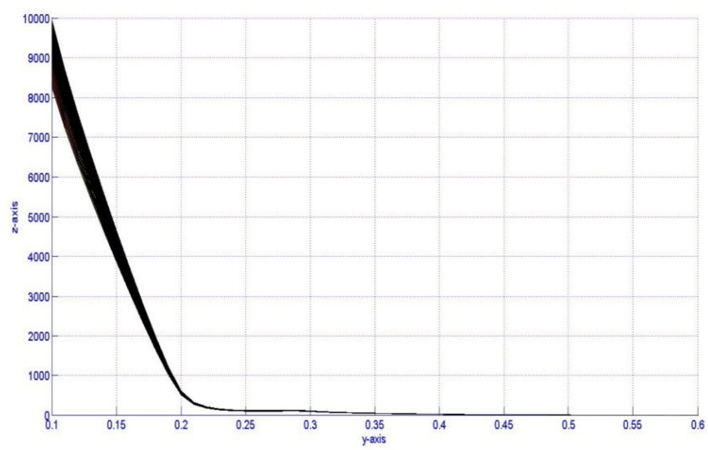


Fig.10. yz-view of Fig.9

7 Concluding Remarks

In this paper, we have extended a C^1 piecewise rational cubic function with three free parameters to rational bi-cubic partially blended function with twelve free parameters in each rectangular patch to preserve the shape of convex data. The free parameters are arranged in such a way that four of them are constrained parameters to preserve the shape of convex data while the remaining are left free for user 's choice to refine the surface as desired. No extra knots are needed in the scheme. The scheme is more flexible in terms of convexity preserving due to free adjustable parameters in the interpolant as compared to existing schemes. The effectiveness of the scheme has been demonstrated through different numerical examples and observed that the scheme is not only local and computationally economical but also visually pleasant.

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Competing Interests

Authors have declared that no competing interests exist.

References

Abbas, M, Majid, A.A., Awang, M.N.H., Ali, J.M. (2012). Local Convexity Shape-Preserving Data Visualization by Spline Function. ISRN Mathematical Analysis, Volume 2012, Article ID 174048, 14 pages, doi:10.5402/2012/174048.

Asaturyan, S. (1990). Shape-preserving surface interpolation scheme. Ph. D. Thesis, Department of Mathematics and Computer Science, University of Dundee, Scotland, UK.

Asaturyan, S., Costantini, P., and Manni, C. (2001). Local shape-preserving interpolation by space curves. IMA Journal of Numerical Analysis, 21(1), 301–325.

Brodlie, K.W., Butt, S. (1991). Preserving convexity using piecewise cubic interpolation. Computer Graphics, 15, 15–23.

Butt, S. (1991). Shape-preserving curves and surfaces for Computer Graphics. Ph.D. Thesis, School of Computer Studies. The University of Leeds, UK.

Carnicer, J.M., Garcia-Esnaol, M., Peña, J.M. (1996). Convexity of rational curves and total positivity. Journal of Computational and Applied Mathematics, 71(2), 365–382.

Casciola, G., Romani, L. (2002). Rational interpolants with tension parameters. Proceedings of Curve and Surface Design, T. Lyche, M. Mazure and L.L. Schumaker(eds.), Saint-Malo, Nashboro Press, Brentwood, TN.41-50.

Clements, J.C. (1992). A convexity preserving C^2 parametric rational cubic interpolant. Numerische Mathemati, 63(1), 165-171.

Costantini, P. (1986). On monotone and convex spline interpolation. Mathematics of Computation, 46(173), 203-214.

Costantini, P., Fontanella, F. (1990). Shape-preserving bivariate interpolation. SIAM Journal of Numerical Analysis, 27(2), 488-506.

DeVore, A., Yan. Z. (1986). Error analysis for piecewise quadratic curve fitting algorithms. Computer Aided Geometric Design, 3, 205–215.

Dodd, S.L., McAllister, D.F., Roulier, J.A. (1983). Shape-preserving spline interpolation for specifying bivariate functions on grids. IEEE Computer Graphics and Applications, 3(6), 70-79.

Farin, G.E. (1996). Curves and Surfaces for Computer-Aided Geometric Design: A Practical guide. Academic Press, Inc.

Floater, M.S. (1994). A weak condition for the convexity of tensor-product Bézier and B-spline surfaces. Advances in Computational Mathematics, 2(1), 67–80.

Gregory, J.A. (1986). Shape-preserving spline interpolation. Computer Aided Design, 18(1), 53–57.

Hoschek, J., Lasser, D. (1993). Fundamentals of computer aided geometric design. Vol. 1. AK peters Wellesley, MA 02181.

Hussain, M.Z., Maria, H., Shakeel, A. (2011). Shape-preserving Surfaces for the Visualization of Positive and Convex Data using Rational Bi-quadratic Splines. International Journal of Computer Applications, 27(10), 12-20

Hussain, M.Z., Maria, H. (2008). Convex surface interpolation. Lecture Notes in Computer Science, 4975, 475-482.

Hussain, M.Z., Maria, H., Shaikh, T.S. (2008). Shape-preserving Convex Surface Data Visualization Using Rational Bi-Quartic Function. European Journal of Scientific Research, 21(2), 319-327

Hussain, M.Z., Maria, H. (2006). Visualization of 3D data preserving convexity. Journal of Applied Mathematics and Computing, 2, 170-186.

Maria, H., Hussain, M.Z. (2008). Convexity-preserving Piecewise Rational Bi-cubic Interpolation. Computer Graphics and CAD/CAM 02, 14–24

McAllister, D.F., Roulier, J.A. (1981). An algorithm for computing a shape-preserving oscillatory quadratic spline. ACM Trans. Math. Software, 7, 331–347.

Passow, E., Roulier, J.A. (1977). Monotone and convex spline interpolation. SIAM Journal of Numerical Analysis, 14, 904–909.

Roulier, J.A. (1987). A convexity preserving grid refinement algorithm for interpolation of bivariate functions. IEEE Computer Graphics and Applications, 57-62.

Schumaker, L.L. (1983). On shape-preserving quadratic spline interpolation. SIAM Journal of Numerical Analysis, 20, 854–864.