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Research Article

Cartesian Products of Some Regular Graphs Admitting Antimagic Labeling for Arbitrary Sets of Real Numbers

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An edge labeling of graph G with labels in A is an injection from $E(G)$ to A, where $E(G)$ is the edge set of G, and A is a subset of $\mathbb{R}$. A graph G is called $\mathbb{R}$-antimagic if for each subset A of $\mathbb{R}$ with $|A| = |E(G)|$, there is an edge labeling with labels in A such that the sums of the labels assigned to edges incident to distinct vertices are different. The main result of this paper is that the Cartesian products of complete graphs (except K_1) and cycles are $\mathbb{R}$-antimagic.

1. Introduction

All graphs considered in this paper are finite, simple, and without isolated vertices. As usual, let $\mathbb{R}$ denote the set of real numbers. For a graph G and a vertex v in G, $V(G)$, $E(G)$, and $E_G(v)$ denote the vertex set of G, the edge set of G, and the set of edges incident to v in G, respectively. In this paper, the following terminologies and notations are used. Let G be a graph. When A is a subset of $\mathbb{R}$ with $|A| = |E(G)|$ and the function $f\colon E(G) \longrightarrow A$ is injective, we say that f is an edge labeling of G with labels in A; in this case, for any vertex v of G, we use $f^+(v)$ to denote $\sum_{e\in E_G(v)} f(e)$. If B is a subset of $\mathbb{R}$ with $|B| \geq |E(G)|$ such that for each subset A of B with $|A| = |E(G)|$, there is an edge labeling of G with labels in A such that $f^+(u)$ is not equal to $f^+(v)$ for any two distinct vertices u, v of G, then we say that G is B-antimagic.

In the literature, a graph G is antimagic if G is $\{1, 2, \ldots, |E(G)|\}$-antimagic. The concept of antimagic graphs was introduced by Hartsfield and Ringel [1] in 1990. They conjectured that every connected graph with at least two edges was antimagic. This conjecture has not been completely solved yet. Some partial results are listed below. The antimagicness for some special types of regular graphs is verified by Cranston [2], Cranston et al. [3], and Liang and Zhu [4]. Then, Chang et al. [5] proved that all regular graphs with degree ≥2 are antimagic.

Some studies have addressed the antimagicness of Cartesian products. In 2008, Wang and Hsiao [6] introduced new classes of antimagic graphs constructed through Cartesian products, and Wang [7] proved that any Cartesian product of two or more cycles is antimagic. The antimagicness of the Cartesian products of two paths and the Cartesian products of two or more regular graphs are proved in [8, 9] by Cheng. Moreover, Zhang and Sun [10] proved that if a regular graph G is antimagic, then for any connected graph H, the Cartesian product $G\Box H$ is antimagic.

Let $\mathbb{R}^+$ denote the set of real numbers. A graph G is universal antimagic if G is $\mathbb{R}^+$-antimagic. Matamala and Zamora [11] proved that paths, cycles, and graphs whose connected components are cycles or paths of odd lengths are universal antimagic in 2020. In this paper, we generalize further and define $\mathbb{R}$-antimagic graphs. The methods of labeling on Cartesian products of cycles used in this paper are similar in [7, 8]. In Section 2, we show that wheels, cycles, and complete graphs of order ≥3 are $\mathbb{R}$-antimagic. In Section 3, we show that Cartesian products $G_1\Box G_2\Box\cdots\Box G_n\,(n\geq 2)$ are $\mathbb{R}$-antimagic, where each G_i is a complete graph of order ≥2 or a cycle.

2. $\mathbb{R}$-Antimagic Graphs and Uniformly $\mathbb{R}$-Antimagic Graphs

Let P_n be a path on n vertices. In [11], it is shown that $P_n\,(n\geq 3)$ is $\mathbb{R}^+$-antimagic, but P_3, P_4, P_5 are not $\mathbb{R}$-antimagic. Shang et al. [12] investigated the antimagicness of star forests. We prove that stars are $\mathbb{R}^+$-antimagic, but not $\mathbb{R}$-antimagic.

Remark 1. Let S_n denote the star with n edges. Then, $S_n\,(n\geq 2)$ is $\mathbb{R}^+$-antimagic, but not $\mathbb{R}$-antimagic.

Proof. Let S_n be the star with $V(S_n)=\{v_1,v_2,\ldots,v_n\}\cup\{v\}$ and $E(S_n)=\{vv_i|i=1,2,\ldots,n\}$.

Let $r_1<r_2<r_3<\cdots<r_n$ be the arbitrarily given positive numbers. We define an edge labeling f of S_n with labels in $\{r_1,r_2,r_3,\ldots,r_n\}$ by $f(vv_i)=r_i$ for $i=1,2,\ldots,n$. Then,

$$f^+(v_i)=r_i<r_{i+1}=f^+(v_{i+1}), \tag{1}$$

for $i=1,2,\ldots,n-1$, and

$$f^+(v_n)=r_n<r_1+r_2+\cdots+r_n=f^+(v). \tag{2}$$

We have

$$f^+(v_1)<f^+(v_2)<\cdots<f^+(v_n)<f^+(v). \tag{3}$$

Hence, S_n is $\mathbb{R}^+$-antimagic.

Let $r_1<r_2<r_3<\cdots<r_n$ be real numbers with $r_1+r_2+\cdots+r_{n-1}=0$. Let f be an arbitrary edge labeling of S_n with labels in $\{r_1,r_2,r_3,\ldots,r_n\}$. Without loss of generality, f is defined by $f(vv_i)=r_i$ for $i=1,2,\ldots,n$. We see that $f^+(v_n)=r_n=r_1+r_2+\cdots+r_{n-1}+r_n=f^+(v)$. Accordingly, S_n is not $\{r_1,r_2,r_3,\ldots,r_n\}$-antimagic, which results in S_n not $\mathbb{R}$-antimagic.

Let K_n denote the complete graph of order n, and C_n the cycle of order n. A wheel $W_n\,(n\geq 3)$ is the graph obtained by connecting a single vertex to every vertex of the cycle C_n. In this section, we prove that wheels, cycles, and complete graphs of order ≥3 are $\mathbb{R}$-antimagic. □

Theorem 1. *Every wheel is $\mathbb{R}$-antimagic.*

Proof. Let W_n be the wheel with $V(W_n)=\{v_1,v_2,\ldots,v_n\}\cup\{v\}$ and $E(W_n)=\{v_1v_2\}\cup\{v_iv_{i+2}|i=1,2,\ldots,n-2\}\cup\{v_n-1v_n\}\cup\{vv_i|i=1,2,\ldots,n\}$. To prove the theorem, let $r_1<r_2<r_3<\cdots<r_{2n}$ be the arbitrarily given real numbers. We distinguish two cases: Case 1, $r_{n-1}+r_n<r_{n+1}+r_{n+2}+\cdots+r_{2n-1}$, and Case 2, $r_{n+1}+r_{n+2}+\cdots+r_{2n-1}\leq r_{n-1}+r_n$. □

Case 1. $r_{n-1}+r_n<r_{n+1}+r_{n+2}+\cdots+r_{2n-1}$.

We define an edge labeling f of W_n with labels in $\{r_1,r_2,r_3,\ldots,r_{2n}\}$ by $f(v_1v_2)=r_1$, $f(v_iv_{i+2})=r_{i+1}$ for $i=1,2,\ldots,n-2$, $f(v_{n-1}v_n)=r_n$, and $f(vv_i)=r_{n+i}$ for $i=1,2,\ldots,n$ (see Figure 1). Then, $f^+(v_1)=r_1+r_2+r_{n+1}$, $f^+(v_i)=r_{i-1}+r_{i+1}+r_{n+i}$ for $i=2,\ldots,n-1$, and $f^+(v_n)=r_{n-1}+r_n+r_{2n}$. Note that

$$\begin{aligned}f^+(v_1)&=r_1+r_2+r_{n+1}<r_1+r_3+r_{n+2}=f^+(v_2),\\ f^+(v_i)&=r_{i-1}+r_{i+1}+r_{n+i}<r_i+r_{i+2}+r_{n+i+1}=f^+(v_{i+1}),\end{aligned} \tag{4}$$

for $i=2,\ldots,n-2$,

$$\begin{aligned}f^+(v_{n-1})&=r_{n-2}+r_n+r_{2n-1}<r_{n-1}+r_n+r_{2n}=f^+(v_n),\\ f^+(v_n)&=(r_{n-1}+r_n)+r_{2n}<(r_{n+1}+r_{n+2}+\cdots+r_{2n-1})\\ &\quad+r_{2n}=f^+(v).\end{aligned} \tag{5}$$

Hence,

$$f^+(v_1)<f^+(v_2)<\cdots<f^+(v_n)<f^+(v). \tag{6}$$

Case 2. $r_{n+1}+r_{n+2}+\cdots+r_{2n-1}\leq r_{n-1}+r_n$.

We define an edge labeling f of W_n with labels in $\{r_1,r_2,r_3,\ldots,r_{2n}\}$ by $f(v_1v_2)=r_{n+1}$, $f(v_iv_{i+2})=r_{n+i+1}$ for $i=1,2,\ldots,n-2$, $f(v_{n-1}v_n)=r_{2n}$, and $f(vv_i)=r_i$ for $i=1,2,\ldots,n$ (see Figure 2). Then, $f^+(v_1)=r_{n+1}+r_{n+2}+r_1$, $f^+(v_i)=r_{n+i-1}+r_{n+i+1}+r_i$ for $i=2,\ldots,n-1$, and $f^+(v_n)=r_{2n-1}+r_{2n}+r_n$. Note that

$$\begin{aligned}f^+(v_1)&=r_{n+1}+r_{n+2}+r_1<r_{n+1}+r_{n+3}+r_2=f^+(v_2),\\ f^+(v_i)&=r_{n+i-1}+r_{n+i+1}+r_i<r_{n+i}+r_{n+i+2}+r_{i+1}=f^+(v_{i+1}),\end{aligned} \tag{7}$$

for $i=2,\ldots,n-2$,

$$\begin{aligned}f^+(v_{n-1})&=r_{2n-2}+r_{2n}+r_{n-1}<r_{2n-1}+r_{2n}+r_n=f^+(v_n),\\ f^+(v)&=r_1+(r_2+r_3+\cdots+r_{n-1}+r_n)\\ &<r_1+(r_{n+1}+r_{n+2}+\cdots+r_{2n-2}+r_{2n-1})\leq r_1\\ &\quad+(r_{n-1}+r_n)=f^+(v_1).\end{aligned} \tag{8}$$

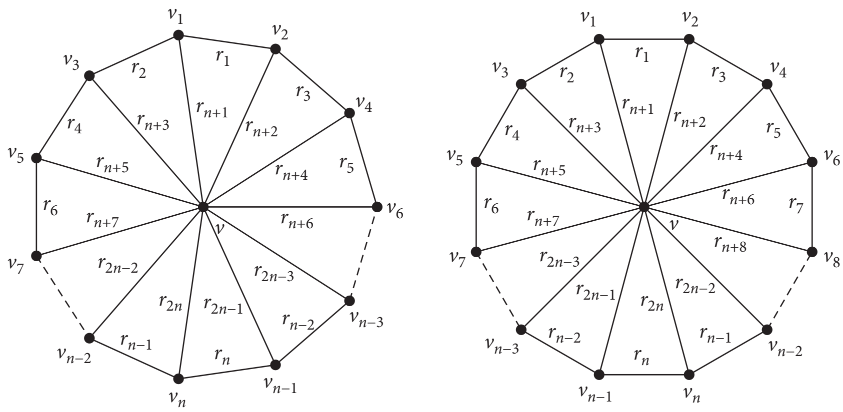
Hence,

$$f^+(v)<f^+(v_1)<f^+(v_2)<\cdots<f^+(v_n). \tag{9}$$

This completes the proof.

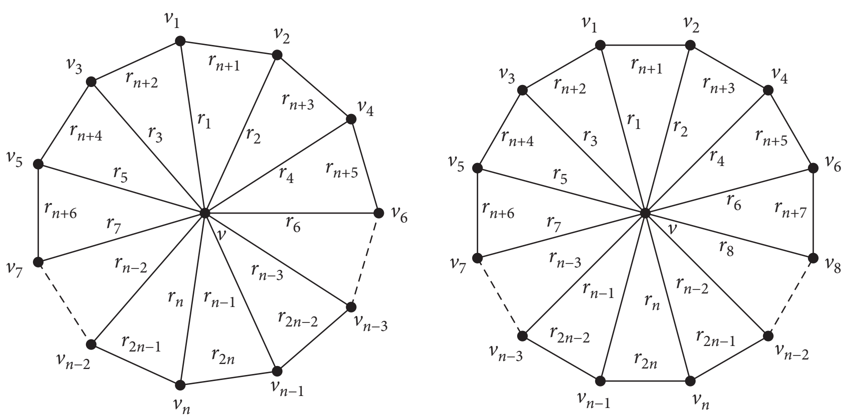
To prove the results in Section 3, we need the concept of uniformly $\mathbb{R}$-antimagic graphs, which is defined below. Let G be a graph. Suppose that all the vertices of G can be listed as $u_1,u_2,\ldots,u_m$ such that for every $A\subseteq\mathbb{R}$ with $|A|=|E(G)|$, there is an edge labeling f of G with labels in A such that $f^+(u_1)<f^+(u_2)<\cdots<f^+(u_m)$. Then, we say that G is uniformly $\mathbb{R}$-antimagic and that the sequence of vertices $u_1,u_2,\ldots,u_m$ has the uniformly $\mathbb{R}$-antimagic property. Note that in this definition, the ordering of the vertices $u_1,u_2,\ldots,u_m$ satisfying the property $f^+(u_1)<f^+(u_2)<\cdots<f^+(u_m)$ is independent of the choice of the subset A of $\mathbb{R}$. Obviously, every uniformly $\mathbb{R}$-antimagic graph is $\mathbb{R}$-antimagic.

Before proving our main result, we first describe uniformly $\mathbb{R}$-antimagic property with cycles and complete graphs.



n is odd *n is even*

Figure 1: Edge labeling of W_n if $r_{n-1} + r_n < r_{n+1} + r_{n+2} + \cdots + r_{2n-1}$.



n is odd *n is even*

Figure 2: Edge labeling of W_n if $r_{n+1} + r_{n+2} + \cdots + r_{2n-1} \le r_{n-1} + r_n$.

Theorem 2. *Every cycle is uniformly $\mathbb{R}$-antimagic.*

Proof. Let C_n be the cycle with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_1v_2\} \cup \{v_iv_{i+2} | i = 1, 2, \ldots, n-2\} \cup \{v_{n-1}v_n\}$. Let $r_1 < r_2 < r_3 < \cdots < r_n$ be the arbitrarily given n real numbers. We define an edge labeling f of C_n with labels in $\{r_1, r_2, \ldots, r_n\}$ by $f(v_1v_2) = r_1$, $f(v_iv_{i+2}) = r_{i+1}$ for $i = 1, 2, \ldots, n-2$, and $f(v_{n-1}v_n) = r_n$ (see Figure 3).

Then, $f^+(v_1) = r_1 + r_2$, $f^+(v_i) = r_{i-1} + r_{i+1}$ for $i = 2, \ldots, n-1$, and $f^+(v_n) = r_{n-1} + r_n$. Since $r_1 + r_2 < r_1 + r_3 < r_2 + r_4 < r_3 + r_5 < r_4 + r_6 < \cdots < r_{n-3} + r_{n-1} < r_{n-2} + r_n < r_{n-1} + r_n$, we have $f^+(v_1) < f^+(v_2) < \cdots < f^+(v_n)$. We see that the listing of vertices $v_1, v_2, \ldots, v_n$ with the property $f^+(v_1) < f^+(v_2) < \cdots < f^+(v_n)$ is independent of the arbitrarily given $r_1 < r_2 < r_3 < \cdots < r_n$. Thus, C_n is uniformly $\mathbb{R}$-antimagic. □

Theorem 3. *The complete graph $K_n (n \ge 3)$ is uniformly $\mathbb{R}$-antimagic.*

Proof. Let K_n be the complete graph with vertex set $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(K_n) = \{v_iv_j | 1 \le i \le j \le n\}$. Let $r_1 < r_2 < r_3 < \cdots < r_{\binom{n}{2}}$ be the arbitrarily given real numbers.

Let f be an edge labeling of K_n with labels in $\{r_1, r_2, r_3, \ldots, r_{\binom{n}{2}}\}$ such that for $i = 1, 2, \ldots, n-2$, $f(v_iv_{i+1}) < f(v_iv_{i+2}) < f(v_iv_{i+3}) < \cdots < f(v_iv_n) < f(v_{i+1}v_{i+2})$. Hence, $f(v_1v_2) < f(v_1v_3) < \cdots < f(v_1v_n) < f(v_2v_3) < f(v_2v_4) < \cdots < f(v_2v_n) < f(v_3v_4) < \cdots < f(v_{n-1}v_n)$.

For $1 \le i \le n-1$, we have

$$\begin{aligned} f^+(v_i) &= \sum_{1 \le k < i} f(v_kv_i) + f(v_iv_{i+1}) + \sum_{i+1 < k \le n} f(v_iv_k) \\ &< \sum_{1 \le k < i} f(v_kv_{i+1}) + f(v_iv_{i+1}) \\ &\quad + \sum_{i+1 < k \le n} f(v_{i+1}v_k) = f^+(v_{i+1}). \end{aligned} \tag{10}$$

Hence, $f^+(v_1) < f^+(v_2) < \cdots < f^+(v_n)$. We see that the listing of vertices $v_1, v_2, \ldots, v_n$ with the property $f^+(v_1) < f^+(v_2) < \cdots < f^+(v_n)$ is independent of the arbitrarily given $r_1 < r_2 < r_3 < \cdots < r_{\binom{n}{2}}$. Thus, K_n is uniformly $\mathbb{R}$-antimagic. □

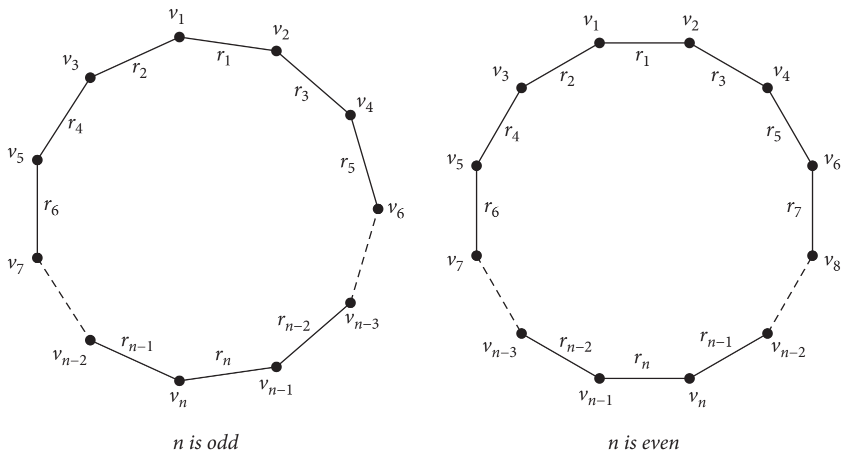


Figure 3: Edge labeling of C_n.

3. Main Results

Let G be a graph and A be a subset of $\mathbb{R}$ with $|A| = |E(G)|$. If g is an edge labeling of G with labels in A and K, L are nonempty subsets of $E(G)$ such that $g(x) < g(y)$ for all $x \in K$, $y \in L$, then we write $K \prec L$ under g. It is easy to see that the relation $\prec$ is transitive (i.e., if K, L, M are nonempty subsets of $E(G)$, and $K \prec L$, $L \prec M$, then $K \prec M$). The following trivial lemma will be used in the proofs of Theorems 4 and 5.

Lemma 1. *Let G be an arbitrary graph and A be a subset of $\mathbb{R}$ with $|A| = |E(G)|$. Let g be an edge labeling of G with labels in A. Suppose that A_1, A_2, B_1, B_2 are pairwise disjoint nonempty subsets of the edge set $E(G)$ with $|A_1| = |B_1|$, $|A_2| = |B_2| = 1$ such that $A_1 \prec B_1 \cup B_2$ and $A_2 \prec B_1$ under g. Then,*

$$\sum_{e\in A_1\cup A_2} g(e) < \sum_{e\in B_1\cup B_2} g(e). \tag{11}$$

Proof. Let $A_2 = \{a\}$ and b be an arbitrary edge in B_1. Since $A_2 \prec B_1$ under g, we have $g(a) < g(b)$. Since $A_1 \prec B_1 \cup B_2$ under g and $|A_1| = |B_2 \cup (B_1 - \{b\})|$, we have

$$\sum_{e\in A_1} g(e) < \sum_{e\in B_2\cup(B_1-\{b\})} g(e). \tag{12}$$

Note that

$$\sum_{e\in A_1\cup A_2} g(e) = g(a) + \sum_{e\in A_1} g(e), \tag{13}$$

and

$$\sum_{e\in B_1\cup B_2} g(e) = g(b) + \sum_{e\in B_2\cup(B_1-\{b\})} g(e). \tag{14}$$

Combining (12)–(14) and $g(a) < g(b)$, we have

$$\sum_{e\in A_1\cup A_2} g(e) < \sum_{e\in B_1\cup B_2} g(e). \tag{15}$$

We need the following notations. Let G be a graph, and A be a subset of $\mathbb{R}$ with $|A| = |E(G)|$. If f is an edge labeling of G with labels in A and D being a nontrivial connected subgraph of G which contains no isolated vertices, then we use $f_{E(D)}$ to denote the restriction of f to $E(D)$ with range $f(E(D))$. Obviously, $f_{E(D)}$ is an edge labeling of D with labels in $f(E(D))$. Moreover, for a vertex $v \in V(D)$, we use $f^+_{E(D)}(v)$ to denote $(f_{E(D)})^+(v)$. Recall that $E_D(v)$ is the set of all edges incident to v in D. Thus, $f^+_{E(D)}(v) = \sum_{e\in E_D(v)} f(e)$.

Let G and H be two graphs with $V(G) = \{u_1, u_2, \ldots, u_m\}$ and $V(H) = \{v_1, v_2, \ldots, v_n\}$, respectively. The Cartesian product of G and H, denoted by $G\square H$, is the graph with vertex set $V(G) \times V(H)$ such that (u_i, v_j) is adjacent to (u_k, v_l) if either $u_i = u_k$ and $v_j v_l \in E(H)$ or $v_j = v_l$ and $u_i u_k \in E(G)$. For the convenience of the following discussions, we will use the following notations in the proofs of Theorems 4 and 5. In the graph $G\square H$, the vertex $(u_i, v_j) \in V(G) \times V(H)$ is represented by $w_{i,j}$. For $j = 1, 2, \ldots, n$, we use G_j to denote the subgraph of $G\square H$ induced by the vertices $w_{i,j}$ $(i = 1, 2, \ldots, m)$. □

Note 1. The graphs $G, G_1, G_2, \ldots, G_n$ are isomorphic, and for each i $(i = 1, 2, \ldots, m)$, the vertices $u_i \in V(G)$, $w_{i,1} \in V(G_1)$, $w_{i,2} \in V(G_2)$, …, $w_{i,n} \in V(G_n)$ are the corresponding vertices under these isomorphisms.

Also, we use E_j to denote $E(G_j)$; i.e., E_j is the set of all edges in G_j. For $1 \le j < l \le n$ and $v_j v_l \in E(H)$, we use $E_{j,l}$ to denote the set $\{w_{i,j} w_{i,l} | i = 1, 2, \ldots, m\}$, i.e., $E_{j,l}$ is the set of all edges joining the vertices in G_j and the vertices in G_l. We see that $E(G\square H)$ is the disjoint union of E_j $(j = 1, 2, \ldots, n)$ and $E_{j,l}$ $(1 \le j < l \le n,\ v_j v_l \in E(H))$.

The notations for the vertices $w_{i,j}$, the subgraphs G_j, and the edge sets $E_j, E_{j,l}$ of $G\square H$ will be used in the proofs of Theorems 4 and 5.

Theorem 4. *Let G be a regular and uniformly $\mathbb{R}$-antimagic graph. Then, $G\square K_n$ $(n \ge 2)$ is also regular and uniformly $\mathbb{R}$-antimagic.*

Proof. Since both G and K_n are regular, it is trivial that $G\square K_n$ is regular. Since G is uniformly $\mathbb{R}$-antimagic, we assume that $u_1, u_2, \ldots, u_m$ $(m \geq 3)$ is the sequence of vertices of G with the uniformly $\mathbb{R}$-antimagic property. We see that the edge set $E(G\square K_n)$ is the union of E_j $(j = 1, 2, \ldots, n)$ and $E_{j,l}$ $(1 \leq j < l \leq n)$.

Now, we prove that $G\square K_n (n \geq 2)$ is uniformly $\mathbb{R}$-antimagic. Let $A \subseteq \mathbb{R}$ with $|A| = |E(G\square K_n)|$ be arbitrarily given. Define g to be an edge labeling of $G\square K_n$ with labels in A by the following three rules:

Rule 1. For $j = 1, 2, \ldots, n-1$, $E_j \prec E_{j,j+1} \prec E_{j,j+2} \prec \cdots \prec E_{j,n} \prec E_{j+1}$.

Rule 2. For $1 \leq j < l \leq n$, and for $i = 1, 2, \ldots, m-1$, $g(w_{i,j}w_{i,l}) < g(w_{i+1,j}w_{i+1,l})$ (i.e., $g(w_{1,j}w_{1,l}) < g(w_{2,j}w_{2,l}) < g(w_{3,j}w_{3,l}) < \cdots < g(w_{m,j}w_{m,l})$).

Rule 3. For $j = 1, 2, \ldots, n$, and for $i = 1, 2, \ldots, m-1$, $g^+_{E_j}(w_{i,j}) < g^+_{E_j}(w_{i+1,j})$ (i.e., $g^+_{E_j}(w_{1,j}) < g^+_{E_j}(w_{2,j}) < g^+_{E_j}(w_{3,j}) < \cdots < g^+_{E_j}(w_{m,j})$).

The edge labeling g with labels in A can have Rule 3 derived from the fact that the sequence of vertices $u_1, u_2, \ldots, u_m$ has the uniformly $\mathbb{R}$-antimagic property in G and the fact stated in Note 1. □

Claim 1. For $j = 1, 2, \ldots, n$, $g^+(w_{1,j}) < g^+(w_{2,j}) < g^+(w_{3,j}) < \cdots < g^+(w_{m,j})$.

Check of Claim 1. We need to show $g^+(w_{i,j}) < g^+(w_{i+1,j})$ for $i = 1, 2, \ldots, m-1$.

Let $J = \{1, 2, \ldots, n\}$. Note that

$$\begin{aligned} g^+(w_{i,j}) &= g^+_{E_j}(w_{i,j}) + \sum_{l\in J-\{j\}} g(w_{i,j}w_{i,l}), \\ g^+(w_{i+1,j}) &= g^+_{E_j}(w_{i+1,j}) + \sum_{l\in J-\{j\}} g(w_{i+1,j}w_{i+1,l}). \end{aligned} \tag{16}$$

By Rule 3, $g^+_{E_j}(w_{i,j}) < g^+_{E_j}(w_{i+1,j})$.

By Rule 2, for $1 \leq j < l \leq n$, $g(w_{i,j}w_{i,l}) < g(w_{i+1,j}w_{i+1,l})$, it implies

$$\sum_{l\in J-\{j\}} g(w_{i,j}w_{i,l}) < \sum_{l\in J-\{j\}} g(w_{i+1,j}w_{i+1,l}). \tag{17}$$

Thus, $g^+(w_{i,j}) < g^+(w_{i+1,j})$, which completes the Check of Claim 1.

Claim 2. For $j = 1, 2, \ldots, n-1$, $g^+(w_{m,j}) < g^+(w_{1,j+1})$.

Check of Claim 2. Let $J = \{1, 2, \ldots, n\}$. Note that

$$\begin{aligned} g^+(w_{m,j}) &= g^+_{E_j}(w_{m,j}) + \sum_{k\in J-\{j\}} g(w_{m,j}w_{m,k}) \\ &= g^+_{E_j}(w_{m,j}) + g(w_{m,j}w_{m,j+1}) + \sum_{k\in J-\{j,j+1\}} g(w_{m,k}w_{m,j}), \\ g^+(w_{1,j+1}) &= g^+_{E_{j+1}}(w_{1,j+1}) + \sum_{k\in J-\{j+1\}} g(w_{1,j+1}w_{1,k}) \\ &= g^+_{E_{j+1}}(w_{1,j+1}) + g(w_{1,j}w_{1,j+1}) + \sum_{k\in J-\{j,j+1\}} g(w_{1,k}w_{1,j+1}). \end{aligned} \tag{18}$$

Let $A_1 = E_{G_j}(w_{m,j}) \subseteq E_j$, $A_2 = \{w_{m,j}w_{m,j+1}\} \subseteq E_{j,j+1}$, $B_1 = E_{G_{j+1}}(w_{1,j+1}) \subseteq E_{j+1}$, $B_2 = \{w_{1,j}w_{1,j+1}\} \subseteq E_{j,j+1}$. Thus,

$$\begin{aligned} \sum_{e\in A_1\cup A_2} g(e) &= g^+_{E_j}(w_{m,j}) + g(w_{m,j}w_{m,j+1}), \\ \sum_{e\in B_1\cup B_2} g(e) &= g^+_{E_{j+1}}(w_{1,j+1}) + g(w_{1,j}w_{1,j+1}). \end{aligned} \tag{19}$$

By Rule 1, $E_j \prec E_{j,j+1} \prec E_{j+1}$. Since $A_1 \subseteq E_j$, $B_1 \subseteq E_{j+1}$, $A_2, B_2 \subseteq E_{j,j+1}$, we have $A_1 \prec B_1 \cup B_2$ and $A_2 \prec B_1$. Also, note $|A_1| = |B_1|$, $|A_2| = |B_2| = 1$. Thus, by Lemma 1, $\sum_{e\in A_1\cup A_2} g(e) < \sum_{e\in B_1\cup B_2} g(e)$. Hence,

$$g^+_{E_j}(w_{m,j}) + g(w_{m,j}w_{m,j+1}) < g^+_{E_{j+1}}(w_{1,j+1}) + g(w_{1,j}w_{1,j+1}). \tag{20}$$

By Rule 1, $E_{k,j} \prec E_{k,j+1}$ if $k < j$, and $E_{j,k} \prec E_{j+1,k}$ if $k > j+1$, and we see that $w_{m,k}w_{m,j} \in E_{k,j}$, $w_{1,k}w_{1,j+1} \in E_{k,j+1}$. Thus, $g(w_{m,k}w_{m,j}) < g(w_{1,k}w_{1,j+1})$, which implies

$$\sum_{k\in J-\{j,j+1\}} g(w_{m,k}w_{m,j}) < \sum_{k\in J-\{j,j+1\}} g(w_{1,k}w_{1,j+1}). \tag{21}$$

Combining (20) and (21), we obtain $g^+(w_{m,j}) < g^+(w_{1,j+1})$. This completes the Check of Claim 2.

From Claims 1 and 2, we obtain

$$\begin{aligned} g^+(w_{1,1}) &< g^+(w_{2,1}) < \cdots < g^+(w_{m,1}) \\ &< g^+(w_{1,2}) < g^+(w_{2,2}) < \cdots < g^+(w_{m,2}) \\ &< g^+(w_{1,3}) < g^+(w_{2,3}) < \cdots < g^+(w_{m,3}) \\ &< \cdots < \cdots < \cdots \\ &< g^+(w_{1,n}) < g^+(w_{2,n}) < \cdots < g^+(w_{m,n}). \end{aligned} \tag{22}$$

We also see that the order of the vertices $w_{1,1}, w_{2,1}, w_{3,1}, \ldots, w_{m,1}, w_{1,2}, w_{2,2}, w_{3,2}, \ldots, w_{m,2}, w_{1,3}, w_{2,3}, w_{3,3}, \ldots, w_{m,3}, w_{1,4}, \ldots, w_{m,n-1}, w_{1,n}, w_{2,n}, w_{3,n}, \ldots, w_{m,n}$ satisfying the aforementioned strict inequalities is independent of the chosen $A \subseteq \mathbb{R}$ with $|A| = |G\square K_n|$. Thus, $G\square K_n (n \geq 2)$ is uniformly $\mathbb{R}$-antimagic.

It has been proved that Cartesian product of two or more cycles is antimagic [7]. We further propose that $G\square C_n$ is (uniformly) $\mathbb{R}$-antimagic where G is a regular and uniformly $\mathbb{R}$-antimagic graph. In $G\square C_n$, the labels we use are in each subset A of real numbers with $|A| = |E(G)|$ and the labels used in [7, 8] are in $\{1, 2, \ldots, |E(G)|\}$. Because of the difference in labels, we have to modify the order of labelings that are different from those in [7, 8]. We use some strategies in the construction of labelings.

Theorem 5. *Let G be a regular and uniformly $\mathbb{R}$-antimagic graph. Then, $G\square C_n$ is also regular and uniformly $\mathbb{R}$-antimagic.*

Proof. Since both G and C_n are regular, it is trivial that $G\square C_n$ is regular. Now, we show that $G\square C_n$ is uniformly $\mathbb{R}$-antimagic. By Theorem 4, $G\square K_3$ is uniformly $\mathbb{R}$-antimagic. Thus, $G\square C_3$ is uniformly $\mathbb{R}$-antimagic. Using Theorem 4 twice, we see that $(G\square K_2)\square K_2$ is uniformly $\mathbb{R}$-antimagic. Thus, $G\square C_4$ is uniformly $\mathbb{R}$-antimagic since $(G\square K_2)\square K_2$ is isomorphic to $G\square C_4$. We assume that $n \geq 5$.

Assume that the cycle C_n has vertex set $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(C_n) = \{v_1 v_2\} \cup \{v_i v_{i+2} | i = 1, 2, \ldots, n-2\} \cup \{v_{n-1} v_n\}$. We use the notations for the vertices, subgraphs, and edge sets of $G\square H$ which are defined in Theorem 4, where H is now taken to be C_n. We see that the edge set $E(G\square C_n)$ is the union of E_j $(j = 1, 2, \ldots, n)$ and $E_{1,2}$, $E_{j,j+2}$ $(j = 1, 2, \ldots, n-2)$, $E_{n-1,n}$.

Now, we prove that $G\square C_n$ is uniformly $\mathbb{R}$-antimagic. Since G is uniformly $\mathbb{R}$-antimagic, we assume that $u_1, u_2, \ldots, u_m$ $(m \geq 3)$ is the sequence of vertices of G with the uniformly $\mathbb{R}$-antimagic property. Let $A \subseteq \mathbb{R}$ with $|A| = |E(G\square C_n)|$ be arbitrarily given. Define g to be an edge labeling of $G\square C_n$ with labels in A by the following three rules:

Rule 4. Rules of $\prec$ on $G\square C_n$.

(a) $E_1 \prec E_{1,2} \prec E_2$,
(b) for $j = 2, 3, \ldots, n-2$, $E_j \prec E_{j-1,j+1} \prec E_{j+1}$,
(c) $E_{n-1} \prec E_{n-2,n} \prec E_{n-1,n} \prec E_n$ (hence $E_1 \prec E_{1,2} \prec E_2 \prec E_{1,3} \prec E_3 \prec E_{2,4} \prec E_4 \prec E_{3,5} \prec E_5 \prec \cdots \prec E_{n-3} \prec E_{n-4,n-2} \prec E_{n-2} \prec E_{n-3,n-1} \prec E_{n-1} \prec E_{n-2,n} \prec E_{n-1,n} \prec E_n$).

Rule 5. For $v_j v_l \in E(C_n), g(w_{1,j} w_{1,l}) < g(w_{2,j} w_{2,l}) < g(w_{3,j} w_{3,l}) < \cdots < g(w_{m,j} w_{m,l})$.

Rule 6. For $j = 1, 2, \ldots, n$, we have $g^+_{E_j}(w_{1,j}) < g^+_{E_j}(w_{2,j}) < g^+_{E_j}(w_{3,j}) < \cdots < g^+_{E_j}(w_{m,j})$.

The edge labeling g with labels in A can have Rule 6 derived from the fact that the sequence of vertices $u_1, u_2, \ldots, u_m$ has the uniformly $\mathbb{R}$-antimagic property in G and the fact stated in Note 1. □

Claim 3. For $j = 1, 2, \ldots, n$, $g^+(w_{1,j}) < g^+(w_{2,j}) < g^+(w_{3,j}) < \cdots < g^+(w_{m,j})$.

Check of Claim 3.

We need to show $g^+(w_{i,j}) < g^+(w_{i+1,j})$ for $i = 1, 2, \ldots, m-1$. Note that

$$\begin{aligned} g^+(w_{i,j}) &= g^+_{E_j}(w_{i,j}) + \sum_{v_j v_l \in E(C_n)} g(w_{i,j} w_{i,l}), \\ g^+(w_{i+1,j}) &= g^+_{E_j}(w_{i+1,j}) + \sum_{v_j v_l \in E(C_n)} g(w_{i+1,j} w_{i+1,l}). \end{aligned} \tag{23}$$

By Rule 6, $g^+_{E_j}(w_{i,j}) < g^+_{E_j}(w_{i+1,j})$.

From Rule 5, we obtain that for fixed i, $i = 1, 2, \ldots, m-1$,

$$\sum_{v_j v_l \in E(C_n)} g(w_{i,j} w_{i,l}) < \sum_{v_j v_l \in E(C_n)} g(w_{i+1,j} w_{i+1,l}). \tag{24}$$

Thus, $g^+(w_{i,j}) < g^+(w_{i+1,j})$. This completes the Check of Claim 3.

Claim 4. For $j = 1, 2, \ldots, n-1$, $g^+(w_{m,j}) < g^+(w_{1,j+1})$.

Check of Claim 4. We distinguish five cases: Case 3, $j = 1$; Case 4, $j = 2$; Case 5, $j = 3, 4, \ldots, n-3$; Case 6, $j = n-2$; and Case 7, $j = n-1$.

Case 3. $j = 1$.

We need to show that $g^+(w_{m,1}) < g^+(w_{1,2})$. Let $A_1 = E_{G_1}(w_{m,1})$ and $A_2 = \{w_{m,1} w_{m,2}\}$. Then,

$$g^+(w_{m,1}) = g(w_{m,1} w_{m,3}) + \sum_{e \in A_1 \cup A_2} g(e). \tag{25}$$

Let $B_1 = E_{G_2}(w_{1,2})$ and $B_2 = \{w_{1,1} w_{1,2}\}$. Then,

$$g^+(w_{1,2}) = g(w_{1,2} w_{1,4}) + \sum_{e \in B_1 \cup B_2} g(e). \tag{26}$$

From Rule 4, $E_1 \prec E_{1,2} \prec E_2 \prec E_{1,3} \prec E_{2,4}$. Since $E_{1,3} \prec E_{2,4}$, we have

$$g(w_{m,1} w_{m,3}) < g(w_{1,2} w_{1,4}). \tag{27}$$

Since $E_1 \prec E_{1,2} \prec E_2$, $A_1 \subseteq E_1$, $A_2, B_2 \subseteq E_{1,2}$, $B_1 \subseteq E_2$, we have $A_1 \prec B_1 \cup B_2$, $A_2 \prec B_1$. Since G is regular, we have $|A_1| = |B_1|$. Trivially, $|A_2| = |B_2| = 1$. Thus, by Lemma 1,

$$\sum_{e \in A_1 \cup A_2} g(e) < \sum_{e \in B_1 \cup B_2} g(e). \tag{28}$$

From the aforementioned, we obtain $g^+(w_{m,1}) < g^+(w_{1,2})$.

Case 4. $j = 2$.

We need to show that $g^+(w_{m,2}) < g^+(w_{1,3})$. Note that

$$\begin{aligned} g^+(w_{m,2}) &= g^+_{E_2}(w_{m,2}) + g(w_{m,1} w_{m,2}) + g(w_{m,2} w_{m,4}), \\ g^+(w_{1,3}) &= g^+_{E_3}(w_{1,3}) + g(w_{1,1} w_{1,3}) + g(w_{1,3} w_{1,5}). \end{aligned} \tag{29}$$

Since $E_2 \prec E_3$ and G_2 and G_3 are regular with the same degree, we have

$$g^+_{E_2}(w_{m,2}) < g^+_{E_3}(w_{1,3}). \tag{30}$$

Since $E_{1,2} \prec E_{1,3}$, we have

$$g\left(w_{m,1}w_{m,2}\right) < g\left(w_{1,1}w_{1,3}\right). \tag{31}$$

Since $E_{2,4} \prec E_{3,5}$, we have

$$g\left(w_{m,2}w_{m,4}\right) < g\left(w_{1,3}w_{1,5}\right). \tag{32}$$

Thus, we obtain $g^+(w_{m,2}) < g^+(w_{1,3})$.

Case 5. $j = 3, 4, \ldots, n-3$.

We need to show that $g^+(w_{m,j}) < g^+(w_{1,j+1})$. For $n = 5$, we do not need to consider this case. Assume that $n \geq 6$. Note that

$$\begin{aligned} g^+\left(w_{m,j}\right) &= g^+_{E_j}\left(w_{m,j}\right) + g\left(w_{m,j-2}w_{m,j}\right) + g\left(w_{m,j}w_{m,j+2}\right), \\ g^+\left(w_{1,j+1}\right) &= g^+_{E_{j+1}}\left(w_{1,j+1}\right) + g\left(w_{1,j-1}w_{1,j+1}\right) + g\left(w_{1,j+1}w_{1,j+3}\right). \end{aligned} \tag{33}$$

From Rule 4(b), we have $E_j \prec E_{j-1,j+1} \prec E_{j+1} \prec E_{j,j+2}$ for $2 \leq j \leq n-3$. Since $E_j \prec E_{j+1}$ and G_j and G_{j+1} are regular with the same degree, we have

$$g^+_{E_j}\left(w_{m,j}\right) < g^+_{E_{j+1}}\left(w_{1,j+1}\right). \tag{34}$$

Since $E_{j-2,j} \prec E_{j-1,j+1}$, we have

$$g\left(w_{m,j-2}w_{m,j}\right) < g\left(w_{1,j-1}w_{1,j+1}\right). \tag{35}$$

Since $E_{j,j+2} \prec E_{j+1,j+3}$, we have

$$g\left(w_{m,j}w_{m,j+2}\right) < g\left(w_{1,j+1}w_{1,j+3}\right). \tag{36}$$

Accordingly, we obtain $g^+(w_{m,j}) < g^+(w_{1,j+1})$.

Case 6. $j = n-2$.

We need to show that $g^+(w_{m,n-2}) < g^+(w_{1,n-1})$. Note that

$$\begin{aligned} g^+\left(w_{m,n-2}\right) &= g^+_{E_{n-2}}\left(w_{m,n-2}\right) + g\left(w_{m,n-4}w_{m,n-2}\right) \\ &\quad + g\left(w_{m,n-2}w_{m,n}\right), \\ g^+\left(w_{1,n-1}\right) &= g^+_{E_{n-1}}\left(w_{1,n-1}\right) + g\left(w_{1,n-3}w_{1,n-1}\right) \\ &\quad + g\left(w_{1,n-1}w_{1,n}\right). \end{aligned} \tag{37}$$

Also note that $E_{n-4,n-2} \prec E_{n-2} \prec E_{n-3,n-1} \prec E_{n-1}$. Since $w_{m,n-4}w_{m,n-2} \in E_{n-4,n-2}$, $w_{1,n-3}w_{1,n-1} \in E_{n-3,n-1}$, we have

$$g\left(w_{m,n-4}w_{m,n-2}\right) < g\left(w_{1,n-3}w_{1,n-1}\right). \tag{38}$$

Since $E_{G_{n-2}}(w_{m,n-2}) \subseteq E_{n-2}$, $E_{G_{n-1}}(w_{1,n-1}) \subseteq E_{n-1}$, we have

$$g^+_{E_{n-2}}\left(w_{m,n-2}\right) < g^+_{E_{n-1}}\left(w_{1,n-1}\right). \tag{39}$$

Furthermore, $E_{n-2,n} \prec E_{n-1,n}$, this implies

$$g\left(w_{m,n-2}w_{m,n}\right) < g\left(w_{1,n-1}w_{1,n}\right). \tag{40}$$

Hence, we obtain $g^+(w_{m,n-2}) < g^+(w_{1,n-1})$.

Case 7. $j = n-1$.

We need to show that $g^+(w_{m,n-1}) < g^+(w_{1,n})$. Let $A_1 = E_{G_{n-1}}(w_{m,n-1})$ and $A_2 = \left\{w_{m,n-1}w_{m,n}\right\}$. Then,

$$g^+\left(w_{m,n-1}\right) = g\left(w_{m,n-3}w_{m,n-1}\right) + \sum_{e\in A_1\cup A_2} g(e). \tag{41}$$

Let $B_1 = E_{G_n}(w_{1,n})$ and $B_2 = \left\{w_{1,n-1}w_{1,n}\right\}$. Then,

$$g^+\left(w_{1,n}\right) = g\left(w_{1,n-2}w_{1,n}\right) + \sum_{e\in B_1\cup B_2} g(e). \tag{42}$$

Note that $E_{n-3,n-1} \prec E_{n-1} \prec E_{n-2,n} \prec E_{n-1,n} \prec E_n$. From $E_{n-3,n-1} \prec E_{n-2,n}$ and $w_{m,n-3}w_{m,n-1} \in E_{n-3,n-1}$, $w_{1,n-2}w_{1,n} \in E_{n-2,n}$, we have

$$g\left(w_{m,n-3}w_{m,n-1}\right) < g\left(w_{1,n-2}w_{1,n}\right). \tag{43}$$

From $E_{n-1} \prec E_{n-1,n} \prec E_n$ and $A_1 \subseteq E_{n-1}$, $A_2 \subseteq E_{n-1,n}$, $B_1 \subseteq E_n$, $B_2 \subseteq E_{n-1,n}$, we have $A_1 \prec B_1 \cup B_2$ and $A_2 \prec B_1$. Since G is regular, we have $|A_1| = |B_1|$. Trivially, $|A_2| = |B_2| = 1$. Thus, by Lemma 1,

$$\sum_{e\in A_1\cup A_2} g(e) < \sum_{e\in B_1\cup B_2} g(e). \tag{44}$$

Therefore, we obtain $g^+(w_{m,n-1}) < g^+(w_{1,n})$.

These complete the Check of Claim 4.

From Claims 3 and 4, we obtain

$$\begin{aligned} g^+\left(w_{1,1}\right) &< g^+\left(w_{2,1}\right) < \cdots < g^+\left(w_{m,1}\right) \\ &< g^+\left(w_{1,2}\right) < g^+\left(w_{2,2}\right) < \cdots < g^+\left(w_{m,2}\right) \\ &< g^+\left(w_{1,3}\right) < g^+\left(w_{2,3}\right) < \cdots < g^+\left(w_{m,3}\right) \\ &< \cdots < \cdots < \cdots \\ &< g^+\left(w_{1,n}\right) < g^+\left(w_{2,n}\right) < \cdots < g^+\left(w_{m,n}\right). \end{aligned} \tag{45}$$

We also see that the order of the vertices $w_{1,1}, w_{2,1}, w_{3,1}, \ldots, w_{m,1}, w_{1,2}, w_{2,2}, w_{3,2}, \ldots, w_{m,2}, w_{1,3}, w_{2,3}, w_{3,3}, \ldots, w_{m,3}, w_{1,4}, \ldots, w_{m,n-1}, w_{1,n}, w_{2,n}, w_{3,n}, \ldots, w_{m,n}$ satisfying the aforementioned strict inequalities is independent of the chosen $A \subseteq \mathbb{R}$ with $|A| = |E(G\square C_n)|$. Thus, $G\square C_n$ is uniformly $\mathbb{R}$-antimagic. This completes the proof of the theorem.

The following corollaries derive directly from Theorems 4 and 5.

Corollary 1. *The graph $G_1\square G_2\square\cdots\square G_n$ $(n \geq 2)$ is uniformly $\mathbb{R}$-antimagic, where G_1 is regular and uniformly $\mathbb{R}$-antimagic, and for $i \geq 2$, each G_i is a complete graph of order ≥ 2 or a cycle.*

Corollary 2. *The graph $G_1\square G_2\square\cdots\square G_n$ $(n \geq 2)$ is uniformly $\mathbb{R}$-antimagic, where each G_i is a complete graph of order ≥ 2 or a cycle.*

Proof. Each G_i is a complete graph of order ≥ 2 or a cycle. □

Case 8. Some $G_i \neq K_2$.

Without loss of generality, assume $G_1 \neq K_2$. Then, G_1 is a cycle or a complete graph of order ≥ 3. By Theorems 2 and 3, G_1 is uniformly $\mathbb{R}$-antimagic. Then, the corollary derives from Corollary 1.

Case 9. $G_i = K_2$ for $i = 1, 2, \ldots, n$.

Since $K_2 \square K_2 \cong C_4$, by Theorem 2, $G_1 \square G_2$ is uniformly $\mathbb{R}$-antimagic. Again, the corollary derives from Corollary 1.

Note that the hypercube Q_n is isomorphic to $G_1 \square G_2 \square \cdots \square G_n$, where each $G_i = K_2$ for $i = 1, 2, \ldots, n$. The following corollary derives from Corollary 2.

Corollary 3. *Hypercube* Q_n $(n \geq 2)$ *is uniformly* $\mathbb{R}$*-antimagic.*

4. Conclusions

In this paper, we propose the notion of $\mathbb{R}$-antimagic graph. This is a generalization of $\mathbb{R}^+$-antimagic graph. Every $\mathbb{R}$-antimagic graph is $\mathbb{R}^+$-antimagic, and every $\mathbb{R}^+$-antimagic is antimagic. Not all $\mathbb{R}^+$-antimagic graphs (e.g., stars and P_n, $n = 3, 4, 5$) are $\mathbb{R}$-antimagic.

In Section 2, we show that wheels, cycles, and complete graphs of order ≥3 are $\mathbb{R}$-antimagic. Let G be a complete graph (except K_1) or a cycle with $V(G) = \{u_1, u_2, \ldots, u_n\}$. We have found that all the vertices of G can be listed as $u_1, u_2, \ldots, u_n$ such that for every $A \subseteq \mathbb{R}$ with $|A| = |E(G)|$, there is an edge labeling f of G with labels in A such that $f^+(u_1) < f^+(u_2) < \cdots < f^+(u_n)$. The property we call uniformly $\mathbb{R}$-antimagic property is independent of the choice of the subset A of $\mathbb{R}$. We have found some graphs with uniformly $\mathbb{R}$-antimagic property.

We use labelings modified from those in [7, 8] and make them more systematic in this paper. The proofs in this paper provide efficient algorithms for finding edge labelings of Cartesian products of cycles and complete graphs. Our contribution is to quickly find the edge labelings of Cartesian products of cycles and complete graphs through the algorithms we constructed. It has been proved the Cartesian products $G_1 \square G_2 \square \cdots \square G_n$ $(n \geq 2)$ of G_1, G_2, ..., G_n are (uniformly) $\mathbb{R}$-antimagic if each G_i is either a complete graph (except K_1) or a cycle in Section 3.

We construct some classes of uniformly $\mathbb{R}$-antimagic graphs through Cartesian products. Some join graphs which are antimagic have been proved in [13, 14]. In [13], they use the way of listing edges in [9] to show that a class of join graphs are antimagic. It makes the method of labelings in this paper more plausible.

We end this paper with the following observation: every $\mathbb{R}^+$-antimagic graph is also $\mathbb{R}$-antimagic if the graph is regular. In further studies, we will propose $\mathbb{R}$-antimagicness of more regular graphs (e.g., Petersen graph). Also, we will generalize the research results in this paper in the proposals of Cartesian product of some other regular graphs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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