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Multiperiodicity Evoked by Periodic External Inputs in Cohen-Grossberg-type BAM Networks with Discrete and Distributed Delays

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Abstract

In this paper, by using contraction mapping theorem, analysis approach and decomposition of solution space, the multiperiodicity issue is discussed for Cohen-Grossberg-type (CG-type) bidirectional associative memory networks (BAMNs) with discrete and distributed delays and a general class of activation functions, where the general class of activation functions consist of nondecreasing functions with saturations including piecewise linear functions with two corner points and standard activation functions as their special cases. It is shown that for any saturation region, if there is a periodic orbit located in it, it must be locally exponentially stable. Then, based on this result, some conditions are derived for ascertaining the $(n + m)$-neuron CG-type BAMNs can have 2 locally exponentially stable limit cycles located in two saturation regions respectively which are symmetrical. Also, taking account of different saturation regions, results about $2(p + q)$ $(p \le m,\ q \le n - 1)$, $2^{\min\{n,m\}}$ locally exponentially stable limit cycles can be obtained, where n is the number of the neurons in one layer, m is the number of the neurons in the other layer. Meanwhile, for every locally exponentially stable limit cycle given, the corresponding saturation region can be expressed concretely. Finally, three examples are given to illustrate the effectiveness of the obtained results.

Keywords: Multiperiodicity; Cohen-Grossberg-type (CG-type) bidirectional associative memory networks (BAMNs); exponential stability; saturation regions.

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1 Introduction

In the past decades, the considerable research interests are focused on bidirectional associative memory (BAM) networks and Cohen-Grossberg neural networks (CGNNs) without or with time delays because of their potential applications in practice such as classification, associative memory (Kosko, 1988; Maundy and El-Masry, 1990; Cohen and Grossberg, 1983; Liao et al., 2001; Cao and Li, 2005; Bao and Cao, 2011; Huang and Cao, 2011). BAM neural network is firstly proposed in (Kosko, 1988), and this class of network generalized the single-layer autoassociative circuits to a two-layer pattern-matched heteroassociative circuits. And for BAM networks, there is no connection between any two neurons which are in the same layer. The circuit diagram and connection pattern implemented for the delayed BAM networks can be found in (Cao and Wang, 2005). In (Bao and Cao, 2011), the problem of robust state estimation for uncertain stochastic bidirectional associative memory networks with time-varying delays is investig- ated. CGNN is proposed by Cohen and Grossberg (Cohen and Grossberg, 1983) in 1983. And this class of networks includes a lot of famous ecological systems and neural networks as special cases such as the Lotka-Volterra system, the Gilpia-Analg competition system, the Eingen-Schuster system and the Hopfield neural networks (Cohen and Grossberg, 1983; Liao et al., 2001). In (Cao and Song 2006), sufficient conditions are obtained ensuring the existence, uniqueness and global exponential stability of the equilibrium point for Cohen-Grossberg-type bidirectional associa- tive memory networks with time-varying delays, which combines CGNNs with BAM networks.

The dynamics of neuron activation states at an equilibrium point or periodic orbit is prerequisite for many applications. In previous works, most authors have studied mono-stability and mono-periodicity of neural network models. However, in some neurodynamics systems, there exist multiple stable equilibria or periodic orbits, which is usually referred to multistability or multiperiodicity, respectively. And it is worth noting that coexistence of many equilibrium points or periodic orbits is necessary in practical applications such as associative memory storage, pattern recognition, decision making, digital selection and analogy amplification (Chua and Yang, 1988; Hahnloser, 1998). And an interesting application of the multiperiodicity and multistability analysis is to design associative memories and store a large number of patterns as stable equilibria or limit cycles such that stored patterns can be retrieved when the initial probes contain sufficient information about the patterns.

To the best of our knowledge, the multiperiodicity is seldom considered for neural networks (Wang and Zou, 2004; Cao et al., 2008). In (Cheng et al., 2006) and (Cheng et al., 2007), by geometrical observation, the multistability is discussed for the Hopfield neural networks with smooth sigmoidal activation functions. In (Zeng and Wang, 2006), the multiperiodicity evoked by external inputs is considered for a class of delayed neural networks with standard activation functions, and in (Zeng et al., 2005), the multistability issue is discussed for cellular neural networks with standard activation functions. In (Zeng et al., 2008), the number of memory patterns which the n-neuron cellular neural network can have are discussed.

In addition, time delay is unavoidable in a network because of the finite speeds of the switching and transmission of signals. And time delay may lead to oscillation, instability, bifurcation or chaos of networks. For delayed BAM networks and CGNNs, there exist lots of works on the mono-stability or mono-periodicity, please refer to (Gopalsamy and He, 1994; Arik, 2005; Ye et al., 1995; Zhang, J. et al., 2005) and the references cited therein. Meanwhile, although the models of delayed feedback with discrete delays are good approximation in simple circuits consisting of a small number of cells, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Thus, there is a distribution of conduction velocities along these pathways and a distribution of propagation delays. Therefore, the models with discrete and continuously distributed delays are more appropriate. In this paper, we will mainly focus on the studies of the multiperiodicity for CG-type BAMNs with discrete and distributed delays and a general class of activation functions, where the general class of activation functions consist of nondecreasing functions with the saturation, including piecewise linear functions with two corner points and standard activation functions as its special case. For the $(n+m)$-neuron CG-type BAMNs where n is the number of the neurons in one layer, m is the number of the neurons in the other layer, we shall show that if there is a periodic orbit located in a saturation region, it must be locally exponentially stable. Based on this result, we shall give some conditions to guarantee that there have $2,\ 2(p+q)(p \le m, q \le n-1),\ 2^{\min\{n,m\}}$ locally exponentially stable limit cycles located in saturation regions, respectively. Meanwhile, for every locally exponentially stable limit cycle given, the corresponding saturation region, i.e., attractive region, can also be given.

The rest of the paper is organized as follows. In Section 2, model description and preliminaries are presented. The main results are stated in Section 3. In Section 4, three examples are given to show the validity of the obtained results. Finally, in Section 5, the conclusions are drawn.

2 Model Description and Preliminaries

In this paper, we consider the following $(n+m)$-neuron CGNN-type BAMNs with discrete and distributed delays:

$$\begin{cases} \frac{du_i(t)}{dt} = -a_i(u_i(t))[u_i(t) - \sum\limits_{j=n+1}^{n+m} h_{ij} f_j(v_j(t)) - \sum\limits_{j=n+1}^{n+m} w_{ij} f_j(v_j(t-\tau_{ij}(t))) \\ - \sum\limits_{j=n+1}^{n+m} b_{ij} \int_{t-\tau_1}^{t} k_{ij}(t-s) f_j(v_j(s))ds - I_i(t)] \quad i = 1, 2, \ldots, n, \\ \frac{dv_j(t)}{dt} = -c_j(v_j(t))[v_j(t) - \sum\limits_{i=1}^{n} h^*_{ji} f_i(u_i(t)) - \sum\limits_{i=1}^{n} w^*_{ji} f_i(u_i(t-\sigma_{ji}(t))) \\ - \sum\limits_{i=1}^{n} b^*_{ji} \int_{t-\tau_1}^{t} k^*_{ji}(t-s) f_i(u_i(s))ds - J_j(t)] \quad j = n+1, n+2, \ldots, n+m, \end{cases} \tag{2.1}$$

where $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T \in \mathbb{R}^n,\ v(t) = (v_{n+1}(t), v_{n+2}(t), \ldots, v_{n+m}(t))^T \in \mathbb{R}^m$; $u_i(t)$ and $v_j(t)$ are the state of the ith neuron from the neural field F_U and the jth neuron from the neural field F_V at time t, respectively; $a_i(u_i(t))$ and $c_j(v_j(t))$ represent amplification functions; $f_j(v_j), f_i(u_i)$ denote the activation functions of the jth neuron from F_V and the ith neuron from F_U, respectively; $I(t) = (I_1(t), I_2(t), \ldots, I_n(t))^T \in \mathbb{R}^n$ and $J(t) = (J_{n+1}(t), J_{n+2}(t), \ldots, J_{n+m}(t))^T \in \mathbb{R}^m$ are periodic input vectors with period ω (i.e., there exists a constant $\omega > 0$ such that $I_i(t+\omega) = I_i(t), J_j(t+\omega) = J_j(t), i = 1, 2, \ldots, n, j = n+1, n+2, \ldots, n+m$); $H = (h_{ij})_{n\times m}, W = (w_{ij})_{n\times m}, B = (b_{ij})_{n\times m}, H^* = (h^*_{ji})_{m\times n}, W^* = (w^*_{ji})_{m\times n}, B^* = (b^*_{ji})_{m\times n}$ are connection weight matrices; $\tau_{ij}(t)$ and $\sigma_{ji}(t)$ correspond to the transmission delays and satisfy $0 \le \tau_{ij}(t) \le \tau_{ij}, 0 \le \sigma_{ji}(t) \le \sigma_{ji}$ (τ_{ij} and σ_{ji} are positive constants), τ_1 is a positive constant, denote $\tau = \max_{1\le i\le n, n+1\le j\le n+m}\{\tau_{ij}, \sigma_{ji}, \tau_1\}$. $k_{ij}(\cdot), k^*_{ji}(\cdot), i = 1, 2, \cdots, n, j = n+1, n+2, \cdots, n+m$ are delay kernels.

Let $C([t_0-\tau,\ t_0],\ D)$ be the space of continuous functions mapping $[t_0-\tau,\ t_0]$ into $D \subset \mathbb{R}^n$ with the norm defined by $\| \xi \|_{t_0} = \max_{1\le i\le n}\{\sup_{s\in[t_0-\tau,\ t_0]} | \xi_i(s) |\}$,

where $\xi(s) = (\xi_1(s), \xi_2(s), \ldots, \xi_n(s))^T$. Denote $\| x \| = \max_{1\le i\le n}\{| x_i |\}$ as the vector norm of the vector $x = (x_1, x_2, \ldots, x_n)^T$.

The initial condition of model (2.1) is

$$u(s) = \phi(s) \quad \text{for } s \in [t_0-\tau,\ t_0],$$

$$v(s) = \varphi(s) \quad \text{for } s \in [t_0-\tau,\ t_0],$$

where $\phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^T \in C([t_0-\tau,\ t_0],\ \mathbb{R}^n), \varphi(s)$ $= (\varphi_{n+1}(s), \varphi_{n+2}(s), \ldots, \varphi_{n+m}(s))^T \in C([t_0-\tau,\ t_0],\ \mathbb{R}^m)$.

We take account of a class of activation functions described as follows for (2.1):

$$f_i \in C, \qquad f_i(\xi) = \begin{cases} l_i & \text{if } -\infty < \xi < p_i, \\ \tilde{f}_i(\xi) & \text{if } p_i \le \xi \le q_i, \\ m_i & \text{if } q_i < \xi < \infty, \end{cases} \tag{2.2}$$

where l_i, m_i, p_i, q_i are constants with $l_i < m_i, p_i < q_i$, and $\tilde{f}_i(\cdot) \in C^1$ is increasing function. This class of activation functions consists of nondecreasing functions with the saturation, including the piecewise linear function with two corner points at p_i, q_i :

$$f_i(\xi) = \begin{cases} l_i, \text{ if } -\infty < \xi < p_i, \\ l_i + \frac{m_i - l_i}{q_i - p_i}(\xi - p_i), \text{ if } p_i \le \xi \le q_i, \\ m_i, \text{ if } q_i < \xi < \infty, \end{cases} \tag{2.3}$$

and the standard activation function:

$$f_i(\xi) = \frac{\mid \xi + 1 \mid - \mid \xi - 1 \mid}{2}. \tag{2.4}$$

Throughout this paper, we make the following assumption:
$(A1)$ Functions $a_i(r)$ and $c_j(r)$ are continuous, $0 < \check{a}_i \le a_i(r) \le \hat{a}_i$ and $0 < \check{c}_j \le c_j(r) \le \hat{c}_j$ for all $r \in \mathbb{R},\ i = 1, 2, \ldots, n,\ j = n+1, n+2, \ldots, n+m.$

The delay kernels $k_{ij}(\cdot), k^*_{ji}(\cdot), i = 1, 2, \cdots, n, j = n+1, n+2, \cdots, n+m$ are assumed to satisfy the following conditions simultaneously:
$(A2)$:(i) $k_{ij}, k^*_{ji} : [0, \infty) \longrightarrow [0, \infty)$;
(ii) k_{ij}, k^*_{ji} is bounded and continuous on $[0, \infty)$;
(iii) $\int_0^{\tau_1} k_{ij}(s)ds = 1, \int_0^{\tau_1} k^*_{ji}(s)ds = 1.$

Denote $(-\infty, p_i] = (-\infty, p_i]^1 \times (p_i, q_i)^0 \times [q_i, +\infty)^0$; $(p_i, q_i) = (-\infty, p_i]^0 \times (p_i, q_i)^1 \times [q_i, +\infty)^0$; $[q_i, +\infty) = (-\infty, p_i]^0 \times (p_i, q_i)^0 \times [q_i, +\infty)^1$; $\mathbb{R} = (-\infty, p_i] \bigcup (p_i, q_i) \bigcup [q_i, +\infty)$. So $\mathbb{R}^{n+m}$ can be divided into 3^{n+m} subspaces:

$$\Omega = \{\prod_{i=1}^{n} (-\infty, p_i]^{\delta_1^{(i)}} \times (p_i, q_i)^{\delta_2^{(i)}} \times [q_i, +\infty)^{\delta_3^{(i)}} \prod_{j=n+1}^{n+m} (-\infty, p_j]^{\delta_1^{(j)}} \times (p_j, q_j)^{\delta_2^{(j)}} \times [q_j, +\infty)^{\delta_3^{(j)}},$$
$$(\delta_1^{(i)}, \delta_2^{(i)}, \delta_3^{(i)}) = (1, 0, 0) \text{ or } (0, 1, 0) \text{ or } (0, 0, 1), i = 1, 2, \ldots, n,$$
$$(\delta_1^{(j)}, \delta_2^{(j)}, \delta_3^{(j)}) = (1, 0, 0) \text{ or } (0, 1, 0) \text{ or } (0, 0, 1), j = n+1, n+2, \ldots, n+m\}. \tag{2.5}$$

Denote saturation regions as

$$\Omega_s = \{\prod_{i=1}^{n} (-\infty, p_i]^{\delta^{(i)}} \times [q_i, +\infty)^{1-\delta^{(i)}} \prod_{j=n+1}^{n+m} (-\infty, p_j]^{\delta^{(j)}} \times [q_j, +\infty)^{1-\delta^{(j)}},$$
$$\delta^{(i)}, \delta^{(j)} = 1 \text{ or } 0, i = 1, 2, \ldots, n, j = n+1, n+2, \ldots, n+m\}.$$

Hence, Ω_s has 2^{n+m} elements.

In the following, let

$$(-\infty, p_i]^{\delta(ik)} = \begin{cases} [q_i, +\infty), & i = k, \\ (-\infty, p_i], & i \ne k, \end{cases}$$

$$[q_i, +\infty)^{\delta(ik)} = \begin{cases} [q_i, +\infty), & i \ne k, \\ (-\infty, p_i], & i = k, \end{cases}$$

Denote $\mathcal{B}^{n+m} = \{s \in \mathbb{R}^{n+m}, s = (s_1, \ldots, s_{n+m})^T, s_k = 1, \text{or } -1, k = 1, 2, \ldots, n+m\}$. For any $(s_1, \ldots, s_{n+m})^T \in \mathcal{B}^{n+m}$, let

$$\mathcal{L}(s_k) = \begin{cases} [q_k, +\infty), & s_k = 1, \\ (-\infty, p_k], & s_k = -1; \end{cases}$$

$$g(s_k) = \begin{cases} l_k, & s_k = -1, \\ m_k, & s_k = 1; \end{cases}$$

$$p(s_k) = \begin{cases} p_k, & s_k = -1, \\ q_k, & s_k = 1. \end{cases}$$

So, $(s_1, \ldots, s_{n+m})^T$ and $\prod_{k=1}^{n+m} \mathcal{L}(s_k)$, $(g(s_1), \ldots, g(s_{n+m}))^T$, $(p(s_1), \ldots, p(s_{n+m}))^T$ represent one-to-one correspondences, respectively.

Definition 2.1: A periodic orbit $(u^*(t)^T, v^*(t)^T)^T$ is said to be a limit cycle of Cohen-Grossberg-type BAM networks if $(u^*(t)^T, v^*(t)^T)^T$ is an isolated periodic orbit; that is, there exists $\omega > 0$ such that $\forall t \geq t_0, (u^*(t+\omega)^T, v^*(t+\omega)^T)^T = (u^*(t)^T, v^*(t)^T)^T$, and there exists $\delta > 0$ such that $\forall (\tilde{u}(t)^T, \tilde{v}(t)^T)^T \in \{(u(t)^T, v(t)^T)^T | \ 0 < \| (u(t)^T, v(t)^T)^T - (u^*(t)^T, v^*(t)^T)^T \| < \delta, \ t \geq t_0\}$, $(\tilde{u}(t)^T, \tilde{v}(t)^T)^T$ is not a point on any periodic orbit of the Cohen-Grossberg-type BAM networks.

Definition 2.2: A periodic orbit $(u^*(t)^T, v^*(t)^T)^T$ of Cohen-Grossberg-type BAM networks is said to be locally exponentially stable in region Ξ if there exist constants $\alpha > 0, \ \beta > 0$ such that $\forall t \geq t_0$

$$\begin{aligned} \| \quad & (u(t;t_0,\phi)^T, v(t;t_0,\varphi)^T)^T - (u^*(t)^T, v^*(t)^T)^T \| \\ \leq \quad & \beta \max\{\| \phi - \phi^* \|_{t_0}, \| \varphi - \varphi^* \|_{t_0}\} \exp\{-\alpha(t-t_0)\}, \end{aligned}$$

where $(u(t;t_0,\phi)^T, v(t;t_0,\varphi)^T)^T$ is the state of the Cohen-Grossberg-type BAM networks with any initial condition $(\phi^T, \varphi^T)^T \in C([t_0-\tau, \ t_0], \ \Xi)$. When $\Xi = \mathbb{R}^{n+m}$, $(u^*(t)^T, v^*(t)^T)^T$ is said to be globally exponentially stable.

Lemma 2.1: (Guo et al., 2003) Let L be a mapping on complete metric space $(C([t_0-\tau, \ t_0], \ D), \| \cdot \|_{t_0})$. If $L(C([t_0-\tau, \ t_0], \ D)) \subset C([t_0-\tau, \ t_0], \ D)$, and there exists a constant $\gamma < 1$ such that $\forall \phi, \varphi \in C([t_0-\tau, \ t_0], \ D)$, $\| L(\phi) - L(\varphi) \|_{t_0} \leq \gamma \| \phi - \varphi \|_{t_0}$, then there exists one unique $\phi^* \in C([t_0-\tau, \ t_0], \ D)$ such that $L(\phi^*) = \phi^*$.

3 Main Results

In this section, we discuss the multiperiodicity for model (2.1) and give some results.

Theorem 3.1. For $\forall (s_1, \ldots, s_{n+m})^T \in \mathcal{B}^{n+m}$, denote $\tilde{\Omega} = \prod_{k=1}^{n+m} \mathcal{L}(s_k)$. Under the assumptions $(A1)$ and $(A2)$, if $(u^*(t)^T, v^*(t)^T)^T \in \tilde{\Omega}$ is a limit cycle of model (2.1), then it is locally exponentially stable, and $\tilde{\Omega}$ is locally exponentially attractive region.

Proof: If for $\forall s \in [t_0-\tau, t]$, $(u(t)^T, v(t)^T)^T, (u(s)^T, v(s)^T)^T \in \tilde{\Omega}$, we have

$$\begin{cases} \frac{du_i(t)}{dt} = -a_i(u_i(t))[u_i(t) - (\sum\limits_{j=n+1}^{n+m} h_{ij} + \sum\limits_{j=n+1}^{n+m} w_{ij} + \sum\limits_{j=n+1}^{n+m} b_{ij}) g(s_j) - I_i(t)], \\ \qquad i = 1, 2, \ldots, n, \\ \frac{dv_j(t)}{dt} = -c_j(v_j(t))[v_j(t) - (\sum\limits_{i=1}^{n} h_{ji}^* + \sum\limits_{i=1}^{n} w_{ji}^* + \sum\limits_{i=1}^{n} b_{ji}^*) g(s_i) - J_j(t)], \\ \qquad j = n+1, n+2, \ldots, n+m, \end{cases} \tag{3.1}$$

Let $(u(t;t_0,\phi)^T, v(t;t_0,\varphi)^T)^T$ be the state of model (3.1) with initial condition $(\phi^T, \varphi^T)^T$. Obviously, there exists a constant $\varepsilon > 0$, such that

$$1 - \frac{\varepsilon}{\breve{a}_i} > 0, \tag{3.2}$$

and

$$1 - \frac{\varepsilon}{\breve{c}_j} > 0. \tag{3.3}$$

Define $z_i(t)$ as follows:

$$z_i(t) = e^{\varepsilon t} \mid \int_{u_i^*(t,t_0,\phi_i^*)}^{u_i(t;t_0,\phi_i)} \frac{1}{a_i(s)} ds \mid . \tag{3.4}$$

Computing the time derivative of z_i, we get

$$\frac{dz_i(t)}{dt} \le e^{\varepsilon t}[\frac{\varepsilon}{\check{a}_i} \mid u_i(t;t_0,\phi_i) - u_i^*(t,t_0,\phi_i^*) \mid - \mid u_i(t;t_0,\phi_i) - u_i^*(t,t_0,\phi_i^*) \mid].$$

From (3.2), we have

$$\frac{dz_i(t)}{dt} < 0.$$

And this means

$$z_i(t) \le z_i(t_0).$$

On the other hand,

$$z_i(t) \ge \frac{1}{\hat{a}_i}\exp\{\varepsilon t\} \mid u_i(t;t_0,\phi_i) - u_i^*(t,t_0,\phi_i^*) \mid,$$

$$z_i(t_0) \le \frac{1}{\check{a}_i}\exp\{\varepsilon t_0\} \mid \phi_i - \phi_i^* \mid.$$

So the following inequality holds:

$$\begin{aligned} &\mid u_i(t;t_0,\phi_i) - u_i^*(t,t_0,\phi_i^*) \mid \le \frac{\hat{a}_i}{\check{a}_i} \mid \phi_i - \phi_i^* \mid \exp\{-\varepsilon(t-t_0)\} \\ &\le \frac{\hat{a}}{\check{a}} \mid \phi_i - \phi_i^* \mid \exp\{-\varepsilon(t-t_0)\}, \end{aligned} \tag{3.5}$$

where $\hat{a} = \max\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n\},\ \check{a} = \min\{\check{a}_1, \check{a}_2, \ldots, \check{a}_n\}$.

Similarly, we have

$$\mid v_j(t;t_0,\varphi_j) - v_j^*(t;t_0,\varphi_j^*) \mid \le \frac{\hat{c}}{\check{c}} \mid \varphi_j - \varphi_j^* \mid \exp\{-\varepsilon(t-t_0)\}, \tag{3.6}$$

where $\hat{c} = \max\{\hat{c}_{n+1}, \hat{c}_{n+2}, \ldots, \hat{c}_{n+m}\},\ \check{c} = \min\{\check{c}_{n+1}, \check{c}_{n+2}, \ldots, \check{c}_{n+m}\}$.
From (3.5) and (3.6), the limit cycle of (3.1) (if any) is globally exponentially stable.
Hence, $(u^*(t)^T, v^*(t)^T)^T$ is locally exponentially stable, and $\tilde{\Omega}$ is locally exponentially attractive region.
Remark 1. From the proof of Theorem 3.1, the exponential convergence rate can be estimated via the maximal allowable value by virtue of inequality $\varepsilon < \min\{\check{a}_i, \check{c}_j\}, i = 1,2,\ldots,n; j = n+1, n+2, \ldots, n+m$. From this, one can see that amplification functions have key effect on the convergence rate of the multiperiodicity for the considered model.
Theorem 3.2. Under the assumptions $(A1)$ and $(A2)$, if there exists $(s_1, \ldots, s_{n+m})^T \in \mathcal{B}^{n+m}$, such that $\forall i \in \{1,2,\ldots,n\}$ and $\forall j \in \{n+1, n+2, \ldots, n+m\}$, $\forall t \ge t_0$,

$$\big(\sum_{j=n+1}^{n+m}(h_{ij} + w_{ij} + b_{ij})g(s_j) + I_i(t)\big)s_i > s_i p(s_i), \tag{3.7}$$

$$\big(\sum_{i=1}^{n}(h_{ji}^* + w_{ji}^* + b_{ji}^*)g(s_i) + J_j(t)\big)s_j > s_j p(s_j), \tag{3.8}$$

$$\big(\sum_{j=n+1}^{n+m}(h_{ij} + w_{ij} + b_{ij})g(-s_j) + I_i(t)\big)(-s_i) > -s_i p(-s_i), \tag{3.9}$$

$$\Big(\sum_{i=1}^{n}(h_{ji}^{*}+w_{ji}^{*}+b_{ji}^{*})g(-s_i)+J_j(t)\Big)(-s_j)>-s_jp(-s_j), \tag{3.10}$$

then model (2.1) has only 2 locally exponentially stable limit cycles located in $\Omega_+=\prod_{k=1}^{n+m}\mathcal{L}(s_k)$ and $\Omega_-=\prod_{k=1}^{n+m}(\mathcal{L}(-s_k))$, respectively.
Proof: If $\forall s\in[t_0-\tau,t_0], (u(s)^T,v(s)^T)^T\in\Omega_+$, one can obtain

$$\begin{cases}\frac{du_i(s)}{ds}\\ =-a_i(u_i(s))[u_i(s)-(\sum\limits_{j=n+1}^{n+m}h_{ij}+\sum\limits_{j=n+1}^{n+m}w_{ij}+\sum\limits_{j=n+1}^{n+m}b_{ij})g(s_j)-I_i(s)], i=1,2,\dots,n,\\ \frac{dv_j(s)}{ds}\\ =-c_j(v_j(s))[v_j(s)-(\sum\limits_{i=1}^{n}h_{ji}^{*}+\sum\limits_{i=1}^{n}w_{ji}^{*}+\sum\limits_{i=1}^{n}b_{ji}^{*})g(s_i)-J_j(s)], j=n+1,n+2,\dots,n+m,\end{cases} \tag{3.11}$$

From (3.7) and (3.8), $(u(t)^T,v(t)^T)^T\in\Omega_+$ for $t\ge t_0$, that is, Ω_+ is an invariant set of the considered model. Hence, the model (2.1) can be rewritten as the equation (3.11).

Let $(u(t;t_0,\phi)^T,v(t;t_0,\varphi)^T)^T$ be the state of model (2.1) with initial condition $(\phi^T,\varphi^T)^T$. Define $((u_\phi^{(t)}(\theta))^T,$
$(v_\varphi^{(t)}(\theta))^T)^T=(u(t+\theta;t_0,\phi)^T,v(t+\theta;t_0,\varphi)^T)^T,\ \theta\in[t_0-\tau,t_0]$. Since Ω_+ is an invariant set of the model (2.1), $\forall(\phi^T,\varphi^T)^T\in C([t_0-\tau,\ t_0],\ \Omega_+)$, we have $((u_\phi^{(t)})^T,\ (v_\varphi^{(t)})^T)^T\in C([t_0-\tau,\ t_0],\ \Omega_+)$. Define a mapping $L:C([t_0-\tau,\ t_0],\ \Omega_+)\longrightarrow C([t_0-\tau,\ t_0],\ \Omega_+)$ by $H((\phi^T,\varphi^T)^T)=((u_\phi^{(\omega)})^T,\ (v_\varphi^{(\omega)})^T)^T$. Then $L(C([t_0-\tau,\ t_0],\ \Omega_+))\subset C([t_0-\tau,\ t_0],\ \Omega_+)$, and $L^k((\phi^T,\varphi^T)^T)=((u_\phi^{(k\omega)})^T,\ (v_\varphi^{(k\omega)})^T)^T$. We can choose a positive integer k such that

$$\max\{\frac{\hat{a}}{\breve{a}},\frac{\hat{c}}{\breve{c}}\}\exp\{-\varepsilon(k\omega\ -\tau)\}\le\gamma<1.$$

And, from (3.5) and (3.6), we have

$$\begin{aligned}&\parallel L^k((\phi^T,\varphi^T)^T)-L^k((\tilde{\phi}^T,\tilde{\varphi}^T)^T)\parallel_{t_0}\\ \le\ &\max\{\frac{\hat{a}}{\breve{a}},\frac{\hat{c}}{\breve{c}}\}\parallel(\phi^T,\varphi^T)^T-(\tilde{\phi}^T,\tilde{\varphi}^T)^T\parallel_{t_0}\exp\{-\varepsilon(k\omega+t_0-\tau-t_0)\}\\ \le\ &\gamma\parallel(\phi^T,\varphi^T)^T-(\tilde{\phi}^T,\tilde{\varphi}^T)^T\parallel_{t_0}\end{aligned}$$

From Lemma 2.1, there exists a unique fixed point $(\phi^{*T},\psi^{*T})^T\in C([t_0-\tau,\ t_0],\ \Omega_+)$ such that $L^k((\phi^{*T},\psi^{*T})^T)=(\phi^{*T},\psi^{*T})^T$.
In addition, $L^k(L((\phi^{*T},\psi^{*T})^T))=L(L^k((\phi^{*T},\psi^{*T})^T))=L((\phi^{*T},\psi^{*T})^T)$, i.e., $L((\phi^{*T},\psi^{*T})^T)$ is also a fixed point of L^k. By the uniqueness of the fixed point of the mapping L^k, $L((\phi^{*T},\psi^{*T})^T)=(\phi^{*T},\psi^{*T})^T$, i.e., $((u_{\phi^*}^{(\omega)})^T,\ (v_{\varphi^*}^{(\omega)})^T)^T=(\phi^{*T},\psi^{*T})^T$. Let $(u^*(t)^T,v^*(t)^T)^T$ be a state of model (2.1) with initial condition $(\phi^{*T},\ \psi^{*T})^T$, we obtain $\forall i\in\{1,2,\dots,n\},\forall j\in\{n+1,n+2,\dots,n+m\},t\ge t_0$,

$$\begin{cases}\frac{du_i^*(t)}{dt}=-a_i(u_i^*(t))[u_i^*(t)-(\sum\limits_{j=n+1}^{n+m}h_{ij}+\sum\limits_{j=n+1}^{n+m}w_{ij}+\sum\limits_{j=n+1}^{n+m}b_{ij})g(s_j)-I_i(t)],\\ \frac{dv_j^*(t)}{dt}=-c_j(v_j^*(t))[v_j(t)-(\sum\limits_{i=1}^{n}h_{ji}^{*}+\sum\limits_{i=1}^{n}w_{ji}^{*}+\sum\limits_{i=1}^{n}b_{ji}^{*})g(s_i)-J_j(t)].\end{cases}$$

Then, $\forall i\in\{1,2,\dots,n\},\forall j\in\{n+1,n+2,\dots,n+m\},t+\omega\ge t_0$,

$$\begin{cases}\frac{du_i^*(t+\omega)}{dt}=-a_i(u_i^*(t+\omega))[u_i^*(t+\omega)-(\sum\limits_{j=n+1}^{n+m}h_{ij}+\sum\limits_{j=n+1}^{n+m}w_{ij}+\sum\limits_{j=n+1}^{n+m}b_{ij})g(s_j)-I_i(t+\omega)]\\ =-a_i(u_i^*(t+\omega))[u_i^*(t+\omega)-(\sum\limits_{j=n+1}^{n+m}h_{ij}+\sum\limits_{j=n+1}^{n+m}w_{ij}+\sum\limits_{j=n+1}^{n+m}b_{ij})g(s_j)-I_i(t)],\\ \frac{dv_j^*(t+\omega)}{dt}=-c_j(v_j^*(t+\omega))[v_j(t+\omega)-(\sum\limits_{i=1}^{n}h_{ji}^{*}+\sum\limits_{i=1}^{n}w_{ji}^{*}+\sum\limits_{i=1}^{n}b_{ji}^{*})g(s_i)-J_j(t+\omega)]\\ =-c_j(v_j^*(t+\omega))[v_j(t+\omega)-(\sum\limits_{i=1}^{n}h_{ji}^{*}+\sum\limits_{i=1}^{n}w_{ji}^{*}+\sum\limits_{i=1}^{n}b_{ji}^{*})g(s_i)-J_j(t)].\end{cases}$$

That is, $(u^*(t+\omega)^T, v^*(t+\omega)^T)^T$ is also a state of the model (2.1) with initial condition $(\phi^{*T},\ \psi^{*T})^T$, here, $((u^{(\omega)}_{\phi^*})^T,\ (v^{(\omega)}_{\varphi^*})^T)^T = (\phi^{*T}, \psi^{*T})^T$. Hence, for $\forall t \geq t_0$,

$$\begin{aligned}&(u^*(t+\omega)^T, v^*(t+\omega)^T)^T = ((u^*(t)^T, v^*(t)^T)^T; t_0, ((u^{(\omega)}_{\phi^*})^T, (v^{(\omega)}_{\varphi^*})^T)^T)\\ &= (u^*(t)^T, v^*(t)^T)^T.\end{aligned} \tag{3.12}$$

Hence, $(u^{*T}, v^{*T})^T$ is an isolated periodic orbit of model (2.1) with period ω located in Ω_+. Similarly, there exists another isolated periodic orbit of model (2.1) with period ω located in Ω_-. From Theorem 1, we can obtain that they are locally exponentially stable. The proof of Theorem 3.2 is completed.

Theorem 3.3. Under the assumptions $(A1)$ and $(A2)$, the model (2.1) has neither more nor less than 2 locally exponentially stable limit cycles located in $\prod_{i=1}^{n}[q_i, +\infty) \prod_{j=n+1}^{n+m}[q_j, +\infty)$ and $\prod_{i=1}^{n}(-\infty, p_i]$ $\prod_{j=n+1}^{n+m}(-\infty, p_j]$, respectively, if $\forall i \in \{1, 2, \ldots, n\}$, $\forall j \in \{n+1, n+2, \ldots, n+m\}$, $\forall t \geq t_0$,

$$\sum_{j=n+1}^{n+m} (h_{ij} + w_{ij} + b_{ij})m_j + I_i(t) > q_i, \tag{3.13}$$

$$\sum_{i=1}^{n} (h^*_{ji} + w^*_{ji} + b^*_{ji})m_i + J_j(t) > q_j, \tag{3.14}$$

$$\sum_{j=n+1}^{n+m} (h_{ij} + w_{ij} + b_{ij})l_j + I_i(t) < p_i, \tag{3.15}$$

$$\sum_{i=1}^{n} (h^*_{ji} + w^*_{ji} + b^*_{ji})l_i + J_j(t) < p_j, \tag{3.16}$$

$\forall k \in \{n+1, n+2, \ldots, n+m\}$,

$$\begin{aligned}&[s_1 \sum_{j=n+1, j\neq k}^{n+m} (\mid h_{1j} \mid + \mid w_{1j} \mid + \mid b_{1j} \mid) \max\{\mid l_j \mid \mid m_j \mid\} + (h_{1k} + w_{1k} + b_{1k})g(s_k) + I_1(t)]s_1\\ < \quad & s_1 p(s_1),\end{aligned} \tag{3.17}$$

where $s_1, s_k = 1$, or -1, and $s_1 s_k = -1$;
$\forall k \in \{2, \ldots, n\}$,

$$\begin{aligned}&[s_1 \sum_{i=1, i\neq k}^{n} (\mid h^*_{ji} \mid + \mid w^*_{ji} \mid + \mid b^*_{ji} \mid) \max\{\mid l_i \mid \mid m_i \mid\}\\ &+ (h^*_{jk} + w^*_{jk} + b^*_{jk})g(s_k) + J_j(t)]s_1 < s_1 p(s_j), j = n+1, \ldots, n+m,\end{aligned} \tag{3.18}$$

where $s_1, s_k, s_j = 1$, or -1, and $s_1 s_k = -1, s_1 s_j = 1$.

Proof: Similar to the proof of Theorem 3.2, from (3.13)-(3.16), the model (2.1) has 2 locally exponentially stable limit cycles which locate in $\prod_{i=1}^{n}[q_i, +\infty) \prod_{j=n+1}^{n+m}[q_j, +\infty)$ and $\prod_{i=1}^{n}(-\infty, p_i] \prod_{j=n+1}^{n+m}(-\infty, p_j]$, respectively.

Assume $(\bar{u}^*(t)^T, \bar{v}^*(t)^T)^T$ is another limit cycle of model (2.1) located in Ω_s, without loss of generality, assume $(1, s_2, \ldots, s_n, s_{n+1}, \ldots, s_{n+m})^T \in \mathcal{B}^{n+m}, (\bar{u}^{*T}, \bar{v}^{*T})^T \in \bar{\Omega} = \mathcal{L}(1) \times \mathcal{L}(s_2) \times \cdots \times \mathcal{L}(s_{n+m})$. From Theorem 3.1, $\bar{\Omega}$ is an invariant set. Then, there at least exists $k \in \{2, \ldots, n+m\}$, such that $s_k = -1$. If there exists $k \in \{n+1, \ldots, n+m\}$, such that $s_k = -1$, then for $\bar{u}^*_1(t) = q_1$, from (3.17), one can have

$$\begin{aligned}\frac{d\bar{u}_1^*(t)}{dt} &= -a_1(q_1)[q_1 - (\sum_{j=n+1, j\neq k}^{n+m}(h_{1j}+w_{1j}+b_{1j})g(s_j) - (h_{1k}+w_{1k}+b_{1k})p_k - I_1(t)] \\ &\leq a_1(q_1)[-q_1 + \sum_{j=n+1, j\neq k}^{n+m}(\mid h_{1j}\mid + \mid w_{1j}\mid + \mid b_{1j}\mid)\max\{\mid l_j\mid \mid m_j\mid\} \\ &\quad +(h_{1k}+w_{1k}+b_{1k})p_k + I_1(t)] \\ &< 0. \end{aligned} \tag{3.19}$$

If else, there must exist $k \in \{2,\ldots,n\}$, such that $s_k = -1$, and $s_j = 1, j = n+1,\ldots,n+m$. For $\bar{v}_j^*(t) = q_j$, From (3.18), one can have

$$\begin{aligned}\frac{d\bar{v}_j^*(t)}{dt} &= -c_j(q_j)[q_j - \sum_{i=1, i\neq k}^{n}(h_{ji}^*+w_{ji}^*+b_{ji}^*)g(s_i) + (h_{jk}^*+w_{jk}^*+b_{jk}^*)p_k - J_j(t)] \\ &\leq c_j(q_{1j})[-q_j + \sum_{i=1, i\neq k}^{n}(\mid h_{ji}^*\mid + \mid w_{ji}^*\mid + \mid b_{ji}^*\mid)\max\{\mid l_i\mid \mid m_i\mid\} \\ &\quad +(h_{jk}^*+w_{jk}^*+b_{jk}^*)p_k + J_j(t)] \\ &< 0. \end{aligned} \tag{3.20}$$

(3.19) and (3.20) contradict that $\bar{\Omega} = \mathcal{L}(1)\times\mathcal{L}(s_2)\times\cdots\times\mathcal{L}(s_{n+m})$ is an invariant set.

For $(-1, s_2,\ldots,s_n,s_{n+1},\ldots,s_{n+m})^T \in \mathcal{B}^{n+m}, (\bar{u}^*(t)^T, \bar{v}^*(t)^T)^T \in \bar{\Omega} = \mathcal{L}(-1)\times\mathcal{L}(s_2)\times\cdots\times\mathcal{L}(s_{n+m})$, similar contradiction is obtained. This completes the proof of Theorem 3.3.

In the following theorem, the conditions are give to ensure that the number of limit cycles is determined by the minimum of n and m. Without loss of generality, let $n \leq m$.

Theorem 3.4. Under the assumptions $(A1)$ and $(A2)$, the model (2.1) has 2^n locally exponentially stable limit cycles located in Ω_s, if $\forall i \in \{1,2,\ldots,n\}$, $\forall j \in \{n+1,n+2,\ldots,n+m\}$, $\forall t \geq t_0$, the following inequalities hold:

$$\begin{aligned}&(h_{in+i}+w_{in+i}+b_{in+i})m_{n+i} \\ &-\sum_{j=n+1, j\neq n+i}^{n+m}(\mid h_{ij}\mid + \mid w_{ij}\mid + \mid b_{ij}\mid)\max\{\mid l_j\mid, \mid m_j\mid\} + I_i(t) > q_i,\end{aligned} \tag{3.21}$$

$$\begin{aligned}&(h_{in+i}+w_{in+i}+b_{in+i})l_{n+i} \\ &+\sum_{j=n+1, j\neq n+i}^{n+m}(\mid h_{ij}\mid + \mid w_{ij}\mid + \mid b_{ij}\mid)\max\{\mid l_j\mid, \mid m_j\mid\} + I_i(t) < p_i,\end{aligned} \tag{3.22}$$

$$\begin{aligned}&(h_{jj-n}^*+w_{jj-n}^*+b_{jj-n}^*)m_{j-n} - \sum_{i=1, i\neq j-n}^{n}(\mid h_{ji}^*\mid + \mid w_{ji}^*\mid + \mid b_{ji}^*\mid)\max\{\mid l_i\mid, \mid m_i\mid\} \\ &+J_j(t) > q_j, j = n+1, n+2,\ldots,2n,\end{aligned} \tag{3.23}$$

$$\begin{aligned}&(h_{jn}^*+w_{jn}^*+b_{jn}^*)m_n - \sum_{i=1}^{n-1}(\mid h_{ji}^*\mid + \mid w_{ji}^*\mid + \mid b_{ji}^*\mid)\max\{\mid l_i\mid, \mid m_i\mid\} + J_j(t) > q_j, \\ &j = 2n+1, 2n+2,\ldots,n+m,\end{aligned} \tag{3.24}$$

$$(h^*_{jj-n} + w^*_{jj-n} + b^*_{jj-n})l_{j-n} + \sum_{i=1, i \neq j-n}^{n} (| h^*_{ji} | + | w^*_{ji} | + | b^*_{ji} |) \max\{| l_i |, | m_i |\} + J_j(t)$$
$$< p_j, j = n+1, n+2, \ldots, 2n, \quad (3.25)$$

$$(h^*_{jn} + w^*_{jn} + b^*_{jn})l_n + \sum_{i=1}^{n-1} (| h^*_{ji} | + | w^*_{ji} | + | b^*_{ji} |) \max\{| l_i |, | m_i |\} + J_j(t) < p_j,$$
$$j = 2n+1, 2n+2, \ldots, n+m. \quad (3.26)$$

Proof: Let $\Omega_n = \mathcal{L}(s_1) \times \mathcal{L}(s_2) \times \cdots \times \mathcal{L}(s_{n+m})$, where $s_{n+i} = s_i, i = 1, 2, \ldots, n; s_{2n+1} = s_{2n+2} = \cdots = s_{n+m} = s_n$. From (3.21)-(3.26) and the proof of Theorem 3.2, one can easily obtain that there exists a locally exponentially stable limit cycle located in Ω_n. Meanwhile, in Ω_s, there have 2^n elements viewed as Ω_n, hence, the model (2.1) has 2^n locally exponentially stable limit cycles located in Ω_s.

Remark 2. From Theorem 3.4, one can see that the number of locally exponentially stable equilibrium points located in Ω_s depends on the minimum of n and m.

Remark 3. The conditions given in Theorem 3.3 and Theorem 3.4 can guarantee that there exist locally exponentially stable limit cycles located in $\prod_{i=1}^{n}[q_i, +\infty) \prod_{j=n+1}^{n+m}[q_j, +\infty)$ and $\prod_{i=1}^{n}(-\infty, p_i] \prod_{j=n+1}^{n+m}(-\infty, p_j]$, respectively.

Theorem 3.5. Under the assumptions $(A1)$ and $(A2)$, the model (2.1) has neither more nor less than $2 \times (p+q)$ locally exponentially stable limit cycles located in Ω_s, if $\forall i \in \{1, 2, \ldots, n\}$, $\forall j \in \{n+1, n+2, \ldots, n+m\}$, $k \in N_1 = \{k_1, \ldots, k_p\} \subset \{n+1, \ldots, n+m\}$, $\lambda \in M_1 = \{\lambda_1, \ldots, \lambda_q\} \subset \{1, 2, \ldots, n\}$, where $p \leq m, q \leq n-1$, $\forall t \geq t_0$, the following inequalities hold:

$$\sum_{j=n+1, j \neq k}^{n+m} (h_{ij} + w_{ij} + b_{ij})m_j + (h_{ik} + w_{ik} + b_{ik})l_k + I_i(t) > q_i, \quad (3.27)$$

$$\sum_{i=1}^{n} (h^*_{ki} + w^*_{ki} + b^*_{ki})m_i + J_k(t) < p_k, \quad (3.28)$$

$$\sum_{i=1}^{n} (h^*_{ji} + w^*_{ji} + b^*_{ji})m_i + J_j(t) > q_j, j \neq k, \quad (3.29)$$

$$\sum_{j=n+1, j \neq k}^{n+m} (h_{ij} + w_{ij} + b_{ij})l_j + (h_{ik} + w_{ik} + b_{ik})m_k + I_i(t) < p_i, \quad (3.30)$$

$$\sum_{i=1}^{n} (h^*_{ki} + w^*_{ki} + b^*_{ki})l_i + J_k(t) > q_k, \quad (3.31)$$

$$\sum_{i=1}^{n} (h^*_{ji} + w^*_{ji} + b^*_{ji})l_i + J_j(t) < p_j, j \neq k, \quad (3.32)$$

$$\sum_{i=1, i \neq \lambda}^{n} (h^*_{ji} + w^*_{ji} + b^*_{ji})m_i + (h^*_{j\lambda} + w^*_{j\lambda} + b^*_{j\lambda})l_\lambda + J_j(t) > q_j, \quad (3.33)$$

$$\sum_{j=n+1}^{n+m}(h_{\lambda j}+w_{\lambda j}+b_{\lambda j})m_j+I_\lambda(t)<p_\lambda, \tag{3.34}$$

$$\sum_{j=n+1}^{n+m}(h_{ij}+w_{ij}+b_{ij})m_j+I_i(t)>q_i, i\neq\lambda, \tag{3.35}$$

$$\sum_{i=1,i\neq\lambda}^{n}(h^*_{ji}+w^*_{ji}+b^*_{ji})l_i+(h^*_{j\lambda}+w^*_{j\lambda}+b^*_{j\lambda})m_\lambda+J_j(t)<p_j, \tag{3.36}$$

$$\sum_{j=n+1}^{n+m}(h_{\lambda j}+w_{\lambda j}+b_{\lambda j})l_j+I_\lambda(t)>q_\lambda, \tag{3.37}$$

$$\sum_{j=n+1}^{n+m}(h_{ij}+w_{ij}+b_{ij})l_j+I_i(t)<p_i, i\neq\lambda, \tag{3.38}$$

$\forall\gamma\in\{n+1,n+2,\ldots,n+m\}\backslash N_1$,

$$\begin{aligned}&[s_1\sum_{j=n+1,j\neq\gamma}^{n+m}(|\,h_{1j}\,|+|\,w_{1j}\,|+|\,b_{1j}\,|)\max\{|\,l_j\,||\,m_j\,|\}+(h_{1\gamma}+w_{1\gamma}\\&+b_{1\gamma})g(s_\gamma)+I_1(t)]s_1<s_1p(s_1),\end{aligned} \tag{3.39}$$

where $s_1,s_\gamma=1$, or -1, and $s_1s_\gamma=-1$;
$\forall\nu\in\{2,\ldots,n\}\backslash M_1$,

$$\begin{aligned}&[s_1\sum_{i=1,i\neq\nu}^{n}(|\,h^*_{ji}\,|+|\,w^*_{ji}\,|+|\,b^*_{ji}\,|)\max\{|\,l_i\,||\,m_i\,|\}+(h^*_{j\nu}+w^*_{j\nu}+b^*_{j\nu})g(s_\nu)+J_j(t)]s_1\\&<s_1p(s_j), j=n+1,\ldots,n+m,\end{aligned} \tag{3.40}$$

where $s_1,s_\nu,s_j=1$, or -1, and $s_1s_\nu=-1,s_1s_j=1$;
and $\forall\gamma^{'},\gamma^{''}\in N_1$,

$$\begin{aligned}&[s_1\sum_{j=n+1,j\neq\gamma^{'},j\neq\gamma^{''}}^{n+m}(|\,h_{1j}\,|+|\,w_{1j}\,|+|\,b_{1j}\,|)\max\{|\,l_j\,||\,m_j\,|\}+(h_{1\gamma^{'}}+w^{'}_{1\gamma}+b^{'}_{1\gamma})g(s^{'}_{\gamma})\\&+(h_{1\gamma^{''}}+w^{''}_{1\gamma}+b^{''}_{1\gamma})g(s^{''}_{\gamma})+I_1(t)]s_1<s_1p(s_1),\end{aligned} \tag{3.41}$$

where $s_1,s_{\gamma^{'}},s_{\gamma^{''}}=1$, or -1, and $s_1s_{\gamma^{'}}=-1,s_1s_{\gamma^{''}}=-1$;
$\forall\nu^{'},\nu^{''}\in M_1$,

$$\begin{aligned}&[s_1\sum_{i=1,i\neq\nu^{'},i\neq\nu^{'}}^{n}(|\,h^*_{ji}\,|+|\,w^*_{ji}\,|+|\,b^*_{ji}\,|)\max\{|\,l_i\,||\,m_i\,|\}+(h^*_{j\nu^{'}}+w^*_{j\nu^{'}}+b^*_{j\nu^{'}})g(s_{\nu^{'}})+(h^*_{j\nu^{''}}\\&+w^*_{j\nu^{''}}+b^*_{j\nu^{''}})g(s_{\nu^{''}})+J_j(t)]s_1<s_1p(s_j), j=n+1,\ldots,n+m,\end{aligned} \tag{3.42}$$

where $s_1,s_{\nu^{'}},s_{\nu^{''}},s_j=1$, or -1, and $s_1s_{\nu^{'}}=-1,s_1s_{\nu^{''}}=-1,s_1s_j=1$.
Proof: For $\prod_{i=1}^{n}(-\infty,p_i]\prod_{j=n+1}^{n+m}(-\infty,p_j]^{\delta(jk)}$, $\prod_{i=1}^{n}[q_i,+\infty)\prod_{j=n+1}^{n+m}[q_j,+\infty)^{\delta(jk)}$, $\prod_{i=1}^{n}(-\infty,p_i]^{\delta(i\lambda)}$ $\prod_{j=n+1}^{n+m}(-\infty,p_j]$, $\prod_{i=1}^{n}[q_i,+\infty)^{\delta(i\lambda)}\prod_{j=n+1}^{n+m}[q_j,+\infty)$, where $k\in N_1,\lambda\in M_1$, similar to the proof of

Theorem 3.3, the model (2.1) has $2\times(1+1)$ locally exponentially stable limit cycles located in them respectively. Since N_1 has p elements and M_1 has q elements, hence, the model (2.1) has neither more nor less than $2\times(p+q)$ locally exponentially stable limit cycles located in $\bigcup_{k\in N_1}(\prod_{i=1}^{n}(-\infty,p_i]$ $\prod_{j=n+1}^{n+m}(-\infty,p_j]^{\delta(jk)}\bigcup\prod_{i=1}^{n}[q_i,+\infty)\prod_{j=n+1}^{n+m}[q_j,+\infty)^{\delta(jk)})\bigcup_{\lambda\in M_1}(\prod_{i=1}^{n}(-\infty,p_i]^{\delta(i\lambda)}$ $\prod_{j=n+1}^{n+m}(-\infty,p_j]\bigcup\prod_{i=1}^{n}[q_i,+\infty)^{\delta(i\lambda)}\prod_{j=n+1}^{n+m}[q_j,+\infty))$.

Remark 4. It is worth noting that Theorem 3.5 is not true for $n=1, m=1$. Because for $n=1, m=1$, one can see

$$\prod_{i=1}^{n}[q_i,+\infty)\prod_{j=n+1}^{n+m}[q_j,+\infty)^{\delta(jk)}=\prod_{i=1}^{n}(-\infty,p_i]^{\delta(i\lambda)}\prod_{j=n+1}^{n+m}(-\infty,p_j];$$

$$\prod_{i=1}^{n}(-\infty,p_i]\prod_{j=n+1}^{n+m}(-\infty,p_j]^{\delta(jk)}=\prod_{i=1}^{n}[q_i,+\infty)^{\delta(i\lambda)}\prod_{j=n+1}^{n+m}[q_j,+\infty).$$

Hence, for $n=m=1$, the model (2.1) has neither more nor less than 2 locally exponentially stable equilibrium points located in $(-\infty,p_1]\times[q_2,+\infty)$ and $[q_1,+\infty)\times(-\infty,p_2]$, respectively.

Remark 5. Comparing with Theorem 3.3 and Theorem 3.4, the conditions given in Theorem 3.5 guarantee there does not exists a limit cycle located in $\prod_{i=1}^{n}[q_i,+\infty)\prod_{j=n+1}^{n+m}[q_j,+\infty)$ and $\prod_{i=1}^{n}(-\infty,p_i]$ $\prod_{j=n+1}^{n+m}(-\infty,p_j]$.

Remark 6. The multiperiodicity analysis of neural networks is more general than multistability analysis since an equilibrium point can be viewed as a special case of oscillation with any arbitrary period. For model (2.1) with constant external inputs, we can easily derive some multistability results similar to Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4 and Theorem 3.5, but are omitted here.

Remark 7. Let $a_i(u_i(t))=c_j(v_j(t))=1, i=1,2,\ldots,n, j=n+1,n+2,\ldots,n+m$, model (2.1) becomes BAMNs. Hence, the results in this paper can be applied to BAMNs. In addition, similar results also can be obtained for Cohen-Grossberg networks which satisfy the assumption $(A1)$.

4 Illustrative Examples

In this section, three examples will be given to illustrate the effectiveness of our results.

Example 1: Consider the following model:

$$\begin{cases} \frac{du_i(t)}{dt}=-a_i(u_i(t))[u_i(t)-h_{i3}\int_{t-\pi/2}^{t}k_{i3}(t-s)f_3(v_3(s))ds-I_i(t)], \quad i=1,2, \\ \frac{dv_3(t)}{dt}=-c_3(v_3(t))[v_3(t)-\sum\limits_{i=1}^{2}h_{3i}^{*}\int_{t-\pi/2}^{t}k_{3i}^{*}(t-s)f_i(u_i(s))ds-J_3(t)], \end{cases} \tag{4.1}$$

where $k_{i3}(t)=k_{3i}^{*}(t)=\cos t$,

$$f_j(\xi)=\begin{cases} -\frac{1}{2} & \text{if } \xi<-\frac{1}{2}, \\ \frac{3}{2}\xi+\frac{1}{4} & \text{if } -\frac{1}{2}\le\xi\le\frac{1}{2}, \\ 1 & \text{if } \xi>\frac{1}{2}, \end{cases} \qquad j=1,2,3,$$

$$H=\begin{pmatrix} h_{13} \\ h_{23} \end{pmatrix}=\begin{pmatrix} 1.6 \\ 1.6 \end{pmatrix},$$

$$H^{*}=\begin{pmatrix} h_{31}^{*} & h_{32}^{*} \end{pmatrix}=\begin{pmatrix} 1.4 & 1.6 \end{pmatrix},$$

$$I=\begin{pmatrix} 0.02\sin t \\ 0.02\cos t \end{pmatrix}, \quad J_3=0.02\cos t,$$

$$a(u(t)) = \begin{pmatrix} 2+0.4\cos(u_1(t)) & 0 \\ 0 & 2+0.4\cos(u_2(t)) \end{pmatrix},$$

$$c_3(v_3(t)) = 2+0.4\cos(v_3(t)).$$

Here, $p_i = p_j = -\frac{1}{2}, q_i = q_j = \frac{1}{2}, l_i = l_j = -\frac{1}{2}, m_i = m_j = 1.$
It is easy to see that the conditions in Theorem 3.3 hold. And so, model (4.1) has neither more nor less than 2 locally exponentially stable limit cycles, which locate in $(-\infty,-1/2)\times(-\infty,-1/2)\times(-\infty,-1/2)$ and $(1/2,+\infty)\times(1/2,+\infty)\times(1/2,+\infty)$ respectively. Simulation results with random initial states are depicted in Figures 1 and 2.

Example 2: Consider the following model:

$$\begin{cases} \frac{du_i(t)}{dt} = -a_i(u_i(t))[u_i(t) - \sum\limits_{j=3}^{4} h_{ij} f(v_j(t)) - I_i(t)], i = 1,2, \\ \frac{dv_j(t)}{dt} = -c_j(v_j(t))[v_j(t) - \sum\limits_{i=1}^{2} h^*_{3i}(u_i(t)) - J_j(t)], j = 3,4, \end{cases} \tag{4.2}$$

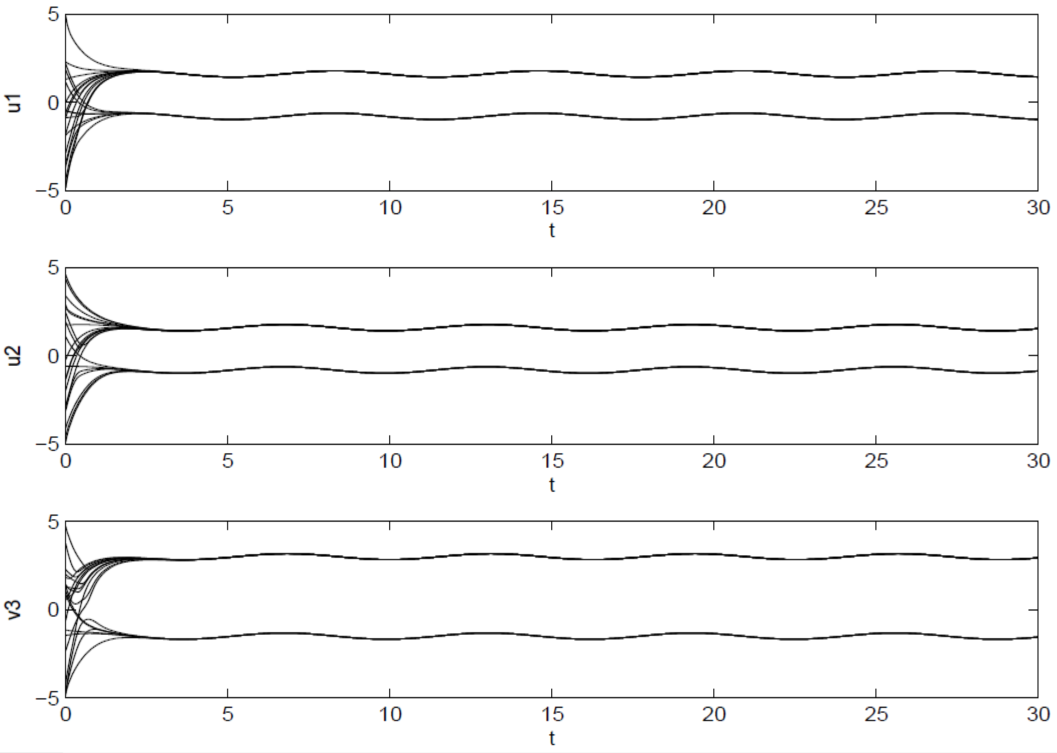

Fig. 1. Transient behavior of u_1, u_2, v_3 in Example 1.

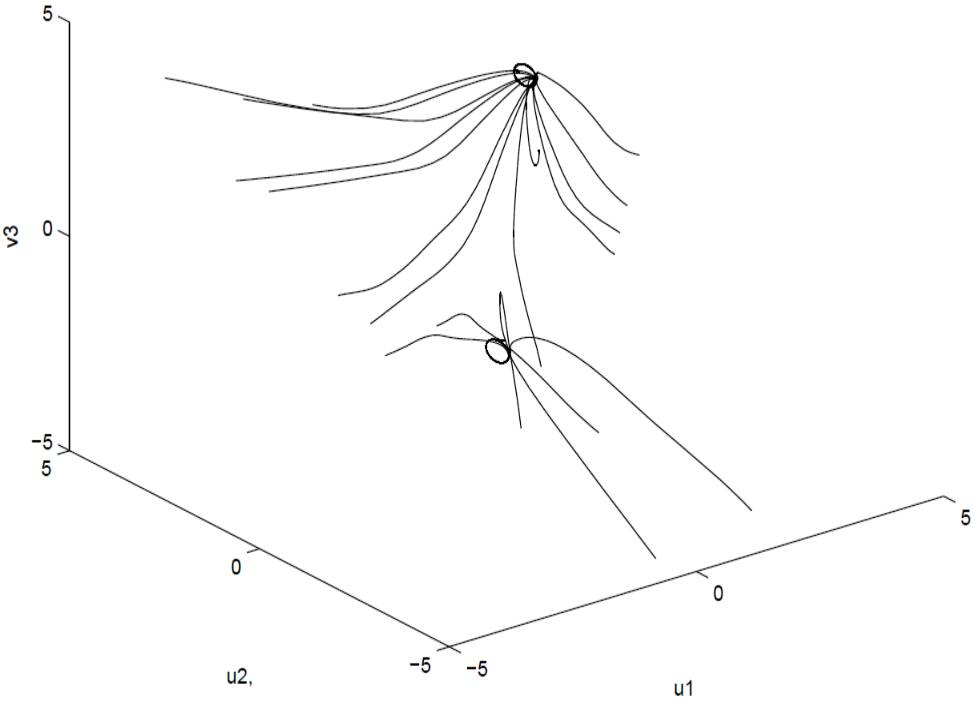

Fig. 2. Phase plots of state variable (u_1, u_2, v_3) in Example 1.

where

$$f(r) = \frac{\mid r+1 \mid - \mid r-1 \mid}{2},$$

$$H = \begin{pmatrix} h_{13} & h_{14} \\ h_{23} & h_{24} \end{pmatrix} = \begin{pmatrix} 2 & -0.4 \\ 0.8 & -2 \end{pmatrix},$$

$$H^* = \begin{pmatrix} h_{31}^* & h_{32}^* \\ h_{41}^* & h_{42}^* \end{pmatrix} = \begin{pmatrix} 2 & -0.4 \\ 0.8 & -2 \end{pmatrix},$$

$$I = \begin{pmatrix} 0.1\sin t \\ 0.1\sin t \end{pmatrix}, \quad J = \begin{pmatrix} 0.1\sin t \\ 0.1\sin t \end{pmatrix},$$

$$a(u(t)) = \begin{pmatrix} 2+0.4\cos(u_1(t)) & 0 \\ 0 & 2+0.4\cos(u_2(t)) \end{pmatrix},$$

$$c(v(t)) = \begin{pmatrix} 2+0.4\cos(v_3(t)) & 0 \\ 0 & 2+0.4\cos(v_4(t)) \end{pmatrix}.$$

one can check that the conditions in Theorem 3.5 hold. According to Theorem 3.5, the model (4.2) has neither more nor less than 4 locally exponentially stable limit cycles, which locate in $(-\infty,-1)\times(-\infty,-1)\times(-\infty,-1)\times(1,+\infty)$, $(1,+\infty)\times(1,+\infty)\times(1,+\infty)\times(-\infty,-1)$, $(1,+\infty)\times(-\infty,-1)\times(1,+\infty)\times(1,+\infty)$ and $(-\infty,-1)\times(1,+\infty)\times(-\infty,-1)\times(-\infty,-1)$, respectively. Simulation results with random initial states are depicted in Figures 3 and 4.

Example 3: Consider the following model:

$$\begin{cases} \frac{du_i(t)}{dt} = -a_i(u_i(t))[u_i(t) - \sum\limits_{j=4}^{6} h_{ij} f_j(v_j(t)) - I_i(t)], i = 1,2,3, \\ \frac{dv_j(t)}{dt} = -c_j(v_j(t))[v_j(t) - \sum\limits_{i=1}^{3} h^*_{ji} f_i(u_i(t)) - J_j(t)], j = 4,5,6, \end{cases} \tag{4.3}$$

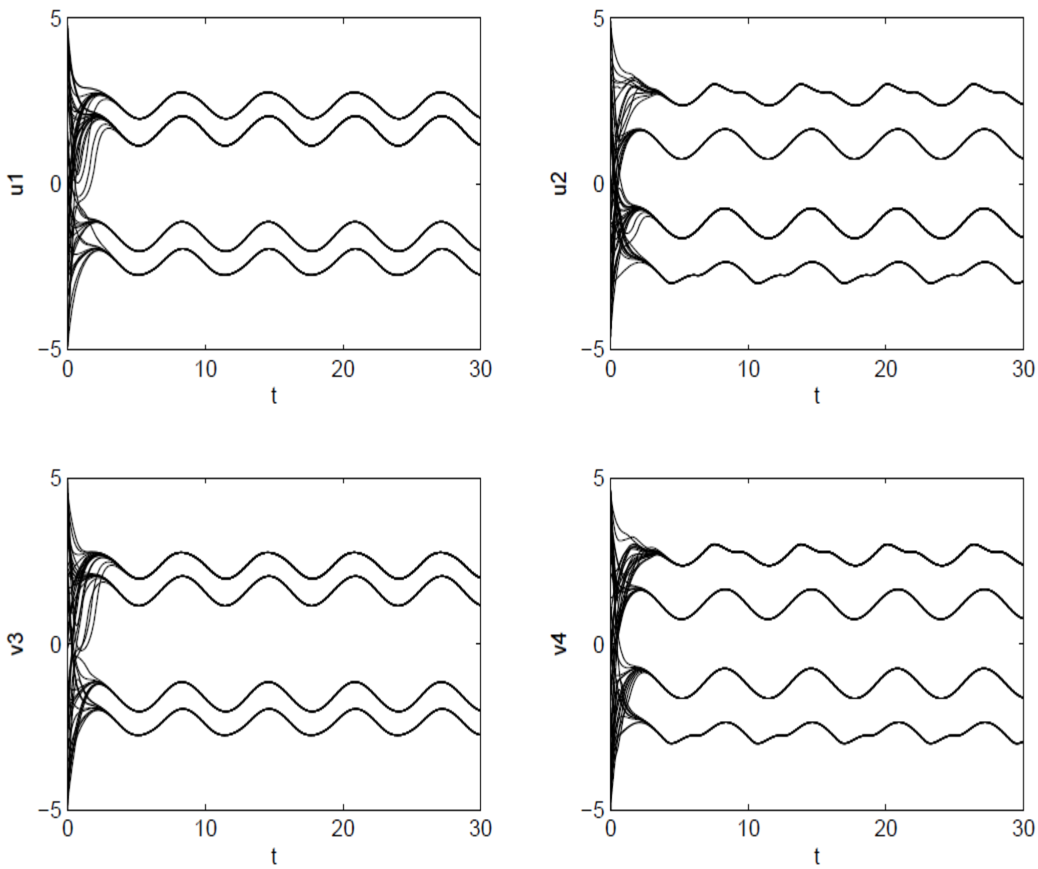

Fig. 3. Transient behavior of u_1, u_2, v_3, v_4.

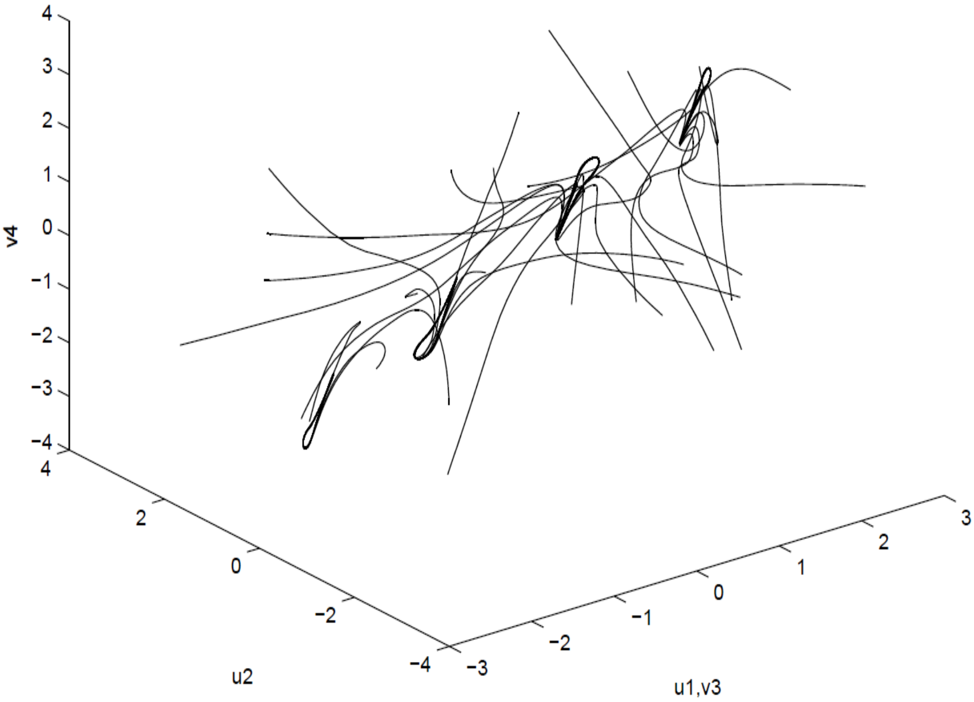

Fig. 4. Phase plots of state variable (u_1, u_2, v_3, v_4).

where

$$f(r) = \frac{| r+1 | - | r-1 |}{2},$$

$$H = H^* = \begin{pmatrix} 5 & 0.4 & 0.2 \\ 0.4 & 5 & 0.2 \\ 0.2 & 0.4 & 5 \end{pmatrix},$$

$$I = \begin{pmatrix} \cos t \\ \cos t \\ \cos t \end{pmatrix}, \quad J = \begin{pmatrix} \cos t \\ \cos t \\ \cos t \end{pmatrix},$$

$$a(u(t)) = \begin{pmatrix} 2+0.4\cos(u_1(t)) & 0 & 0 \\ 0 & 2+0.4\cos(u_2(t)) & 0 \\ 0 & 0 & 2+0.4\cos(u_3(t)) \end{pmatrix}$$

$$c(v(t)) = \begin{pmatrix} 2+0.4\cos(v_4(t)) & 0 & 0 \\ 0 & 2+0.4\cos(v_5(t)) & 0 \\ 0 & 0 & 2+0.4\cos(v_6(t)) \end{pmatrix}$$

By simple computation, the conditions in Theorem 3.4 hold. Hence, model (4.3) has 2^3 locally exponentially stable equilibrium points located in Ω_s. Simulation results with random initial states are depicted in Figures 5 and 6.

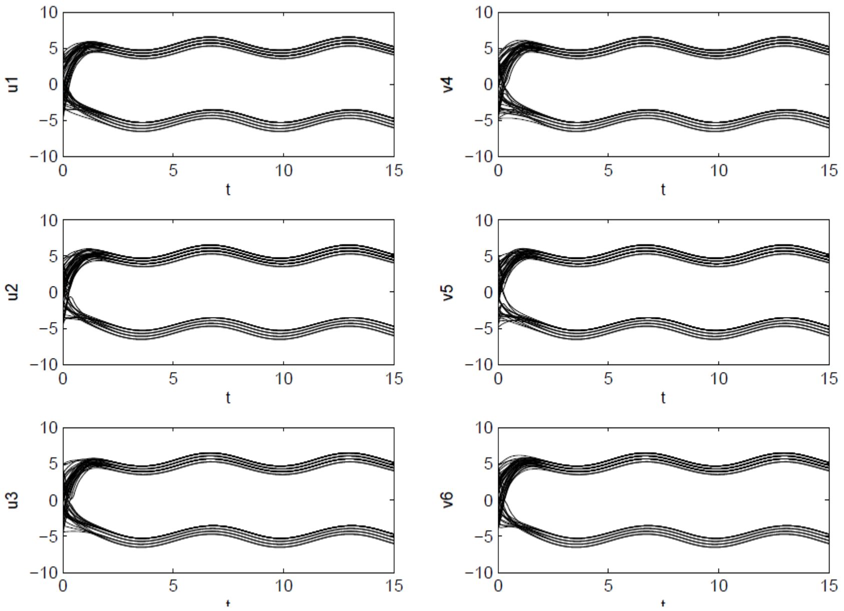

Fig. 5. Transient behavior of $u_1, u_2, u_3, v_4, v_5, v_6$.

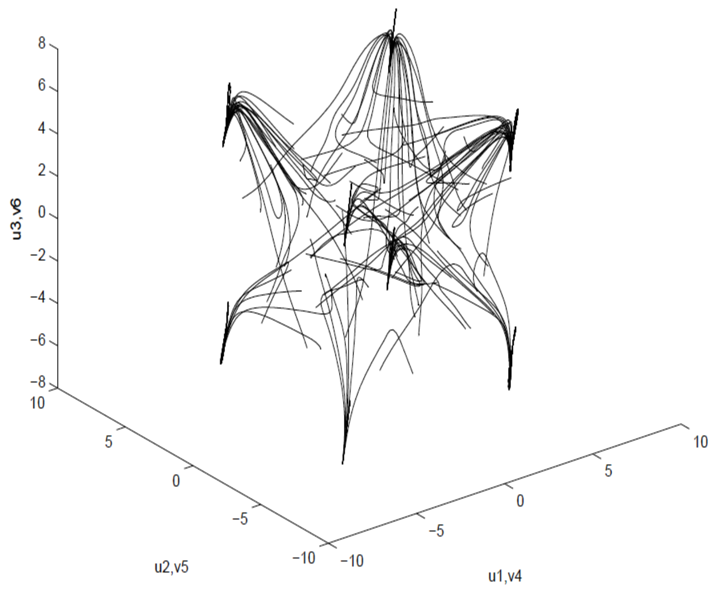

Fig. 6. Phase plots of state variable $(u_1, u_2, u_3, v_4, v_5, v_6)$.

5 Conclusions

In this paper, the multiperiodicity has been considered for Cohen-Grossberg-type bidirectional associative memory networks with discrete and distributed delays. For the $(n+m)$-neuron CG-type BAMNs where n is the number of the neurons in one layer, m is the number of the neurons in the other layer, it is shown that if there is a periodic orbit located in a saturation region, it must be locally exponentially stable. Based on this result, four results about the number of locally exponentially stable limit cycles located in saturation regions are derived. These results provide some conditions ensuring that there have $2,\ 2(p+q)(p \le m, q \le n-1),\ 2^{\min\{n,m\}}$ locally exponentially stable limit cycles located in saturation regions, respectively. Also, the conditions which we obtain are easy to be verified and checked in practice.

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