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Research Article

New S-Type Bounds of M-Eigenvalues for Elasticity Tensors with Applications

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In this paper, based on the extreme eigenvalues of the matrices arisen from the given elasticity tensor, S-type upper bounds for the M-eigenvalues of elasticity tensors are established. Finally, S-type sufficient conditions are introduced for the strong ellipticity of elasticity tensors based on the S-type M-eigenvalue inclusion sets.

1. Introduction

Let $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$; a real tensor $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ is called an elasticity tensor, if

$$a_{ijkl} = a_{kjil} = a_{ilkj}, \quad i, k \in M, j, l \in N. \tag{1}$$

Consider the following optimization problem with an elasticity tensor $\mathscr{A} = (a_{ijkl})$ [1, 2]:

$$\begin{aligned} \max f(x, y) &= \sum_{i,k=1}^{m} \sum_{j,l=1}^{n} a_{ijkl} x_i y_j x_k y_l, \\ s.t.\ x^T x &= 1, y^T y = 1, \\ x &\in \mathbb{R}^m, y \in \mathbb{R}^n. \end{aligned} \tag{2}$$

Qi et al. introduced the following definition of M-eigenvalues of an elasticity tensor [3, 4].

Definition 1. (see [3, 4]). Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be an elasticity tensor, if there exist nonzero vectors, $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} = \in \mathbb{R}^n$, and a real number $\lambda \in \mathbb{R}$, such that

$$\begin{cases} \mathscr{A}\mathbf{yxy} = \lambda \mathbf{x}, \\ \mathscr{A}\mathbf{xyx} = \lambda \mathbf{y}, \\ \mathbf{x}^T\mathbf{x} = 1, \mathbf{y}^T\mathbf{y} = 1, \end{cases} \tag{3}$$

where

$$(\mathscr{A}\mathbf{yxy})_i = \sum_{k\in M} \sum_{j,l=1^n} a_{ijkl} y_j x_k y_l, (\mathscr{A}\mathbf{xyx})_l = \sum_{i,k\in M} \sum_{j=1^n} a_{ijkl} x_i y_j x_k. \tag{4}$$

Then, λ is called an M-eigenvalue of $\mathscr{A}$, and the nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ are called the corresponding M-eigenvectors.

Qi in [3, 5, 6] presented some basic studies for tensor computations and approximations. Li et al. [7–10], Bu et al. [11], Che et al. [12], and Zhao et al. [13, 14] worked on analyzing the M-eigenvalues for various elasticity tensors. The authors in [15] proposed a tensor-based FTV model for the three-dimensional image deblurring problem, and some properties for Z-eigenvalues of tensor are given in [16–18]. Let

$$\begin{aligned} f(\mathbf{x}, \mathbf{y}) &= \sum_{i,k=1}^{m} \sum_{j,l=1}^{n} a_{ijkl} x_i y_j x_k y_l = \sum_{i,k=1}^{m} x_i x_k \mathbf{y}^T B_{ik} \mathbf{y} \\ &= \sum_{j,l=1}^{n} y_j y_l x^T C_{jl} x, \end{aligned} \tag{5}$$

where $B_{ik} \in \mathbb{R}^{n\times n}$ and $C_{jl} \in \mathbb{R}^{m\times m}$ are symmetric matrices with entries

$$(B_{ik})_{st} = a_{iskt},\ \left(C_{jl}\right)_{st} = a_{sjtl}. \tag{6}$$

And, assume that $\lambda_{\min}(A)$ is the minimal eigenvalue of a matrix A, $\lambda_{\max}(A)$ is the maximal eigenvalue of a matrix A, and $\rho(A)$ is the spectral radius of a matrix A. In 2021, Li et al. established the following bounds for M-eigenvalues of an elasticity tensor.

Theorem 1 (see [19]). *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be an elasticity tensor and λ be an M-eigenvalue of $\mathscr{A}$. Then,*

$$\max\{\delta_1, \delta_2\} \le \lambda \le \min\{\theta_1, \theta_2\}, \tag{7}$$

where

$$\begin{aligned}\delta_1 &=: \min_{l\in N}\{\lambda_{\min}(C_{ll}) - g_1(l)\},\ \theta_1 =: \min_{l\in N}\{\lambda_{\max}(C_{ll}) + g_1(l)\},\\ \delta_2 &=: \min_{i\in M}\{\lambda_{\min}(B_{ii}) - g_2(i)\},\ \theta_2 =: \min_{i\in M}\{\lambda_{\max}(B_{ii}) + g_2(i)\},\end{aligned} \tag{8}$$

and

$$g_1(l) = \sum_{j\in N, j\ne l} \rho\left(C_{jl}\right),\ g_2(i) = \sum_{k\in M, k\ne i} \rho(B_{ik}). \tag{9}$$

Theorem 2 (see [19]). *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be an elasticity tensor and $\rho_M(\mathscr{A})$ be the M-spectral radius of $\mathscr{A}$. Then,*

$$\rho_M(\mathscr{A}) \le \gamma =: \min\{\gamma_1, \gamma_2\}, \tag{10}$$

where

$$\begin{aligned}\gamma_1 &= \max_{j,l\in N, j\ne l} \frac{1}{2}\left\{\rho(C_{ll}) + \sqrt{\rho^2(C_{ll}) + 4g_1(l)\left(g_1(j) + \rho\left(C_{jj}\right)\right)}\right\},\\ \gamma_2 &= \max_{i,k\in M, k\ne i} \frac{1}{2}\left\{\rho(B_{ii}) + \sqrt{\rho^2(B_{ii}) + 4g_2(i)\left(g_2(k) + \rho(B_{kk})\right)}\right\}.\end{aligned} \tag{11}$$

The following necessary and sufficient condition for strong ellipticity for general anisotropic elastic materials is presented by Han et al. [20].

Theorem 3 (see [20]). *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be an elasticity tensor. The strong ellipticity condition holds, i.e.,*

$$f(\mathbf{x}, \mathbf{y}) = \sum_{i,k=1}^{m} \sum_{j,l=1}^{n} a_{ijkl} x_i y_j x_k y_l > 0, \tag{12}$$

for all nonzero vectors $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ if and only if the smallest M-eigenvalue of $\mathscr{A}$ is positive.

One application of the lower bound in Theorem 1 is to identify the strong ellipticity condition of an elasticity tensor, and the upper bound in Theorem 2 is given to accelerate convergence of the WQZ-algorithm [19]. In this paper, by breaking N into disjoint subsets S and its complement, new S-type upper bounds for the M-spectral radius of an elasticity tensor are given in Section 2. In Section 3, S-type sufficient conditions are also given to identify the strong ellipticity condition of an elasticity tensor.

2. S-Type Upper Bounds

In this section, we give S-type upper bounds for the largest M-eigenvalues of an elasticity tensor, and the relationship between the S-type upper bounds and existed upper bounds is also established. The sets S_m, $\overline{S}_m$, S_n, and $\overline{S}_n$ are defined by $M = S_m \bigcup \overline{S}_m$ and $S_m \cap \overline{S}_m = \varnothing$, $N = S_n \bigcup \overline{S}_n$, and $S_n \cap \overline{S}_n = \varnothing$.

Theorem 4. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be an elasticity tensor and $\rho_M(\mathscr{A})$ be the M-spectral radius of $\mathscr{A}$. Then,*

$$\rho_M(\mathscr{A}) \le \tau =: \min\{\tau_1, \tau_2\}, \tag{13}$$

where

$$\begin{aligned}\tau_1 &= \max_{i\in S_m, k\in \overline{S}_m} \frac{1}{2}\left\{g_2^{S_m}(i) + g_2^{\overline{S}_m}(k) + \sqrt{\left(g_2^{S_m}(i) - g_2^{\overline{S}_m}(k)\right)^2 + 4g_2^{\overline{S}_m}(i) g_2^{S_m}(k)}\right\},\\ \tau_2 &= \max_{j\in S_n, l\in \overline{S}_n} \frac{1}{2}\left\{g_1^{S_n}(j) + g_1^{\overline{S}_n}(l) + \sqrt{\left(g_1^{S_n}(j) - g_1^{\overline{S}_n}(l)\right)^2 + 4g_1^{\overline{S}_n}(j) g_1^{S_n}(l)}\right\},\end{aligned}$$

$$\begin{aligned} g_1^{S_n}(l) &= \sum_{j\in S_n} \rho\left(C_{jl}\right),\ g_1^{\overline{S}_n}(l) = \sum_{j\in \overline{S}_n} \rho\left(C_{jl}\right),\\ g_2^{S_m}(i) &= \sum_{k\in S_m} \rho\left(B_{ik}\right),\ g_2^{\overline{S}_m}(i) = \sum_{k\in \overline{S}_m} \rho\left(B_{ik}\right). \end{aligned} \tag{14}$$

Proof. Let λ be an M-eigenvalue of $\mathscr{A}$ with the M-eigenvectors $\mathbf{x}$, $\mathbf{y}$,

$$\left|x_p\right| = \max_{k\in S_m}\left\{\left|x_k\right|\right\}, \left|x_s\right| = \max_{k\in \overline{S}_m}\left\{\left|x_k\right|\right\}. \tag{15}$$

Obviously, at least one of $|x_p|$ and $|x_s|$ is nonzero.

Case I. If $|x_p||x_s| \neq 0$, from the p-th equation of $\lambda\mathbf{x} = \mathscr{A}\mathbf{yxy}$, we have

$$\lambda x_p = \sum_{k=1}^{m}\sum_{j,l=1}^{n} a_{pjkl} y_j x_k y_l. \tag{16}$$

Then, we can get

$$\begin{aligned} \lambda x_p &= \sum_{k\in S_m}\sum_{j,l\in N} a_{pjkl} y_j x_k y_l + \sum_{k\in \overline{S}_m}\sum_{j,l\in N} a_{pjkl} y_j x_k y_l\\ &= \sum_{k\in S_m} x_k\left(\sum_{j,l\in N} a_{pjkl} y_j y_l\right) + \sum_{k\in \overline{S}_m} x_k\left(\sum_{j,l\in N} a_{pjkl} y_j y_l\right)\\ &= \sum_{k\in S_m} x_k y^T B_{pk}\mathbf{y} + \sum_{k\in \overline{S}_m} x_k \mathbf{y}^T B_{pk}\mathbf{y}. \end{aligned} \tag{17}$$

Taking modulus in the above equation, we have

$$\begin{aligned} |\lambda|\left|x_p\right| &\leq \sum_{k\in S_m}\left|x_k\right|\left|\mathbf{y}^T B_{pk} y\right| + \sum_{k\in \overline{S}_m}\left|x_k\right|\left|\mathbf{y}^T B_{pk}\mathbf{y}\right|\\ &\leq g_2^{S_m}(p)\left|x_p\right| + g_2^{\overline{S}_m}(p)\left|x_s\right|. \end{aligned} \tag{18}$$

Then,

$$\left(|\lambda| - g_2^{S_m}(p)\right)\left|x_p\right| \leq g_2^{\overline{S}_m}(p)\left|x_s\right|. \tag{19}$$

If $|\lambda| - g_2^{S_m}(p) > 0$, similarly we can get

$$\left(|\lambda| - g_2^{\overline{S}_m}(s)\right)\left|x_s\right| \leq g_2^{S_m}(s)\left|x_p\right|. \tag{20}$$

Multiplying (20) with (21), we have

$$\left(|\lambda| - g_2^{S_m}(p)\right)\left(|\lambda| - g_2^{\overline{S}_m}(s)\right) \leq g_2^{\overline{S}_m}(p) g_2^{S_m}(s). \tag{21}$$

Therefore,

$$|\lambda| \leq \frac{1}{2}\left\{ g_2^{S_m}(p) + g_2^{\overline{S}_m}(s) + \sqrt{\left(g_2^{S_m}(p) - g_2^{\overline{S}_m}(s)\right)^2 + 4 g_2^{\overline{S}_m}(p) g_2^{S_m}(s)} \right\}. \tag{22}$$

If $|\lambda| - g_2^{S_m}(p) < 0$, then

$$|\lambda| < g_2^{S_m}(p), \tag{23}$$

which means that (23) also holds.

Case II. $|x_p||x_s| = 0$. If $|x_s| = 0$, by inequality (5), then $|\lambda| - g_2^{S_m}(p) \leq 0$; it yields that (7) also holds. If $|x_p| = 0$, by inequality (6), then $|\lambda| - g_2^{\overline{S}_m}(s) \leq 0$; it yields that (7) also holds.

Let $|y_q| = \max_{j\in S_n}\{|y_j|\}$ and $|y_t| = \max_{j\in \overline{S}_n}\{|y_j|\}$, from the q-th equation of $\lambda\mathbf{y} = \mathscr{A}\mathbf{xyx}$, we have

$$\lambda y_q = \sum_{i,k\in M}\sum_{j\in S_n} a_{ijkq} x_i y_j x_k + \sum_{i,k\in M}\sum_{j\in \overline{S}_n} a_{ijkq} x_i y_j x_k, \tag{24}$$

and similarly, we can get

$$|\lambda| \leq \frac{1}{2}\left\{ g_1^{S_n}(q) + g_1^{\overline{S}_n}(t) + \sqrt{\left(g_1^{S_n}(q) - g_1^{\overline{S}_n}(t)\right)^2 + 4 g_1^{\overline{S}_n}(q) g_1^{S_n}(t)} \right\}. \tag{25}$$

□

We compare the S-type upper bounds in Theorem 4 with the results in [19], which shows that our new S-type upper bounds are always tighter than the results in [19].

Theorem 5. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be an elasticity tensor. Then,*

$$\rho_M(\mathscr{A}) \leq \tau \leq \gamma. \tag{26}$$

Proof. If $\rho_M(\mathscr{A}) \le \tau$, then

$$\rho_M(\mathscr{A}) \le \frac{1}{2}\left\{ g_2^{S_m}(i) + g_2^{\overline{S}_m}(k) + \sqrt{\left(g_2^{S_m}(i) - g_2^{\overline{S}_m}(k)\right)^2 + 4 g_2^{\overline{S}_m}(i) g_2^{S_m}(k)} \right\} \tag{27}$$

or

$$\rho_M(\mathscr{A}) \le \frac{1}{2}\left\{ g_1^{S_n}(j) + g_1^{\overline{S}_n}(l) + \sqrt{\left(g_1^{S_n}(j) - g_1^{\overline{S}_n}(l)\right)^2 + 4 g_1^{\overline{S}_n}(j) g_1^{S_n}(l)} \right\}. \tag{28}$$

We only proof the following case, and the other case can be proved similarly. If

$$\rho_M(\mathscr{A}) \le \frac{1}{2}\left\{ g_2^{S_m}(i) + g_2^{\overline{S}_m}(s) + \sqrt{\left(g_2^{S_m}(i) - g_2^{\overline{S}_m}(s)\right)^2 + 4 g_2^{\overline{S}_m}(i) g_2^{S_m}(s)} \right\}, \tag{29}$$

from the proof of Theorem 4,

$$\left(|\lambda| - g_2^{S_m}(i)\right)\left(|\lambda| - g_2^{\overline{S}_m}(s)\right) \le g_2^{\overline{S}_m}(i) g_2^{S_m}(s). \tag{30}$$

Let $S_m = i$, $\overline{S}_m = M\backslash i$, then

$$(|\lambda| - \rho(B_{ii}))(|\lambda| - g_2(s)) \le g_2(i)\rho(B_{ss}). \tag{31}$$

From inequalities (20) or (21), there is an $i \in M$ with $|\lambda| - \rho(B_{ii}) \le g_2(i)$; for this index i, we have

$$\begin{aligned}(|\lambda| - \rho(B_{ii}))|\lambda| &\le (|\lambda| - \rho(B_{ii}))g_2(s) + g_2(i)\rho(B_{ss}) \\ &\le g_2(i)(g_2(s) + \rho(B_{ss})),\end{aligned} \tag{32}$$

and therefore, $\rho_M(\mathscr{A}) \le \gamma$. □

In 2009, the following WQZ-algorithm was presented to compute the largest M-eigenvalue of an elasticity tensor [4].

$$\begin{cases} e_{ijkl} = 1, & \text{if } i = k \text{ and } j = l, \\ e_{ijkl} = 0, & \text{otherwise.} \end{cases} \tag{33}$$

$$\begin{aligned} \overline{\mathbf{x}}_{t+1} &= \overline{\mathscr{A}}\mathbf{y}_t\mathbf{x}_t\mathbf{y}_t, \\ \mathbf{x}_{t+1} &= \frac{\overline{\mathbf{x}}_{t+1}}{\|\overline{\mathbf{x}}_{t+1}\|}, \\ \overline{\mathbf{y}}_{t+1} &= \overline{\mathscr{A}}\mathbf{x}_{t+1}\mathbf{y}_t\mathbf{x}_{t+1}, \\ \mathbf{y}_{t+1} &= \frac{\overline{\mathbf{y}}_{t+1}}{\|\overline{\mathbf{y}}_{t+1}\|}, \\ t &= t+1. \end{aligned} \tag{34}$$

$$\lambda_{\max}(\mathscr{A}) = f(\mathbf{x}^*, \mathbf{y}^*) - \upsilon, \tag{35}$$

$$f(\mathbf{x}^*, \mathbf{y}^*) = \sum_{i,k=1^m}\sum_{j,l=1^n} \overline{a}_{ijkl} x_i^* y_j^* x_k^* y_l^*, \upsilon = \sum_{1\le s\le t\le mn} |A_{st}|. \tag{36}$$

The following example in [4] is taken to show that the tighter upper bound can accelerate convergence of the WQZ-algorithm.

Example 1. Consider the tensor $\mathscr{A} = (a_{ijkl})$ of Example 4.1 in [4, 21], where

$$\mathscr{A}(:,:,1,1) = \begin{bmatrix} -0.9727 & 0.3169 & -0.3437 \\ -0.6332 & -0.7866 & 0.4257 \\ -0.3350 & -0.9896 & -0.4323 \end{bmatrix},$$

$$\mathscr{A}(:,:,2,1) = \begin{bmatrix} -0.6332 & -0.7866 & 0.4257 \\ 0.7387 & 0.6873 & -0.3248 \\ -0.7986 & -0.5988 & -0.9485 \end{bmatrix},$$

$$\mathscr{A}(:,:,3,1) = \begin{bmatrix} -0.3350 & -0.9896 & -0.4323 \\ -0.7986 & -0.5988 & -0.9485 \\ 0.5853 & 0.5921 & 0.6301 \end{bmatrix},$$

$$\mathscr{A}(:,:,1,2) = \begin{bmatrix} 0.3169 & 0.6158 & -0.0184 \\ -0.7866 & 0.0160 & 0.0085 \\ -0.9896 & -0.6663 & 0.2559 \end{bmatrix},$$

$$\mathscr{A}(:,:,2,2)=\begin{bmatrix}-0.7866 & 0.0160 & 0.0085\\ 0.6873 & 0.5160 & -0.0216\\ -0.5988 & 0.0411 & 0.9857\end{bmatrix},$$
$$\mathscr{A}(:,:,3,2)=\begin{bmatrix}-0.9896 & -0.6663 & 0.2559\\ -0.5988 & 0.0411 & 0.9857\\ 0.5921 & -0.2907 & -0.3881\end{bmatrix},$$
$$\mathscr{A}(:,:,1,3)=\begin{bmatrix}-0.3437 & -0.0184 & 0.5649\\ 0.4257 & 0.0085 & -0.1439\\ -0.4323 & 0.2559 & 0.6162\end{bmatrix}, \tag{37}$$
$$\mathscr{A}(:,:,2,3)=\begin{bmatrix}0.4257 & 0.0085 & -0.1439\\ -0.3248 & -0.0216 & -0.0037\\ -0.9485 & 0.9857 & -0.7734\end{bmatrix},$$
$$\mathscr{A}(:,:,3,3)=\begin{bmatrix}-0.4323 & 0.2559 & 0.6162\\ -0.9485 & 0.9857 & -0.7734\\ 0.6301 & -0.3881 & -0.8526\end{bmatrix}.$$

In [4], v is taken as follows:

$$\sum_{1\le s\le t\le mn}\left|A_{st}\right| = 23.3503. \tag{38}$$

Let $S_m = S_n = \{1\}$; by Corollary 2 in [22], we have

$$\rho(\mathscr{A}) \le 11.7253. \tag{39}$$

By Theorem 2, we have

$$\rho(\mathscr{A}) \le 4.2523. \tag{40}$$

Let $S_n = \{1,3\}$; by Theorem 4, we have

$$\rho(\mathscr{A}) \le 4.1528. \tag{41}$$

Example 2. Consider the elasticity tensor $\mathscr{A} = (a_{ijkl})$ of CaMg(CO3)2-dolomite [21], whose nonzero entries are

$$\begin{aligned}
&a_{2222} = a_{1111} = 196.6,\ a_{3311} = a_{2233} = 83.2,\ a_{3333} = 110,\\
&a_{2323} = a_{3232} = a_{1313} = a_{3131} = 54.7,\ a_{1212} = a_{2121} = 64.4,\\
&a_{2223} = a_{2232} = -a_{1213} = -a_{2131} = -31.7,\ a_{1122} = 132.2,\\
&a_{2132} = a_{1223} = -35.84,\ a_{3112} = a_{1321} = 44.8,\\
&a_{2321} = a_{1232} = -a_{1311} = -a_{1131} = -25.3.
\end{aligned} \tag{42}$$

In [4], v is taken as follows:

$$\sum_{1\le s\le t\le mn}\left|A_{st}\right| = 23.3503. \tag{43}$$

Let $S_m = S_n = \{1\}$; by Corollary 2 in [22], we have

$$\rho(\mathscr{A}) \le 491.7400. \tag{44}$$

By Theorem 2, we have

$$\rho(\mathscr{A}) \le 462.2316. \tag{45}$$

Let $S_m = \{2,3\}$; by Theorem 4, we have

$$\rho(\mathscr{A}) \le 211.4729. \tag{46}$$

In Figure 1, we can find that, when taking $v = 211.4729$, the sequence generated in the WQZ-algorithm converges to the largest M-eigenvalue more rapidly than taking $v = 1998.6000$ and $v = 462.2316$.

3. S-Type M-Eigenvalue Inclusion Sets and Strong Ellipticity Conditions

In this section, based on the S-type M-eigenvalue inclusion sets of an elasticity tensor, S-type sufficient conditions for strong ellipticity conditions are given. Let $(\mathscr{A}\mathbf{x}^2)_{jl} = \sum_{j,l=1}^{n} a_{ijkl}x_i x_k$ and $(\mathscr{A}\mathbf{y}^2)_{ik} = \sum_{i,k=1}^{m} a_{ijkl}y_j y_l$, we need the following lemma.

Lemma 1 (see [23]). *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be an elasticity tensor. Then, the strong ellipticity condition holds if and only if the matrix $\mathscr{A}\mathbf{x}^2 \in \mathbb{R}^{n\times n}$ (or $\mathscr{A}\mathbf{y}^2 \in \mathbb{R}^{m\times m}$) is positive definite for each nonzero $\mathbf{x} \in \mathbb{R}^m$ (or $\mathbf{y} \in \mathbb{R}^n$).*

Theorem 6. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be an elasticity tensor and λ be an M-eigenvalue of $\mathscr{A}$ with the M-eigenvectors $\mathbf{x}$, $\mathbf{y}$. Then,*

$$\lambda \in \Delta_1(\mathscr{A}) \cap \Delta_2(\mathscr{A}), \tag{47}$$

where

$$\Delta_1(\mathscr{A}) = \left(\bigcup_{j\in S_n, l\in \overline{S}_n}\left\{z\in\mathbb{R}:\ \left(\left|z - x^T C_{jj}x\right| - h_1^{S_n}(j)\right)\left(\left|z - x^T C_{ll}x\right| - h_1^{\overline{S}_n}(l)\right) \le g_1^{\overline{S}_m}(j) g_1^{S_m}(l)\right\}\right)$$
$$\cup\left(\bigcup_{j\in S_n}\left\{z\in\mathbb{R}:\ |z - x^T C_{jj}x| \le h_1^{S_n}(j)\right\}\right),$$

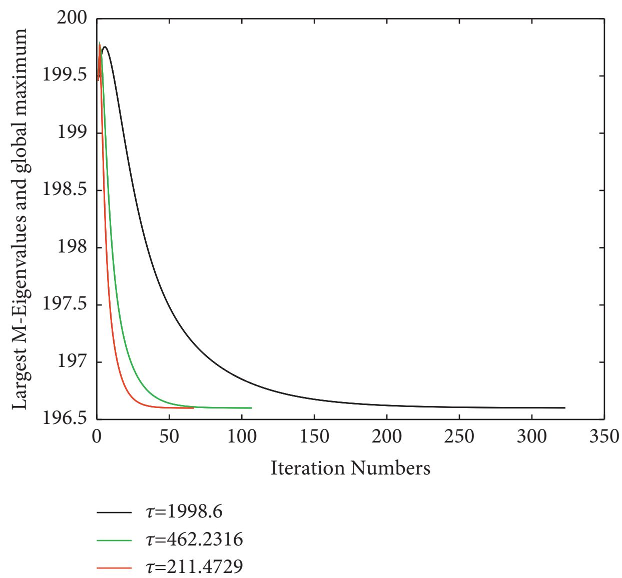

Figure 1: Numerical results for the WQZ-algorithm with different τ.

Step 0: given a tensor $\mathscr{A} = (a_{ijkl})$, vectors $\mathbf{x}_0 \in \mathbb{R}^m$ and $\mathbf{y}_0 \in \mathbb{R}^n$. Set $t = 0$ and $\overline{\mathscr{A}} = v\mathscr{I} + \mathscr{A}$, where $\mathscr{I} = (e_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ with the entries as follows:
Step 1: compute
Output $\mathbf{x}^*$, $\mathbf{y}^*$.
Step 2: find the largest M-eigenvalue $\lambda_{\max}(\mathscr{A})$ of the tensor $\mathscr{A}$:
where

Algorithm 1: WQZ-algorithm.

$$\begin{aligned}
\Delta_2(\mathscr{A}) &= \left(\bigcup_{i\in S_m,k\in\overline{S}_m}\left\{z\in\mathbb{R}\colon \left(\left|z-y^TB_{ii}y\right|-h_2^{S_m}(i)\right)\left(\left|z-y^TB_{kk}y\right|-h_2^{\overline{S}_m}(k)\right)\le g_2^{\overline{S}_m}(i)g_2^{S_m}(k)\right\}\right)\\
&\quad\cup\left(\bigcup_{i\in S_m}\left\{z\in\mathbb{R}\colon \left|z-y^TB_{ii}y\right|\le h_2^{S_m}(i)\right\}\right),\\
h_1^{S_n}(l) &= \sum_{j\in S_n,j\ne l}\rho\left(C_{jl}\right),\ h_1^{\overline{S}_n}(l)=\sum_{j\in\overline{S}_n,j\ne l}\rho\left(C_{jl}\right),\\
h_2^{S_m}(i) &= \sum_{k\in S_m,k\ne i}\rho\left(B_{ik}\right),\ h_2^{\overline{S}_m}(i)=\sum_{k\in\overline{S}_m,k\ne i}\rho\left(B_{ik}\right).
\end{aligned} \tag{48}$$

Proof. Let λ be an M-eigenvalue of $\mathscr{A}$ with the M-eigenvectors $\mathbf{x}$ and $\mathbf{y}$,

$$\left|x_p\right| = \max_{k\in S_m}\left\{\left|x_k\right|\right\},\ \left|x_s\right| = \max_{k\in\overline{S}_m}\left\{\left|x_k\right|\right\}. \tag{49}$$

Obviously, at least one of $|x_p|$ and $|x_s|$ is nonzero.

Case I. If $|x_p\|x_s|\ne 0$, from the p-th equation of $\lambda\mathbf{x} = \mathscr{A}\mathbf{yxy}$, we have

$$\lambda x_p = \sum_{k=1}^{m}\sum_{j,l=1}^{n} a_{pjkl}y_jx_ky_l. \tag{50}$$

Then, we can get

$$\lambda x_p - \mathbf{y}^TB_{pp}\mathbf{y}x_p = \sum_{k\in S_m,k\ne p} x_k\mathbf{y}^TB_{pk}\mathbf{y} + \sum_{k\in\overline{S}_m} x_k\mathbf{y}^TB_{pk}\mathbf{y}. \tag{51}$$

Taking modulus in the above equation, we have

$$\begin{aligned}
\left|\lambda-\mathbf{y}^TB_{pp}\mathbf{y}\|x_p\right| &\le \sum_{k\in S_m,k\ne p}\left|x_k\|\mathbf{y}^TB_{pk}\mathbf{y}\right| + \sum_{k\in\overline{S}_m}\left|x_k\|\mathbf{y}^TB_{pk}\mathbf{y}\right|\\
&\le h_2^{S_m}(p)\left|x_p\right| + g_2^{\overline{S}_m}(p)\left|x_s\right|.
\end{aligned} \tag{52}$$

Then,

$$\left(\left|\lambda - \mathbf{y}^T B_{pp}\mathbf{y}\right| - h_2^{S_m}(p)\right)\left|x_p\right| \le g_2^{\overline{S}_m}(p)\left|x_s\right|. \quad (53)$$

If $|\lambda - \mathbf{y}^T B_{pp}\mathbf{y}| - h_2^{S_m}(p) > 0$, similarly we can get

$$\left(\left|\lambda - \mathbf{y}^T B_{ss}\mathbf{y}\right| - h_2^{\overline{S}_m}(s)\right)\left|x_s\right| \le g_2^{S_m}(s)\left|x_p\right|. \quad (54)$$

Multiplying (53) with (54), we have

$$\left(\left|\lambda - \mathbf{y}^T B_{pp}\mathbf{y}\right| - h_2^{S_m}(p)\right)\left(\left|\lambda - \mathbf{y}^T B_{ss}\mathbf{y}\right| - h_2^{\overline{S}_m}(s)\right) \le g_2^{\overline{S}_m}(p) g_2^{S_m}(s), \quad (55)$$

so that $\lambda \in \Delta_2(\mathscr{A})$. If $|\lambda - \mathbf{y}^T B_{pp}\mathbf{y}| - h_2^{S_m}(p) \le 0$; then,

$$\left|\lambda - \mathbf{y}^T B_{pp}\mathbf{y}\right| \le h_2^{S_m}(p), \quad (56)$$

which means that $\lambda \in \Delta_2(\mathscr{A})$.

Case II. $|x_p||x_s| = 0$. Without loss of generality, let $|x_s| = 0$, by inequality (8), then $|\lambda - \mathbf{y}^T B_{pp}\mathbf{y}| - h_2^{S_m}(p) \le 0$; it yields that $\lambda \in \Delta_2(\mathscr{A})$.

Let $|y_q| = \max_{j\in S_n}\{|y_j|\}$ and $|y_t| = \max_{j\in \overline{S}_n}\{|y_j|\}$, from the q-th equation of $\lambda\mathbf{y} = \mathscr{A}\mathbf{xyx}$, similarly we can get $\lambda \in \Delta_1(\mathscr{A})$. □

Theorem 7. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be an elasticity tensor. If there exists S_m or S_n such that*

$$\begin{aligned} &\lambda_{\min}(B_{ii}) > h_2^{S_m}(i) \quad \text{for all } i \in S_m,\\ &\left(\lambda_{\min}(B_{ii}) - h_2^{S_m}(i)\right)\left(\lambda_{\min}(B_{kk}) - h_2^{\overline{S}_m}(k)\right) > g_2^{\overline{S}_m}(i) g_2^{S_m}(k) \quad \text{for all } i \in S_m, k \in \overline{S}_m, \end{aligned} \quad (57)$$

or

$$\begin{aligned} &\lambda_{\min}\left(C_{jj}\right) > h_1^{S_n}(j) \quad \text{for all } j \in S_n,\\ &\left(\lambda_{\min}\left(C_{jj}\right) - h_1^{S_n}(j)\right)\left(\lambda_{\min}(C_{ll}) - h_1^{\overline{S}_n}(l)\right) > g_2^{\overline{S}_n}(j) g_2^{S_n}(l) \quad \text{for all } j \in S_n, l \in \overline{S}_n, \end{aligned} \quad (58)$$

then the strong ellipticity condition holds.

Proof. Let λ be an M-eigenvalue of $\mathscr{A}$ and $\lambda \le 0$. From Theorem 6, we obtain $\lambda \in \Delta(\mathscr{B})$. If $\lambda \in \Delta_2(\mathscr{B})$, there are $i \in S_m$ and $k \in \overline{S}_m$ such that

$$\left(\left|\lambda - \mathbf{y}^T B_{ii}\mathbf{y}\right| - h_2^{S_m}(i)\right)\left(\left|\lambda - \mathbf{y}^T B_{kk}\mathbf{y}\right| - h_2^{\overline{S}_m}(k)\right) \le g_2^{\overline{S}_m}(i) g_2^{S_m}(k) \quad (59)$$

or

$$\left|\lambda - \mathbf{y}^T B_{ii}\mathbf{y}\right| \le h_2^{S_m}(i). \quad (60)$$

Then,

$$\begin{aligned} &\left(\left|\lambda - \mathbf{y}^T B_{ii}\mathbf{y}\right| - h_2^{S_m}(i)\right)\left(\left|\lambda - \mathbf{y}^T B_{kk}\mathbf{y}\right| - h_2^{\overline{S}_m}(k)\right)\\ &\ge \left(\mathbf{y}^T B_{ii}\mathbf{y} - h_2^{S_m}(i)\right)\left(y^T B_{kk}\mathbf{y} - h_2^{\overline{S}_m}(k)\right)\\ &\ge \left(\lambda_{\min}(B_{ii}) - h_2^{S_m}(i)\right)\left(\lambda_{\min}(B_{kk}) - h_2^{\overline{S}_m}(k)\right)\\ &> g_2^{\overline{S}_m}(i) g_2^{S_m}(k) \end{aligned} \quad (61)$$

and

$$\left|\lambda - \mathbf{y}^T B_{ii}\mathbf{y}\right| \ge \mathbf{y}^T B_{ii}\mathbf{y} \ge \lambda_{\min}(B_{ii}) > h_2^{S_m}(i), \quad (62)$$

which contradicts $\lambda \in \Delta(\mathscr{B})$. Therefore, $\lambda > 0$. Then, by Theorem 3, the strong ellipticity condition holds for the elasticity tensor $\mathscr{A}$.

If $\lambda \in \Delta_1(\mathscr{B})$, the second conclusion can be obtained similarly. □

The following sufficient conditions for strong ellipticity are given by Li et al. [19].

Theorem 8. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be an elasticity tensor. If*

$$\lambda_{\min}(B_{ii}) > g_2(i), \quad \text{for all } i \in M, \quad (63)$$

or

$$\lambda_{\min}(C_{ll}) > g_1(l), \quad \text{for all } l \in N, \quad (64)$$

then the strong ellipticity condition holds.

Based on the above theorems, we introduce the definitions strictly diagonally dominated (M-SDD) and S-type strictly diagonally dominated (M-SSDD) elasticity tensors, which are based on the eigenvalues of matrices of B_{ik} and C_{jl}.

Definition 2. Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be an elasticity tensor. If

$$\lambda_{\min}(B_{ii}) > g_2(i), \quad \text{for all } i \in M, \tag{65}$$

or

$$\lambda_{\min}(C_{ll}) > g_1(l), \quad \text{for all } l \in N, \tag{66}$$

then the elasticity tensor $\mathscr{A}$ is called strictly diagonally dominated(M-SDD).

Definition 3. Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be an elasticity tensor. If there exists S_m or S_n such that

$$\begin{aligned}&\lambda_{\min}(B_{ii}) > h_2^{S_m}(i) \quad \text{for all} \quad i \in S_m,\\&\left(\lambda_{\min}(B_{ii}) - h_2^{S_m}(i)\right)\left(\lambda_{\min}(B_{kk}) - h_2^{\overline{S}_m}(k)\right) > g_2^{\overline{S}_m}(i)g_2^{S_m}(k) \quad \text{for all } i \in S_m, k \in \overline{S}_m,\end{aligned} \tag{67}$$

or

$$\begin{aligned}&\lambda_{\min}\left(C_{jj}\right) > h_1^{S_n}(j) \quad \text{for all } j \in S_n,\\&\left(\lambda_{\min}\left(C_{jj}\right) - h_1^{S_n}(j)\right)\left(\lambda_{\min}(C_{ll}) - h_1^{\overline{S}_n}(l)\right) > g_2^{\overline{S}_n}(j)g_2^{S_n}(l) \quad \text{for all } j \in S_n, l \in \overline{S}_n,\end{aligned} \tag{68}$$

then the elasticity tensor $\mathscr{A}$ is called S-type strictly diagonally dominated(M-SSDD).

Next, we give the relationships between the M-SDD elasticity tensor and the M-SSDD elasticity tensor.

Theorem 9. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{m\times n\times m\times n}$ be an elasticity tensor. If $\mathscr{A}$ is an M-SDD elasticity tensor, then $\mathscr{A}$ is an M-SSDD elasticity tensor.*

Proof. If $\mathscr{A}$ is an M-SDD elasticity tensor, we only prove the following case; the other case can be proved similarly. For all $i \in M$,

$$\lambda_{\min}(B_{ii}) > g_2(i). \tag{69}$$

Then, for all $i \in S_m$ and $k \in \overline{S}_m$,

$$\begin{aligned}&\lambda_{\min}(B_{ii}) > g_2(i) > h_2^{S_m}(i),\\&\lambda_{\min}(B_{ii}) - h_2^{S_m}(i) > g_2^{\overline{S}_m}(i), \lambda_{\min}(B_{kk}) - h_2^{\overline{S}_m}(k) > g_2^{S_m}(k),\end{aligned} \tag{70}$$

which imply that

$$\begin{aligned}&\lambda_{\min}(B_{ii}) > h_2^{S_m}(i),\\&\left(\lambda_{\min}(B_{ii}) - h_2^{S_m}(i)\right)\left(\lambda_{\min}(B_{kk}) - h_2^{\overline{S}_m}(k)\right) > g_2^{\overline{S}_m}(i)g_2^{S_m}(k),\end{aligned} \tag{71}$$

and then $\mathscr{A}$ is an M-SSDD elasticity tensor. □

Now, the following example is explored to show the efficiency of the results in Theorems 8 and 9.

Example 3. Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{2\times 2\times 2\times 2}$ be an elasticity tensor, where

$$\begin{aligned}a_{1111} = a_{1212} = 2.5,\ a_{2121} = a_{2222} = 4,\\a_{1112} = a_{2122} = a_{1122} = -a_{1121} = -a_{1222} = 1,\end{aligned} \tag{72}$$

and other $a_{ijkl} = 0$.

Obviously, we have

$$\begin{aligned}B_{11} &= \begin{bmatrix} 2.5 & 1 \\ 1 & 2.5 \end{bmatrix},\\B_{22} &= \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix},\\B_{12} = B_{21} &= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},\\C_{11} = C_{22} &= \begin{bmatrix} 2.5 & -1 \\ -1 & 4 \end{bmatrix},\\C_{12} = C_{21} &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.\end{aligned} \tag{73}$$

Let $S_m = \{1\}$, by direct computation, we have

$$\lambda_{\min}(B_{11}) = 1.5 > 0 = h_2^{S_m}(1),\ \lambda_{\min}(B_{22}) = 3 > 0 = h_2^{\overline{S}_m}(2), \tag{74}$$

and

$$\left(\lambda_{\min}(B_{11}) - h_2^{S_m}(1)\right)\left(\lambda_{\min}(B_{22}) - h_2^{\overline{S}_m}(2)\right) = 4.5 > 4 = g_2^{\overline{S}_m}(1)g_2^{S_m}(2). \tag{75}$$

Then, $\mathscr{A}$ satisfies the sufficient conditions of Theorem 7, and the conditions of Theorem 8 do not hold by $\lambda_{\min}(B_{11}) = 1.5 < 2 = g_2(1)$ and $\lambda_{\min}(C_{11}) = 1.6707 < 2 = g_1(1)$. Therefore, the strong ellipticity condition holds for the elasticity tensor $\mathscr{A}$ by Theorem 7. In fact, the smallest M-eigenvalue of $\mathscr{A}$ is 3.5.

Let $S_m = S_n = \{1\}$; by Theorem 11 in [22], we have

$$\left(\alpha_1 - \overline{r}_1^1(\mathscr{A})\right)\left(\alpha_2 - \overline{r}_2^2(\mathscr{A})\right) = 7.5 < \overline{r}_1^2(\mathscr{A})\overline{r}_2^1(\mathscr{A}) = 20, \quad (76)$$

where α_1, $\overline{r}_1^1(\mathscr{A})$, α_2, $\overline{r}_2^2(\mathscr{A})$, $\overline{r}_1^2(\mathscr{A})$, $\overline{r}_2^1(\mathscr{A})$ are defined in [22], which shows that the conditions of Theorem 11 in [22] do not hold.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] G. Dahl, J. Leinaas, J. Myrheim, and E. Ovrum, "A tensor product matrix approximation problem in quantum physics," *Linear Algebra and Its Applications*, vol. 420, no. 2-3, pp. 711–725, 2007.

[2] C. Ling, J. Nie, L. Qi, and Y. Ye, "Bi-quadratic optimization over unit spheres and semidefinite programming relaxations," *SIAM Journal on Optimization*, vol. 20, no. 3, pp. 286–1310, 2009.

[3] L. Qi, H.-H. Dai, and D. Han, "Conditions for strong ellipticity and M-eigenvalues," *Frontiers of Mathematics in China*, vol. 4, no. 2, pp. 349–364, 2009.

[4] Y. Wang, L. Qi, and X. Zhang, "A practical method for computing the largestM-eigenvalue of a fourth-order partially symmetric tensor," *Numerical Linear Algebra with Applications*, vol. 16, no. 7, pp. 589–601, 2009.

[5] L. Qi, "Eigenvalues of a real supersymmetric tensor," *Journal of Symbolic Computation*, vol. 40, no. 6, pp. 1302–1324, 2005.

[6] L. Qi, "The best rank-one approximation ratio of a tensor space," *SIAM Journal on Matrix Analysis and Applications*, vol. 32, no. 2, pp. 430–442, 2011.

[7] C. Li, Y. Li, and X. Kong, "New eigenvalue inclusion sets for tensors," *Numerical Linear Algebra with Applications*, vol. 21, no. 1, pp. 39–50, 2014.

[8] C. Li, F. Wang, J. Zhao, Y. Zhu, and Y. Li, "Criterions for the positive definiteness of real supersymmetric tensors," *Journal of Computational and Applied Mathematics*, vol. 255, no. 1, pp. 1–14, 2014.

[9] C. Li and Y. Li, "Double B-tensors and quasi-double B-tensors," *Linear Algebra and Its Applications*, vol. 466, no. 1, pp. 343–356, 2015.

[10] C. Li, J. Zhou, and Y. Li, "A new Brauer-type eigenvalue localization set for tensors," *Linear and Multilinear Algebra*, vol. 64, no. 4, pp. 727–736, 2016.

[11] C. Bu, Y. Wei, L. Sun, and J. Zhou, "Brualdi-type eigenvalue inclusion sets of tensors," *Linear Algebra and Its Applications*, vol. 480, no. 1, pp. 168–175, 2015.

[12] H. Che, H. Chen, and Y. Wang, "On the M-eigenvalue estimation of fourthorder partially symmetric tensors," *Journal of Industrial and Management Optimization*, vol. 16, no. 1, pp. 09–324, 2020.

[13] J. Zhao and C. Li, "Singular value inclusion sets for rectangular tensors," *Linear and Multilinear Algebra*, vol. 66, no. 7, pp. 1333–1350, 2018.

[14] J. Zhao and C. Sang, "An S-type upper bound for the largest singular value of nonnegative rectangular tensors," *Open Mathematics*, vol. 14, no. 1, pp. 925–933, 2016.

[15] L. Guo, X. Zhao, X. Gu, Y. Zhao, Y. Zheng, and T. Huang, "Three-dimensional fractional total variation regularized tensor optimized model for image deblurring," *Applied Mathematics and Computation*, vol. 404, no. 1, Article ID 126224, 2021.

[16] G. Wang, Y. Wang, and Y. Wang, "Some Ostrowski-type bound estimations of spectral radius for weakly irreducible nonnegative tensors," *Linear and Multilinear Algebra*, vol. 68, no. 9, pp. 1–18, 2020.

[17] G. Wang, G. Zhou, and L. Caccetta, "Z-eigenvalue inclusion theorems for tensors," *Discrete and Continuous Dynamical Systems - Series B*, vol. 22, no. 1, pp. 1187–1198, 2017.

[18] C. Sang, "A new Brauer-type Z-eigenvalue inclusion set for tensors," *Numerical Algorithms*, vol. 80, no. 1, pp. 781–794, 2019.

[19] S. Li, Z. Chen, Q. Liu, and L. Lu, "Bounds of M-eigenvalues and strong ellipticity conditions for elasticity tensors," *Linear and Multilinear Algebra*, pp. 1–14, 2021.

[20] D. Han, H. H. Dai, and L. Qi, "Conditions for strong ellipticity of anisotropic elastic materials," *Journal of Elasticity*, vol. 97, no. 1, pp. 1–13, 2009.

[21] S. Li, C. Li, and Y. Li, "M-eigenvalue inclusion intervals for a fourth-order partially symmetric tensor," *Journal of Computational and Applied Mathematics*, vol. 356, no. 1, pp. 391–401, 2019.

[22] J. He, Y. Liu, and G. Xu, "New S-type inclusion theorems for the M-eigenvalues of a 4th-order partially symmetric tensor with applications," *Applied Mathematics and Computation*, vol. 398, no. 1, Article ID 125992, 2021.

[23] W. Ding, J. Liu, L. Qi, and H. Yan, "Elasticity M-tensors and the strong ellipticity condition," *Applied Mathematics and Computation*, vol. 373, no. 15, Article ID 24982, 2020.