Physical Science International Journal

Volume 27, Issue 4, Page 31-72, 2023; Article no.PSIJ.105168
ISSN: 2348-0130

Periodic, Transition and Escape Trajectories for 3D-Kepler 2-Body Problem of Classical Electrodynamics

Vasil G. Angelov [a*]

[a] *University of Mining and Geology, "St. I. Rilski", 1700 Sofia, Bulgaria.*

Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/PSIJ/2023/v27i4797

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/105168

Original Research Article

Received: 12/06/2023
Accepted: 18/08/2023
Published: 30/08/2023

ABSTRACT

In a previous paper we studied the Kepler problem for the extended Synge's 2-body problem of classical electrodynamics. We have used the radiation terms introduced in our previous papers and prove an existence–uniqueness of a periodic orbit in polar coordinates which confirmed the Bohr's hypothesis of the existence of the stationary states in the frame of classical electrodynamics. Our main aim here is to show the existence of trajectories of transition of the particle orbiting the nucleus from one stationary state to another excited state. We also prove the existence of escape trajectories. This is made by a choice of suitable function space and applying fixed point method.

Keywords: Periodic; transition and escape trajectories; 3D-Kepler problem; 2-body problem of classical electrodynamics; radiation term; fixed point method.

1. INTRODUCTION

In [1] J.L. Synge proposed a model of the interaction of two charged particles within the framework of classical electrodynamics. It is based on the results of V. Pauli [2], who derived the relativistic form of the Lienard-Wiechert retarded potentials. The Synge's equations of

**Corresponding author: E-mail: angelov@mgu.bg;*

motion describe the interaction between two particles by accounting for the finite speed of light propagation. This leads to the necessity of using retarded potentials.

The formulation of differential equations with delays was given by R. Driver [3]. The delays are not of the previously known type but depend on the unknown trajectories. They are now called state-dependent delays. Let us note that the Synge's approach describing the interaction is different from the conventional one, where one considers free fields, then includes the interaction (via the S-matrix), and then again has free fields (cf. for instance [4], [5]). In [6], [7] we have considered the existence of solution for the Kepler (plane) orbits of the 2-body problem. Since the most natural way to consider two particles in electrodynamics is in a reference frame external to them, we have investigated this problem in [8-10]. We have proved the existence-uniqueness of a periodic solution (orbit). Our research is based on the extended Synge’s equations, which we introduced by using the generalized, but relativistic invariant, the Dirac radiation term [11]. On the other hand, the Kepler problem cannot be ignored, because it is the same problem, only formally mathematically in a different formulation. That is why, in [12] we have considered the 3D-Kepler problem with radiation terms and proved an existence of periodic solution. In the present paper we improve the results from [12] simplifying radiation terms which look like their Lorentz form. Here we also prove the existence-uniqueness of periodic solutions, including the hitherto neglected group of equations whose arguments lie to the left of the initial one.

The existence of transition trajectories for the plane Kepler problem are obtained in [13]. Here we prove the existence of transition trajectories from the ground state to an excited state and vice versa in the real 3D-case. We also present conditions for the existence of escape trajectories. In this manner we obtain conditions for the existence of all possible cases of behavior of both particles in 3D-Kepler formulation.

Qualitative characteristics of the solution are obtained by appropriate choice of the function spaces and using the fixed-point method [14].

We recall the basic facts and denotations from [8-10]. The system of equations of motion with radiation terms in the Minkowski space consists of eight equations:

$$\begin{aligned} \frac{d\lambda_r^{(1)}}{ds_1} &= \frac{e_1}{m_1 c^2}\left(F_{rs}^{(2)}\,\lambda_s^{(1)} + F_{rs}^{(1)rad}\lambda_s^{(1)}\right), \\ \frac{d\lambda_r^{(2)}}{ds_2} &= \frac{e_2}{m_2 c^2}\left(F_{rs}^{(1)}\,\lambda_s^{(2)} + F_{rs}^{(2)rad}\lambda_s^{(2)}\right),\ (r=1,2,3,4) \end{aligned} \tag{1}$$

with Einstein summation convention. We recall the usually accepted denotations: $\langle .,. \rangle_4$ is the dot product in the Minkowski space, e_1, e_2 are charges of the particles, m_1, m_2 - their masses, c – the vacuum speed of the light, $F_{rs}^{(p)}(p=1,2)$ – the elements of the electromagnetic tensors, $F_{rs}^{(p),rad}(p=1,2)$ – the corresponding elements of the radiation terms, $\lambda_s^{(p)}(p=1,2)$ – the unit tangent vectors to the world lines.

In the previous papers we have proved that every fourth equation is a consequence of the first three ones. In this way, after some transformation, we reach the system:

$$\begin{aligned} \frac{du_1^{(1)}}{dt} &= \frac{\Delta_1^2}{c^2}\frac{\left(c^2 - u_1^{(1)}u_1^{(1)}\right)\left(G_1^{(12)} + G_1^{(1)rad}\right) - u_1^{(1)}u_2^{(1)}(G_2^{(12)} + G_2^{(1)rad}) - u_1^{(1)}u_3^{(1)}\left(G_3^{(12)} + G_3^{(1)rad}\right)}{\Delta_1^2}, \\ \frac{du_2^{(1)}}{dt} &= \frac{\Delta_1^2}{c^2}\frac{-u_2^{(1)}u_1^{(1)}\left(G_1^{(12)} + G_1^{(1)rad}\right) + \left(c^2 - u_2^{(1)}u_2^{(1)}\right)\left(G_2^{(12)} + G_2^{(1)rad}\right) - u_2^{(1)}u_3^{(1)}\left(G_3^{(12)} + G_3^{(1)rad}\right)}{\Delta_1^2}, \\ \frac{du_3^{(1)}}{dt} &= \frac{\Delta_1^2}{c^2}\frac{-u_3^{(1)}u_1^{(1)}\left(G_1^{(12)} + G_1^{(1)rad}\right) - u_3^{(1)}u_2^{(1)}\left(G_2^{(12)} + G_2^{(1)rad}\right) + \left(c^2 - u_3^{(1)}u_3^{(1)}\right)\left(G_3^{(12)} + G_3^{(1)rad}\right)}{\Delta_1^2}, \end{aligned} \tag{2}$$

$$\frac{du_1^{(2)}}{dt}=\frac{\Delta_2^2}{c^2}\frac{\left(c^2-u_1^{(2)}u_1^{(2)}\right)\left(G_1^{(21)}+G_1^{(2)rad}\right)-u_1^{(2)}u_2^{(2)}\left(G_2^{(21)}+G_2^{(2)rad}\right)-u_1^{(2)}u_3^{(2)}\left(G_3^{(21)}+G_3^{(2)rad}\right)}{\Delta_p^2},$$
$$\frac{du_2^{(2)}}{dt}=\frac{\Delta_2^2}{c^2}\frac{-u_2^{(2)}u_1^{(2)}\left(G_1^{(21)}+G_1^{(2)rad}\right)+\left(c^2-u_2^{(2)}u_2^{(2)}\right)\left(G_2^{(21)}+G_2^{(2)rad}\right)-u_2^{(2)}u_3^{(2)}\left(G_3^{(21)}+G_3^{(2)rad}\right)}{\Delta_2^2}, \quad (3)$$
$$\frac{du_3^{(2)}}{dt}=\frac{\Delta_2^2}{c^2}\frac{-u_3^{(2)}u_1^{(2)}\left(G_1^{(21)}+G_1^{(2)rad}\right)-u_3^{(2)}u_2^{(2)}\left(G_2^{(21)}+G_2^{(2)rad}\right)G_2^{(21)}+\left(c^2-u_3^{(2)}u_3^{(2)}\right)\left(G_3^{(21)}+G_3^{(2)rad}\right)}{\Delta_2^2},$$

where $\vec{u}^{(1)}(t)=\left(u_1^{(1)}(t),u_2^{(1)}(t),u_3^{(1)}(t)\right)$, $\vec{u}^{(2)}(t)=\left(u_1^{(2)}(t),u_2^{(2)}(t),u_3^{(2)}(t)\right)$ are the unknown velocities of the moving charged particles. By angular brackets we denote $\left\langle \vec{a},\vec{b}\right\rangle = a_1b_1+a_2b_2+a_3b_3$ the dot product in 3-dimensional Eucleadian subspace of the Minkowski space. The relativistic factors are:

$$\Delta_1=\sqrt{c^2-\left\langle \vec{u}^{(1)}(t),\vec{u}^{(1)}(t)\right\rangle}\,;\Delta_2=\sqrt{c^2-\left\langle \vec{u}^{(2)}(t),\vec{u}^{(2)}(t)\right\rangle},$$
$$\Delta_{12}=\sqrt{c^2-\left\langle \vec{u}^{(2)}(t-\tau_{12}),\vec{u}^{(2)}(t-\tau_{12})\right\rangle}\,;\ \Delta_{21}=\sqrt{c^2-\left\langle \vec{u}^{(1)}(t-\tau_{21}),\vec{u}^{(1)}(t-\tau_{21})\right\rangle}.$$

The isotropic 4-vectors $\xi^{(pq)}=\left(\xi_1^{(pq)},\xi_2^{(pq)},\xi_3^{(pq)},ic\tau_{pq}\right)$, $\left\langle \xi^{(pq)},\xi^{(pq)}\right\rangle_4=0$;

$$\vec{\xi}^{(pq)}=\left(\xi_1^{(pq)},\xi_2^{(pq)},\xi_3^{(pq)}\right)=(x_1^{(p)}(t)-x_1^{(q)}(t-\tau_{pq}),\ x_2^{(p)}(t)-x_2^{(q)}(t-\tau_{pq}),\ x_3^{(p)}(t)-x_3^{(q)}(t-\tau_{pq}))=$$
$$=\vec{x}^{(p)}(t)-\vec{x}^{(q)}(t-\tau_{pq})$$

yield the functional equations for the delays $\tau_{pq}\ (pq)=(12),(21)$:

$$c\tau_{pq}=\left\langle \vec{\xi}^{(pq)},\vec{\xi}^{(pq)}\right\rangle=\sqrt{\left\langle \vec{x}^{(p)}(t)-\vec{x}^{(q)}(t-\tau_{pq}),\vec{x}^{(p)}(t)-\vec{x}^{(q)}(t-\tau_{pq})\right\rangle}$$ ([8] - [10]).

So, the system (2) - (3) is a neutral type one with respect to the unknown velocities. The delays τ_{pq} , $(pq)=(12),(21)$ depend on the unknown trajectories.

This paper consists of two sections and an Appendix. The first section is Introduction**,** where we give the basic formulation of the 2-body problem in Cartesian coordinates obtained in previous paper [8]. Section 2 consists of 10 subsections and includes the main results. First, we formulate the 3D-Kepler problem in Cartesian coordinates in subsection 2.1 and then in subsection 2.2 we introduce spherical coordinates and derive the Initial equations in final form (cf. (7)). This form will be considered on the interval $[-T,0]$ because the arguments of the unknown trajectories (velocities) take values at this interval. We note that T is the period of the solution. That is why we call the first group (2) *Initial equations,* while the second one (3) – *Basic equations.* In the second group of equations (3), the arguments of the unknown velocities are $t\in[0,\infty)$. In fact, what they have in common is the initial point 0. We first solve the Initial system on the initial interval $[-T,0]$ and then solve the Basic system on the interval $[0,\infty)$ as the initial conditions of the Basic system, should be the value of the solution of the first system at $t=0$. In subsection 2.3 we derive the explicit form of the Basic equation in spherical coordinates. Subsection 2.4 is devoted to the derivation of the radiation terms in spherical coordinates. In subsection 2.5 an existence-uniqueness of solution of periodic boundary value problem for the Initial system is proved. In subsection 2.6 we formulate the periodic problem for the Basic system and in subsection 2.7 we obtain an existence-uniqueness theorem for this system. In subsection 2.8 we obtain transition trajectories and in subsection 2.9 we obtain an existence-uniqueness theorem for the Basic system. Finally, in subsection 2.10 we obtain existence-uniqueness of escape trajectories.

2. RESULTS AND DISCUSSION

2.1 The 3D-Kepler Problem

In this section we consider the above problem but, in a 3D-Kepler formulation. This leads to different type equations and the delays manifest themselves in another form. In accordance with the 3D-Kepler formulation of the 2-body problem the first particle P_1 is fixed at the origin $O(0,0,0)$, that is, $P_1\left(x_1^{(1)}(t)=0, x_2^{(1)}(t)=0, x_3^{(1)}(t)=0\right),\ t\in(-\infty,\infty)$ which implies:

$$u_1^{(1)}(t)=0,\ u_2^{(1)}(t)=0,\ u_3^{(1)}(t)=0;\ \ \dot{u}_1^{(1)}(t)=0,\ \dot{u}_2^{(1)}(t)=0,\ \dot{u}_3^{(1)}(t)=0\ . \tag{4}$$

Then the space part of the isotropic vectors $\left\langle \xi^{(pq)},\xi^{(pq)}\right\rangle_4=0,\ (pq)=(12),(21)$ take the form:

$$\vec{\xi}^{(12)}=\left(\xi_1^{(12)},\xi_2^{(12)},\xi_3^{(12)}\right)=(-x_1^{(2)}(t-\tau_{12}),\ -x_2^{(2)}(t-\tau_{12}),\ -x_3^{(2)}(t-\tau_{12}))=-\vec{x}^{(2)}(t-\tau_{12})\ ,$$

$$\vec{\xi}^{(21)}=\left(\xi_1^{(21)},\xi_2^{(21)},\xi_3^{(21)}\right)=(x_1^{(2)}(t),\ x_2^{(2)}(t),\ x_3^{(2)}(t))\ \text{and}\ \ \Delta_1=c\ ,\Delta_{21}=c\ .$$

Since the coordinates of the first particle are 0, we omit the superscripts of the second particle, namely $\vec{x}=\vec{x}^{(2)},\vec{u}=\vec{u}^{(2)}$. We recall the assumption **(C)**: $\sqrt{\left\langle \vec{u}^{(2)}(t),\vec{u}^{(2)}(t)\right\rangle}\le\bar{c}<c$. Then the value of the Sommerfeld fine structure constant becomes $\beta^2=\bar{c}^2/c^2=1/137^2\approx 0$ and consequently $\left\langle \vec{u},\vec{u}\right\rangle/c^2=1/137^2\approx 0$. We follow denotations from [8] - [10]:

$$D_{12}=\frac{c\sqrt{\left\langle \vec{\xi}^{(12)},\vec{\xi}^{(12)}\right\rangle}-\left\langle \vec{\xi}^{(12)},\vec{u}^{(2)}\right\rangle}{c\sqrt{\left\langle \vec{\xi}^{(12)},\vec{\xi}^{(12)}\right\rangle}-\left\langle \vec{\xi}^{(12)},\vec{u}^{(1)}\right\rangle}=\frac{c^2\tau_{pq}+\left\langle \vec{x}(t-\tau_{12}),\vec{u}(t-\tau_{12})\right\rangle}{c^2\tau_{pq}}=\frac{c^2\tau_{pq}+\left\langle \vec{x},\vec{u}\right\rangle}{c^2\tau_{pq}}=\frac{\tau_{pq}+\left\langle \vec{x},\frac{\vec{u}}{c^2}\right\rangle}{\tau_{pq}}\approx 1\,;$$

$$D_{21}=\frac{c\sqrt{\left\langle \vec{\xi}^{(21)},\vec{\xi}^{(21)}\right\rangle}-\left\langle \vec{\xi}^{(12)},\vec{u}^{(1)}\right\rangle}{c\sqrt{\left\langle \vec{\xi}^{(21)},\vec{\xi}^{(21)}\right\rangle}-\left\langle \vec{\xi}^{(21)},\vec{u}^{(2)}\right\rangle}=\frac{c^2\tau_{pq}}{c^2\tau_{pq}-\left\langle \vec{x}(t),\vec{u}(t)\right\rangle}=\frac{\tau_{pq}}{\tau_{pq}-\left\langle \vec{x},\frac{\vec{u}}{c^2}\right\rangle}\approx 1\,;$$

$$M_2=1+\left\langle \xi^{(12)},\frac{d\lambda^{(2)}}{ds_2}\right\rangle_4=1+D_{12}\frac{\left\langle \vec{\xi}^{(12)},\dot{\vec{u}}(t-\tau_{12})\right\rangle\Delta_{12}^2+\left(\left\langle \vec{\xi}^{(12)},\vec{u}(t-\tau_{12})\right\rangle-c^2\tau_{12}\right)\left\langle \vec{u}(t-\tau_{12}),\dot{\vec{u}}(t-\tau_{12})\right\rangle}{\Delta_{12}^4}\approx$$

$$\approx 1-\frac{\left\langle \vec{x},\dot{\vec{u}}\right\rangle\Delta_{12}^2+\left(c^2\tau_{12}+\left\langle \vec{x},\vec{u}\right\rangle\right)\left\langle \vec{u},\dot{\vec{u}}\right\rangle}{\Delta_{12}^4}\approx 1-\frac{\tau_{pq}+\left\langle \vec{x},\frac{\vec{u}}{c^2}\right\rangle}{\tau_{pq}}\frac{\left\langle \vec{x},\dot{\vec{u}}\right\rangle c^2+c^2\left(\tau_{pq}+\left\langle \vec{x},\frac{\vec{u}}{c^2}\right\rangle\right)\left\langle \vec{u},\dot{\vec{u}}\right\rangle}{c^4}\approx 1;$$

$$M_1=1+\left\langle \xi^{(21)},\frac{d\lambda^{(1)}}{ds_1}\right\rangle_4=1+\frac{c^2\tau_{pq}+\left\langle \vec{x}(t-\tau_{12}),\vec{u}(t-\tau_{12})\right\rangle}{c^2\tau_{pq}}\left(\frac{\left\langle \vec{\xi}^{(21)},\dot{\vec{u}}^{(1)}\right\rangle}{\Delta_{21}^2}+\frac{\left\langle \vec{\xi}^{(21)},\vec{u}^{(1)}\right\rangle-c^2\tau_{21}}{\Delta_{21}^4}\left\langle \vec{u}^{(1)},\dot{\vec{u}}^{(1)}\right\rangle\right)=1\,;$$

$$G_\alpha^{(12)}=\frac{e_1e_2\Delta_1}{c^2m_1}\left[\Delta^2{}_{12}\frac{\left(\left\langle \vec{u}^{(1)},\vec{u}^{(2)}\right\rangle-c^2\right)\xi_\alpha{}^{(12)}+\left(\left\langle \vec{\xi}^{(12)},\vec{u}^{(1)}\right\rangle-c^2\tau_{12}\right)u_\alpha{}^{(2)}}{\left(\left\langle \vec{\xi}^{(12)},\vec{u}^{(2)}\right\rangle-c^2\tau_{12}\right)^3}-\right.$$

$$\left.-\frac{\Delta_{12}^2\left\langle \vec{u}^{(1)},\dot{\vec{u}}^{(2)}\right\rangle\xi_\alpha{}^{(12)}+\left(\left\langle \vec{u}^{(1)},\vec{u}^{(2)}\right\rangle-c^2\right)\left\langle \vec{u}^{(2)},\dot{\vec{u}}^{(2)}\right\rangle\xi_\alpha{}^{(12)}-\left(\left\langle \vec{\xi}^{(12)},\vec{u}^{(1)}\right\rangle-c^2\tau_{12}\right)\left(\Delta_{12}^2\dot{u}_\alpha{}^{(2)}+u_\alpha{}^{(2)}\left\langle \vec{u}^{(2)},\dot{\vec{u}}^{(2)}\right\rangle\right)}{\Delta_{12}^2\left(\left\langle \vec{\xi}^{(12)},\vec{u}^{(2)}\right\rangle-c^2\tau_{12}\right)^2}\right]=$$

$$=\frac{e_1e_2\Delta_1}{m_1}\left[\frac{c^4(\tau_{12}u_\alpha-x_\alpha)}{c^2\left(c^2\tau_{12}+\left\langle \vec{x},\vec{u}\right\rangle\right)^3}-\frac{\tau_{12}c^2\dot{u}_\alpha+\left(x_\alpha+\tau_{12}u_\alpha\right)\left\langle \vec{u},\dot{\vec{u}}\right\rangle}{c^2\left(c^2\tau_{12}+\left\langle \vec{x},\vec{u}\right\rangle\right)^2}\right](\alpha=1,2,3);$$

$$G_\alpha^{(21)} = \frac{e_2 e_1 \Delta_2}{c^2 m_2}\left[c^2 \frac{-\left(\left\langle \vec{u}^{(2)},\vec{u}^{(1)}\right\rangle - c^2\right) x_\alpha{}^{(2)} + \left(-\left\langle \vec{x}^{(2)},\vec{u}^{(2)}\right\rangle - c^2\tau_{21}\right) u_\alpha{}^{(1)}}{\left(-\left\langle \vec{x}^{(2)},\vec{u}^{(1)}\right\rangle - c^2\tau_{21}\right)^3} - \right.$$

$$\left. - \frac{c^2\left\langle \vec{u}^{(2)},\dot{\vec{u}}^{(1)}\right\rangle \xi_\alpha{}^{(21)} - \left(\left\langle \vec{u}^{(2)},\vec{u}^{(1)}\right\rangle - c^2\right)\left\langle \vec{u}^{(1)},\dot{\vec{u}}^{(1)}\right\rangle x_\alpha{}^{(2)} - \left(-\left\langle \vec{x}^{(2)},\vec{u}^{(2)}\right\rangle - c^2\tau_{pq}\right)\left(c^2 \dot{u}_\alpha{}^{(1)} + u_\alpha{}^{(1)}\left\langle \vec{u}^{(1)},\dot{\vec{u}}^{(1)}\right\rangle\right)}{c^2\left(-\left\langle \vec{x}^{(2)},\vec{u}^{(1)}\right\rangle - c^2\tau_{21}\right)^2}\right] =$$

$$= \frac{e_2 e_1}{m_2}\frac{x_\alpha}{c^3 \tau_{21}{}^3} \quad (\alpha = 1,2,3).$$

Principal remark 1. We notice that in the 3D Kepler formulation of the system (2) - (3) split into two different groups. The first group (2) contains unknown functions with retarded argument $t-\tau_{12}(t) \in [-T,0]$, while the second one (3) contains unknown functions with argument $t \in [0,\infty)$. Both intervals have a common point, namely 0. The first group we call Initial equations while the second group – Basic equations. In this way we obtain two groups equations without delays, The delay caused by finite velocity of the propagation of interaction manifest in the fact that the first group of equations should be considered on interval $[-T,0]$, and second group – on the next interval $[0,\infty)$.

2.2 Initial Equations

The first group equations (2) take the form

$$G_1^{(12)} + G_1^{(1)rad} = 0, \; G_2^{(12)} + G_2^{(1)rad} = 0, \; G_3^{(12)} + G_3^{(1)rad} = 0.$$

But the particle P_1 is fixed (at the origin) which implies $G_\alpha^{(1)rad} = 0 \, (\alpha = 1,2,3)$ and then (2) become $G_1^{(12)} = 0$, $G_2^{(12)} = 0$, $G_3^{(12)} = 0$.

We notice that assumption **(C)** implies

$$\Delta_{12}^2 \ge c^2 - \bar{c}^2 = c^2(1-\beta^2) > 0 \text{ and } c^2\tau_{12} + \left\langle \vec{x}^{(2)},\vec{u}^{(2)}\right\rangle \ge c^2\tau_{12} - c\tau_{12}\bar{c} = \tau_{12}c(c-\bar{c}) > 0.$$

Therefore $G_\alpha^{(12)} = 0$ becomes

$$\frac{c^4(\tau_{12}u_\alpha - x_\alpha)}{c^2\left(c^2\tau_{12} + \langle \vec{x},\vec{u}\rangle\right)^3} + \left(c^2\tau_{12} + \langle \vec{x},\vec{u}\rangle\right)\frac{-\left\langle \vec{u},\dot{\vec{u}}\right\rangle x_\alpha - \tau_{12}c^2\dot{u}_\alpha - \tau_{12}u_\alpha\left\langle \vec{u},\dot{\vec{u}}\right\rangle}{c^2\left(c^2\tau_{12} + \langle \vec{x},\vec{u}\rangle\right)^3} = 0$$

or

$$c^4(\tau_{12}u_\alpha - x_\alpha) - \left(c^2\tau_{12} + \langle \vec{x},\vec{u}\rangle\right)\left(\tau_{12}c^2\dot{u}_\alpha + \left(x_\alpha + \tau_{12}u_\alpha\right)\left\langle \vec{u},\dot{\vec{u}}\right\rangle\right) = 0.$$

But $c^2\tau_{12} + \langle \vec{x},\vec{u}\rangle \approx c^2\left(\tau_{12} + \left\langle \vec{x},\frac{\vec{u}}{c^2}\right\rangle\right) \approx c^2\tau_{12}$ because $-\frac{\langle \vec{u},\vec{u}\rangle}{c^2} \le \frac{\vec{u}}{c^2} \le \frac{\langle \vec{u},\vec{u}\rangle}{c^2} \Rightarrow \frac{\vec{u}}{c^2} \approx 0$.

Therefore (2) takes the form:

$$\tau_{12}c^2\dot{u}_1 + \left(x_1 + \tau_{12}u_1\right)\left\langle \vec{u},\dot{\vec{u}}\right\rangle = c^2(\tau_{12}u_1 - x_1)/\tau_{12},$$
$$\tau_{12}c^2\dot{u}_2 + \left(x_2 + \tau_{12}u_2\right)\left\langle \vec{u},\dot{\vec{u}}\right\rangle = c^2(\tau_{12}u_2 - x_2)/\tau_{12},$$
$$\tau_{12}c^2\dot{u}_3 + \left(x_3 + \tau_{12}u_3\right)\left\langle \vec{u},\dot{\vec{u}}\right\rangle = c^2(\tau_{12}u_3 - x_3)/\tau_{12}.$$

Since $\left\langle \vec{u},\dot{\vec{u}}\right\rangle = u_1\dot{u}_1 + u_2\dot{u}_2 + u_3\dot{u}_3$, we rewrite the last system in the form:

$$(c^2\tau_{12}+x_1u_1+\tau_{12}u_1^2)\dot{u}_1+(x_1u_2+\tau_{12}u_1u_2)\dot{u}_2+(x_1u_3+\tau_{12}u_1u_3)\dot{u}_3=c^2(\tau_{12}u_1-x_1)/\tau_{12},$$
$$(x_2u_1+\tau_{12}u_1u_2)\dot{u}_1+(c^2\tau_{12}+x_2u_2+\tau_{12}u_2^2)\dot{u}_2+(x_2u_3+\tau_{12}u_2u_3)\dot{u}_3=c^2(\tau_{12}u_2-x_2)/\tau_{12},$$
$$(x_3u_1+\tau_{12}u_1u_3)\dot{u}_1+(x_3u_2+\tau_{12}u_2u_3)\dot{u}_2+(c^2\tau_{12}+x_3u_3+\tau_{12}u_3^2)\dot{u}_3=c^2(\tau_{12}u_3-x_3)/\tau_{12}.$$

We solve it with respect to $\dot{u}_1,\dot{u}_2,\dot{u}_3$. Indeed, since $c^2\tau_{12}+\langle\vec{x},\vec{u}\rangle\approx c^2\tau_{12}$, then

$$\delta=\begin{vmatrix} c^2\tau_{12}+x_1u_1+\tau_{12}u_1^2 & x_1u_2+\tau_{12}u_1u_2 & x_1u_3+\tau_{12}u_1u_3 \\ x_2u_1+\tau_{12}u_1u_2 & c^2\tau_{12}+x_2u_2+\tau_{12}u_2^2 & x_2u_3+\tau_{12}u_2u_3 \\ x_3u_1+\tau_{12}u_1u_3 & x_3u_2+\tau_{12}u_2u_3 & c^2\tau_{12}+x_3u_3+\tau_{12}u_3^2 \end{vmatrix}=$$
$$=c^4\tau_{12}^2(c^2\tau_{12}+x_3u_3+\tau_{12}u_3^2+x_2u_2+\tau_{12}u_2^2+x_1u_1+\tau_{12}u_1^2)=$$
$$=c^4\tau_{12}^2(c^2\tau_{12}+\tau_{12}\langle\vec{u},\vec{u}\rangle+\langle\vec{x},\vec{u}\rangle)=c^2c^4\tau_{12}^2\left(\tau_{12}+\tau_{12}\frac{\langle\vec{u},\vec{u}\rangle}{c^2}+\frac{\langle\vec{x},\vec{u}\rangle}{c^2}\right)\approx c^4\tau_{12}^2\left(c^2\tau_{12}+\langle\vec{x},\vec{u}\rangle\right)\approx c^6\tau_{12}^3>0;$$

$$\delta_1=\frac{c^2}{\tau_{12}}\begin{vmatrix} \tau_{12}u_1-x_1 & x_1u_2+\tau_{12}u_1u_2 & x_1u_3+\tau_{12}u_1u_3 \\ \tau_{12}u_2-x_2 & c^2\tau_{12}+x_2u_2+\tau_{12}u_2^2 & x_2u_3+\tau_{12}u_2u_3 \\ \tau_{12}u_3-x_3 & x_3u_2+\tau_{12}u_2u_3 & c^2\tau_{12}+x_3u_3+\tau_{12}u_3^2 \end{vmatrix}=$$
$$=c^6\tau_{12}(\tau_{12}u_1-x_1)+2c^4\tau_{12}u_3(x_3u_1-x_1u_3)+2c^4\tau_{12}u_2(x_2u_1-x_1u_2)=$$
$$=c^6\left[\tau_{12}(\tau_{12}u_1-x_1)+2\tau_{12}\left(x_3\frac{u_1u_3}{c^2}-x_1\frac{u_3^2}{c^2}+x_2\frac{u_1u_2}{c^2}-x_1\frac{u_2^2}{c^2}\right)\right]\approx c^6\tau_{12}(\tau_{12}u_1-x_1);$$

$$\delta_2=\frac{c^2}{\tau_{12}}\begin{vmatrix} c^2\tau_{12}+x_1u_1+\tau_{12}u_1^2 & \tau_{12}u_1-x_1 & x_1u_3+\tau_{12}u_1u_3 \\ x_2u_1+\tau_{12}u_1u_2 & \tau_{12}u_2-x_2 & x_2u_3+\tau_{12}u_2u_3 \\ x_3u_1+\tau_{12}u_1u_3 & \tau_{12}u_3-x_3 & c^2\tau_{12}+x_3u_3+\tau_{12}u_3^2 \end{vmatrix}\approx c^6\tau_{12}(\tau_{12}u_2-x_2);$$

$$\delta_3=\frac{c^2}{\tau_{12}}\begin{vmatrix} c^2\tau_{12}+x_1u_1+\tau_{12}u_1^2 & x_1u_2+\tau_{12}u_1u_2 & \tau_{12}u_1-x_1 \\ x_2u_1+\tau_{12}u_1u_2 & c^2\tau_{12}+x_2u_2+\tau_{12}u_2^2 & \tau_{12}u_2-x_2 \\ x_3u_1+\tau_{12}u_1u_3 & x_3u_2+\tau_{12}u_2u_3 & \tau_{12}u_3-x_3 \end{vmatrix}\approx c^6\tau_{12}(\tau_{12}u_3-x_3).$$

Then we obtain *Initial equations*:

$$\dot{u}_1=\frac{\tau_{12}u_1-x_1}{\tau_{12}^2},\dot{u}_2=\frac{\tau_{12}u_2-x_2}{\tau_{12}^2},\dot{u}_3=\frac{\tau_{12}u_3-x_3}{\tau_{12}^2}. \tag{5}$$

We look for a solution on the initial set for $t\in[-T,0]$, where T is the period of the solution. Indeed, we notice that all arguments of unknown function are retarded ones $t-\tau_{12}$, that is, $\vec{u}=\vec{u}^{(2)}(t-\tau_{12})$. Therefore, we must look for a solution on the initial set, that is, for $t-\tau_{12}(t)\in[\tau_0;0]$, where $\tau_0=\min\{t-\tau_{12}(t):t\in[0,T]\}$. Since $t-\tau_{12}(t)$ is increasing function because $1-d\tau_{12}(t)/dt>0$, we have proved that if the trajectories are *T*-periodic then $\tau_{12}(t)$ is *T*-periodic, too. It follows $-T-\tau_{12}(-T)\le 0-\tau_{12}(0)\Leftrightarrow -T-\tau_{12}(0)\le -\tau_{12}(0)\Leftrightarrow$ $\tau_{12}(0)\le T-\tau_{12}(0)$ and then $0-\tau_{12}(0)\le t-\tau_{12}(t)\le T-\tau_{12}(T)=T-\tau_{12}(0)=0$.

Consequently $-T\le t-\tau_{12}(t)\le 0$ which means $\tau_0=-T$ and put $t-\tau_{12}(t)=\theta\in[-T,0]$. So the argument of the unknown functions $\vec{u}=\vec{u}(\theta)$ belongs to $\theta\in[-T,0]$.

We pass to the spherical coordinates, that is, the second particle P_2 is located at the point:

$$x_1(\theta)=\rho(\theta)\cos\varphi(\theta)\cos\lambda(\theta);\ x_2(\theta)=\rho(\theta)\sin\varphi(\theta)\cos\lambda(\theta);x_3(\theta)=\rho(\theta)\sin\lambda(\theta)$$
$$\rho(\theta)\ge 0;\ \varphi(\theta)\ge 0;\ \lambda(\theta)\in\left[-\frac{\pi}{2}+\delta,\frac{\pi}{2}-\delta\right],0<\delta<\frac{\pi}{2}$$

and then the velocities and accelerations are:

$$u_1(\theta) = \dot{\rho}\cos\varphi\cos\lambda - \rho\dot{\varphi}\sin\varphi\cos\lambda - \rho\dot{\lambda}\cos\varphi\sin\lambda$$
$$u_2(\theta) = \dot{\rho}\sin\varphi\cos\lambda + \rho\dot{\varphi}\cos\varphi\cos\lambda - \rho\dot{\lambda}\sin\varphi\sin\lambda \ ,$$
$$u_3(\theta) = \dot{\rho}\sin\lambda + \rho\,\dot{\lambda}\cos\lambda$$

where $\dot{\rho} = d\rho / d\theta$ and

$$\dot{u}_1 = \ddot{\rho}\cos\varphi\cos\lambda - \rho\ddot{\varphi}\sin\varphi\cos\lambda - \rho\ddot{\lambda}\cos\varphi\sin\lambda -$$
$$-2\dot{\rho}\dot{\varphi}\sin\varphi\cos\lambda - 2\dot{\rho}\dot{\lambda}\cos\varphi\sin\lambda + 2\rho\dot{\varphi}\dot{\lambda}\sin\varphi\sin\lambda - \rho\dot{\varphi}^2\cos\varphi\cos\lambda - \rho\dot{\lambda}^2\cos\varphi\cos\lambda$$
$$\dot{u}_2 = \ddot{\rho}\sin\varphi\cos\lambda + \rho\ddot{\varphi}\cos\varphi\cos\lambda - \rho\ddot{\lambda}\sin\varphi\sin\lambda +$$
$$+2\dot{\rho}\dot{\varphi}\cos\varphi\cos\lambda - 2\dot{\rho}\dot{\lambda}\sin\varphi\sin\lambda - 2\rho\dot{\varphi}\dot{\lambda}\cos\varphi\sin\lambda - \rho\dot{\varphi}^2\sin\varphi\cos\lambda - \rho\dot{\lambda}^2\sin\varphi\cos\lambda$$
$$\dot{u}_3 = \ddot{\rho}\sin\lambda + \rho\,\ddot{\lambda}\cos\lambda + 2\dot{\rho}\dot{\lambda}\cos\lambda - \rho\,\dot{\lambda}^2\sin\lambda \ . \qquad (6)$$

$\vec{x} = (x_1, x_2, x_3),\ \vec{u} = (u_1, u_2, u_3)\ ;\ \tau_{12} = \rho / c,\ \langle \vec{x}, \vec{u} \rangle = \rho\dot{\rho}\ ,\ \langle \vec{u}, \vec{u} \rangle = \dot{\rho}^2 + \rho^2\dot{\varphi}^2\cos^2\lambda + \rho^2\dot{\lambda}^2\ ;\ \langle \vec{u}, \dot{\vec{u}} \rangle = \dot{\rho}\ddot{\rho} + \rho\dot{\rho}\dot{\varphi}^2 + \rho^2\dot{\varphi}\ddot{\varphi}\ ,$
$\langle \vec{u}, \vec{u} \rangle / c^2 \approx 0\ .$

Then (5) take the form:

$$\dot{u}_1 = c\frac{\dot{\rho}\cos\varphi\cos\lambda - \rho\dot{\varphi}\sin\varphi\cos\lambda - \rho\dot{\lambda}\cos\varphi\sin\lambda - c\cos\varphi\cos\lambda}{\rho} \approx -\frac{c^2\cos\varphi\cos\lambda}{\rho} \equiv Q_1,$$
$$\dot{u}_2 = \frac{(\rho / c)u_2 - x_2}{\rho^2 / c^2} = c\frac{\dot{\rho}\sin\varphi\cos\lambda + \rho\dot{\varphi}\cos\varphi\cos\lambda - \rho\dot{\lambda}\sin\varphi\sin\lambda - c\sin\varphi\cos\lambda}{\rho} \approx -\frac{c^2\sin\varphi\cos\lambda}{\rho} \equiv Q_2,$$
$$\dot{u}_3 = \frac{(\rho / c)u_3 - x_3}{\rho^2 / c^2} = c\frac{\dot{\rho}\sin\lambda + \rho\,\dot{\lambda}\cos\lambda - c\sin\lambda}{\rho} \approx -\frac{c^2\sin\lambda}{\rho} \equiv Q_3.$$

In view of (6) we introduce denotations:

$$P_1 = 2\dot{\rho}\dot{\varphi}\sin\varphi\cos\lambda + 2\dot{\rho}\dot{\lambda}\cos\varphi\sin\lambda - 2\rho\dot{\varphi}\dot{\lambda}\sin\varphi\sin\lambda + \rho\dot{\varphi}^2\cos\varphi\cos\lambda + \rho\dot{\lambda}^2\cos\varphi\cos\lambda\ ;$$
$$P_2 = -2\dot{\rho}\dot{\varphi}\cos\varphi\cos\lambda + 2\dot{\rho}\dot{\lambda}\sin\varphi\sin\lambda + 2\rho\dot{\varphi}\dot{\lambda}\cos\varphi\sin\lambda + \rho\dot{\varphi}^2\sin\varphi\cos\lambda + \rho\dot{\lambda}^2\sin\varphi\cos\lambda\ ;$$
$$P_3 = -2\dot{\rho}\dot{\lambda}\cos\lambda + \rho\,\dot{\lambda}^2\sin\lambda\ .$$

Then from (5) we obtain:

$$\ddot{\rho}\cos\varphi\cos\lambda - \ddot{\varphi}\rho\sin\varphi\cos\lambda - \ddot{\lambda}\rho\cos\varphi\sin\lambda = P_1 + Q_1$$
$$\ddot{\rho}\sin\varphi\cos\lambda + \rho\ddot{\varphi}\cos\varphi\cos\lambda - \rho\ddot{\lambda}\sin\phi\sin\lambda = P_2 + Q_2;$$
$$\ddot{\rho}\sin\lambda + \rho\,\ddot{\lambda}\cos\lambda = P_3 + Q_3\ .$$

To solve the last system with respect to $\ddot{\rho}, \ddot{\varphi}, \ddot{\lambda}$ we calculate:

$$\Delta = \begin{vmatrix} \cos\varphi\cos\lambda & -\rho\sin\varphi\cos\lambda & -\rho\cos\varphi\sin\lambda \\ \sin\varphi\cos\lambda & \rho\cos\varphi\cos\lambda & -\rho\sin\varphi\sin\lambda \\ \sin\lambda & 0 & \rho\cos\lambda \end{vmatrix} = \rho^2\cos\lambda\ ;$$

$$\Delta_1 = \rho^2\cos\lambda\begin{vmatrix} P_1+Q_1 & -\sin\varphi & -\cos\varphi\sin\lambda \\ P_2+Q_2 & \cos\varphi & -\sin\varphi\sin\lambda \\ P_3+Q_3 & 0 & \cos\lambda\end{vmatrix} =$$

$$= \rho^2\cos\lambda\left[\left(P_1+Q_1\right)\cos\varphi\cos\lambda + \left(P_2+Q_2\right)\sin\varphi\cos\lambda + \left(P_3+Q_3\right)\sin\lambda\right];$$

$$\Delta_2 = \rho^2\begin{vmatrix} \cos\varphi\cos\lambda & P_1+Q_1 & -\cos\varphi\sin\lambda \\ \sin\varphi\cos\lambda & P_2+Q_2 & -\sin\varphi\sin\lambda \\ \sin\lambda & P_3+Q_3 & \cos\lambda\end{vmatrix} = \rho\left[\left(P_2+Q_2\right)\cos\varphi - \left(P_1+Q_1\right)\sin\varphi\right];$$

$$\Delta_3 = \rho\cos\lambda\begin{vmatrix} \cos\varphi\cos\lambda & -\sin\varphi & P_1+Q_1 \\ \sin\varphi\cos\lambda & \cos\varphi & P_2+Q_2 \\ \sin\lambda & 0 & P_3+Q_3\end{vmatrix} =$$

$$= \rho\cos\lambda\left[\left(P_3+Q_3\right)\cos\lambda - \left(P_2+Q_2\right)\sin\varphi\sin\lambda - \left(P_1+Q_1\right)\cos\varphi\sin\lambda\right].$$

But

$$P_1\cos\varphi\cos\lambda + P_2\sin\varphi\cos\lambda + P_3\sin\lambda = \rho\dot{\varphi}^2\cos^2\lambda + \rho\dot{\lambda}^2;$$
$$P_2\cos\varphi - P_1\sin\varphi = 2\rho\dot{\varphi}\dot{\lambda}\sin\lambda - 2\dot{\rho}\dot{\varphi}\cos\lambda;$$
$$P_3\cos\lambda - P_2\sin\varphi\sin\lambda - P_1\cos\varphi\sin\lambda = -2\dot{\rho}\dot{\lambda} - \rho\dot{\varphi}^2\sin\lambda\cos\lambda;$$
$$Q_1\cos\varphi\cos\lambda + Q_2\sin\varphi\cos\lambda + Q_3\sin\lambda = -c^2/\rho;\ Q_2\cos\varphi - Q_1\sin\varphi = 0;$$
$$Q_3\cos\lambda - Q_2\sin\varphi\sin\lambda - Q_1\cos\varphi\sin\lambda = 0$$

and then

$$\ddot{\rho} = \frac{\rho^2\cos\lambda\left(\rho\dot{\varphi}^2\cos^2\lambda + \rho\dot{\lambda}^2 - \dfrac{c^2}{\rho}\right)}{\rho^2\cos\lambda};\ \ddot{\varphi} = \frac{\rho\left(2\rho\dot{\varphi}\dot{\lambda}\sin\lambda - 2\dot{\rho}\dot{\varphi}\cos\lambda\right)}{\rho^2\cos\lambda};\ \ddot{\lambda} = \frac{\rho\cos\lambda\left(-2\dot{\rho}\dot{\lambda} - \rho\dot{\varphi}^2\sin\lambda\cos\lambda\right)}{\rho^2\cos\lambda}.$$

Consequently, the *Initial equations* take the form:

$$\ddot{\rho} = \rho\dot{\varphi}^2\cos^2\lambda + \rho\dot{\lambda}^2 - \frac{c^2}{\rho},\ \ddot{\varphi} = 2\dot{\varphi}\dot{\lambda}\,tg\lambda - \frac{2\dot{\rho}\dot{\varphi}}{\rho},\ \ddot{\lambda} = -\frac{2\dot{\rho}\dot{\lambda} + \rho\dot{\varphi}^2\sin\lambda\cos\lambda}{\rho}. \quad (7)$$

We solve (7) on the initial interval $[-T,0]$.

2.3 Equations in Spherical Coordinates

After substituting (6) in (3) we obtain

$$\ddot{\rho}\cos\varphi\cos\lambda - \ddot{\varphi}\rho\sin\varphi\cos\lambda - \ddot{\lambda}\rho\cos\varphi\sin\lambda = P_1 + \frac{e_1e_2\Delta_2}{m_2c^3}\frac{c^2\cos\varphi\cos\lambda - u_1^{(2)}\dot{\rho}}{\rho^2} - \frac{e_2^2}{m_2c^2}\frac{\ddot{u}_1^{(2)}}{\Delta_2};$$
$$\ddot{\rho}\sin\varphi\cos\lambda + \ddot{\varphi}\rho\cos\varphi\cos\lambda - \ddot{\lambda}\rho\sin\varphi\sin\lambda = P_2 + \frac{e_1e_2\Delta_2}{m_2c^3}\frac{c^2\sin\varphi\cos\lambda - u_2^{(2)}\dot{\rho}}{\rho^2} - \frac{e_2^2}{m_2c^2}\frac{\ddot{u}_2^{(2)}}{\Delta_2}; \quad (8)$$
$$\ddot{\rho}\sin\lambda + \ddot{\lambda}\rho\cos\lambda = P_3 - \frac{e_1e_2\Delta_2}{m_2c^3}\frac{-c^2\sin\lambda + u_3^{(2)}\dot{\rho}}{\rho^2} - \frac{e_2^2}{m_2c^2}\frac{\ddot{u}_3^{(2)}}{\Delta_2}.$$

To solve (8) with respect to $\ddot{\rho},\ddot{\varphi},\ddot{\lambda}$ we calculate

$$\Lambda = \begin{vmatrix} \cos\varphi\cos\lambda & -\rho\sin\varphi\cos\lambda & -\rho\cos\varphi\sin\lambda \\ \sin\varphi\cos\lambda & \rho\cos\varphi\cos\lambda & -\rho\sin\varphi\sin\lambda \\ \sin\lambda & 0 & \rho\cos\lambda \end{vmatrix} = \rho^2\cos\lambda > 0 \text{ for } \rho > \rho_0 > 0,\ |\lambda| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}.$$

To simplify system (8) we denote by

$$B_1 = \frac{e_1e_2\Delta_2}{m_2c^3}\frac{c^2\cos\varphi\cos\lambda - u_1^{(2)}\dot{\rho}}{\rho^2} - \frac{e_2^2}{m_2c^2}\frac{\ddot{u}_1^{(2)}}{\Delta_2};\ B_2 = \frac{e_1e_2\Delta_2}{m_2c^3}\frac{c^2\sin\varphi\cos\lambda - u_2^{(2)}\dot{\rho}}{\rho^2} - \frac{e_2^2}{m_2c^2}\frac{\ddot{u}_2^{(2)}}{\Delta_2};$$
$$B_3 = \frac{e_1e_2\Delta_2}{m_2c^3}\frac{c^2\sin\lambda - u_3^{(2)}\dot{\rho}}{\rho^2} - \frac{e_2^2}{m_2c^2}\frac{\ddot{u}_3^{(2)}}{\Delta_2}.$$

Then system (8) can be rewritten in a compact form:

$$\begin{aligned} &\ddot{\rho}\cos\varphi\cos\lambda - \ddot{\varphi}\rho\sin\varphi\cos\lambda - \ddot{\lambda}\rho\cos\varphi\sin\lambda = P_1 + B_1; \\ &\ddot{\rho}\sin\varphi\cos\lambda + \ddot{\varphi}\rho\cos\varphi\cos\lambda - \ddot{\lambda}\rho\sin\varphi\sin\lambda = P_2 + B_2; \\ &\ddot{\rho}\sin\lambda + \ddot{\lambda}\rho\cos\lambda = P_3 + B_3. \end{aligned} \tag{9}$$

Applying the Cramer's formulas we obtain:

$$\begin{aligned} &\ddot{\rho} = \Lambda_1/\Lambda = (P_1+B_1)\cos\varphi\cos\lambda + (P_2+B_2)\sin\varphi\cos\lambda + (P_3+B_3)\sin\lambda, \\ &\ddot{\varphi} = \Lambda_2/\Lambda = \left[(P_2+B_2)\cos\varphi - (P_1+B_1)\sin\varphi\right]/\rho\cos\lambda, \\ &\ddot{\lambda} = \Lambda_3/\Lambda = \left[-(P_1+B_1)\cos\varphi\sin\lambda - (P_2+B_2)\sin\varphi\sin\lambda + (P_3+B_3)\cos\lambda\right]/\rho. \end{aligned}$$

Consequently

$$\begin{aligned} &\ddot{\rho} = P_1\cos\varphi\cos\lambda + P_2\sin\varphi\cos\lambda + P_3\sin\lambda + B_1\cos\varphi\cos\lambda + B_2\sin\varphi\cos\lambda + B_3\sin\lambda; \\ &\ddot{\varphi} = \frac{P_2\cos\varphi - P_1\sin\varphi + B_2\cos\varphi - B_1\sin\varphi}{\rho\cos\lambda}; \\ &\ddot{\lambda} = \frac{-P_1\cos\varphi\sin\lambda - P_2\sin\varphi\sin\lambda + P_3\cos\lambda - B_1\cos\varphi\sin\lambda - B_2\sin\varphi\sin\lambda + B_3\cos\lambda}{\rho}. \end{aligned} \tag{10}$$

To simplify the right-hand sides of the last equations we calculate

$$\begin{aligned} &P_1\cos\varphi\cos\lambda + P_2\sin\varphi\cos\lambda + P_3\sin\lambda = \rho\dot{\varphi}^2\cos^2\lambda + \rho\dot{\lambda}^2, \\ &P_2\cos\varphi - P_1\sin\varphi = -2\dot{\rho}\dot{\varphi}\cos\lambda + 2\rho\dot{\varphi}\dot{\lambda}\sin\lambda, \\ &-P_1\cos\varphi\sin\lambda - P_2\sin\varphi\sin\lambda + P_3\cos\lambda = -2\dot{\rho}\dot{\lambda} - \frac{\rho\dot{\varphi}^2\sin 2\lambda}{2}. \end{aligned}$$

Then (10) becomes:

$$\begin{aligned} &\ddot{\rho} = \rho\dot{\varphi}^2\cos^2\lambda + \rho\dot{\lambda}^2 + B_1\cos\varphi\cos\lambda + B_2\sin\varphi\cos\lambda + B_3\sin\lambda; \\ &\ddot{\varphi} = \frac{-2\dot{\rho}\dot{\varphi}\cos\lambda}{\rho\cos\lambda} + \frac{2\rho\dot{\varphi}\dot{\lambda}\sin\lambda}{\rho\cos\lambda} + \frac{B_2\cos\varphi - B_1\sin\varphi}{\rho\cos\lambda}; \\ &\ddot{\lambda} = -\frac{2\dot{\rho}\dot{\lambda}}{\rho} - \frac{\rho\dot{\varphi}^2\sin 2\lambda}{2\rho} + \frac{-B_1\cos\varphi\sin\lambda - B_2\sin\varphi\sin\lambda + B_3\cos\lambda}{\rho}. \end{aligned} \tag{11}$$

But, since

$$c^2 - u_1^{(2)}\dot\rho\cos\varphi\cos\lambda - u_2^{(2)}\dot\rho\sin\varphi\cos\lambda - u_3^{(2)}\dot\rho\sin\lambda = c^2 - \dot\rho^2 ;$$
$$u_1^{(2)}\dot\rho\sin\varphi - u_2^{(2)}\dot\rho\cos\varphi = -\dot\rho\rho\dot\varphi\cos\lambda ;$$
$$u_1^{(2)}\cos\varphi\sin\lambda + u_2^{(2)}\sin\varphi\sin\lambda - u_3^{(2)}\cos\lambda = -\rho\dot\lambda$$

we obtain:

$$B_1\cos\varphi\cos\lambda + B_2\sin\varphi\cos\lambda + B_3\sin\lambda =$$
$$= \frac{e_1e_2\Delta_2}{m_2c^3}\frac{c^2 - u_1^{(2)}\dot\rho\cos\varphi\cos\lambda - u_2^{(2)}\dot\rho\sin\varphi\cos\lambda - u_3^{(2)}\dot\rho\sin\lambda}{\rho^2} -$$
$$-\frac{e_2^2}{m_2c^2}\frac{\ddot u_1^{(2)}\cos\varphi\cos\lambda + \ddot u_2^{(2)}\sin\varphi\cos\lambda + \ddot u_3^{(2)}\sin\lambda}{\Delta_2} =$$
$$= \frac{e_1e_2\Delta_2}{m_2c^3}\frac{c^2-\dot\rho^2}{\rho^2} - \frac{e_2^2}{m_2c^2}\frac{\ddot u_1^{(2)}\cos\varphi\cos\lambda + \ddot u_2^{(2)}\sin\varphi\cos\lambda + \ddot u_3^{(2)}\sin\lambda}{\Delta_2};$$
$$B_2\cos\varphi - B_1\sin\varphi = \frac{e_1e_2\Delta_2}{m_2c^3}\frac{u_1^{(2)}\dot\rho\sin\varphi - u_2^{(2)}\dot\rho\cos\varphi}{\rho^2} + \frac{e_2^2}{m_2c^2}\frac{\ddot u_1^{(2)}\sin\varphi - \ddot u_2^{(2)}\cos\varphi}{\Delta_2} =$$
$$= -\frac{e_1e_2\Delta_2}{m_2c^3}\frac{\rho\dot\rho\dot\varphi\cos\lambda}{\rho^2} + \frac{e_2^2}{m_2c^2}\frac{\ddot u_1^{(2)}\sin\varphi - \ddot u_2^{(2)}\cos\varphi}{\Delta_2}$$
$$-B_1\cos\varphi\sin\lambda - B_2\sin\varphi\sin\lambda + B_3\cos\lambda =$$
$$= -\frac{e_1e_2\Delta_2}{m_2c^3}\frac{u_1^{(2)}\dot\rho\cos\varphi\sin\lambda + u_2^{(2)}\dot\rho\sin\varphi\sin\lambda - u_3^{(2)}\dot\rho\cos\lambda}{\rho^2} +$$
$$+\frac{e_2^2}{m_2c^2}\frac{\ddot u_1^{(2)}\cos\varphi\sin\lambda + \ddot u_2^{(2)}\sin\varphi\sin\lambda - \ddot u_3^{(2)}\cos\lambda}{\Delta_2} =$$
$$= -\frac{e_1e_2\Delta_2}{m_2c^3}\frac{-\rho\dot\lambda}{\rho^2} + \frac{e_2^2}{m_2c^2}\frac{\ddot u_1^{(2)}\cos\varphi\sin\lambda + \ddot u_2^{(2)}\sin\varphi\sin\lambda - \ddot u_3^{(2)}\cos\lambda}{\Delta_2}.$$

Then (11) becomes:

$$\begin{aligned}
\ddot\rho &= \rho\dot\varphi^2\cos^2\lambda + \rho\dot\lambda^2 + \frac{e_1e_2\Delta_2}{m_2c^3}\frac{c^2-\dot\rho^2}{\rho^2} - \frac{e_2^2}{m_2c^2}\frac{\ddot u_1^{(2)}\cos\varphi\cos\lambda + \ddot u_2^{(2)}\sin\varphi\cos\lambda + \ddot u_3^{(2)}\sin\lambda}{\Delta_2} \\
\ddot\varphi &= \frac{-2\dot\rho\dot\varphi\cos\lambda}{\rho\cos\lambda} + \frac{2\dot\varphi\dot\lambda\sin\lambda}{\cos\lambda} + \frac{1}{\rho\cos\lambda}\left(-\frac{e_1e_2\Delta_2}{m_2c^3}\frac{\rho\dot\rho\dot\varphi\cos\lambda}{\rho^2} + \frac{e_2^2}{m_2c^2}\frac{\ddot u_1^{(2)}\sin\varphi - \ddot u_2^{(2)}\cos\varphi}{\Delta_2}\right), \\
\ddot\lambda &= -\frac{2\dot\rho\dot\lambda}{\rho} - \frac{\rho\dot\varphi^2\sin 2\lambda}{2\rho} + \frac{1}{\rho}\left(\frac{e_1e_2\Delta_2}{m_2c^3}\frac{\rho\dot\lambda}{\rho^2} + \frac{e_2^2}{m_2c^2}\frac{\ddot u_1^{(2)}\cos\varphi\sin\lambda + \ddot u_2^{(2)}\sin\varphi\sin\lambda - \ddot u_3^{(2)}\cos\lambda}{\Delta_2}\right).
\end{aligned} \tag{12}$$

2.4 Final Form of the Basic Equations

To make the last step in the derivations of the equations of motion we derive the explicit form of the radiation term in spherical coordinates. Indeed, since

$$\rho(\theta+\tau)\approx\rho(\theta),\ \dot\rho(\theta+\tau)\approx\dot\rho(\theta),\ \varphi(\theta+\tau)\approx\varphi(\theta),\ \dot\varphi(\theta+\tau)\approx\dot\varphi(\theta),\ \lambda(\theta+\tau)\approx\lambda(\theta),\ \dot\lambda(\theta+\tau)\approx\dot\lambda(\theta)$$

and using the Schwartz derivative we obtain

$$\frac{\dot u_1(\theta+\tau)-\dot u_1(\theta-\tau)}{2\tau} \approx \frac{\ddot\rho(\theta+\tau)-\ddot\rho(\theta-\tau)}{2\tau}\cos\varphi\cos\lambda - \frac{\ddot\phi(\theta+\tau)-\ddot\phi(\theta-\tau)}{2\tau}\rho\sin\varphi\cos\lambda - \frac{\ddot\lambda(\theta+\tau)-\ddot\lambda(\theta-\tau)}{2\tau}\rho\cos\varphi\sin\lambda \approx$$
$$\approx \dddot\rho\cos\varphi\cos\lambda - \dddot\varphi\rho\sin\varphi\cos\lambda - \dddot\lambda\rho\cos\varphi\sin\lambda ;$$

$$\frac{\dot{u}_2(\theta+\tau)-\dot{u}_2(\theta-\tau)}{2\tau} \approx \frac{\dot{\rho}(\theta+\tau)-\dot{\rho}(\theta-\tau)}{2\tau}\sin\varphi\cos\lambda + \frac{\dot{\varphi}(\theta+\tau)-\dot{\varphi}(\theta-\tau)}{2\tau}\rho\cos\varphi\cos\lambda - \frac{\ddot{\lambda}(\theta+\tau)-\ddot{\lambda}(\theta-\tau)}{2\tau}\rho\sin\varphi\sin\lambda \approx$$
$$\approx \ddot{\rho}\sin\varphi\cos\lambda + \ddot{\varphi}\rho\cos\varphi\cos\lambda - \dddot{\lambda}\rho\sin\varphi\sin\lambda;$$
$$\frac{\dot{u}_3(\theta+\tau)-\dot{u}_3(\theta-\tau)}{2\tau} \approx \frac{\dot{\rho}(\theta+\tau)-\dot{\rho}(\theta-\tau)}{2\tau}\sin\lambda + \frac{\dot{\lambda}(\theta+\tau)-\dot{\lambda}(\theta-\tau)}{2\tau}\rho\cos\lambda \approx \ddot{\rho}\sin\lambda + \dddot{\lambda}\rho\cos\lambda.$$

Therefore

$$\ddot{u}_1^{(2)} \approx \ddot{\rho}\cos\varphi\cos\lambda - \ddot{\varphi}\rho\sin\varphi\cos\lambda - \dddot{\lambda}\rho\cos\varphi\sin\lambda$$
$$\ddot{u}_2^{(2)} \approx \ddot{\rho}\sin\varphi\cos\lambda + \ddot{\varphi}\rho\cos\varphi\cos\lambda - \dddot{\lambda}\rho\sin\varphi\sin\lambda$$
$$\ddot{u}_3^{(2)} \approx \ddot{\rho}\sin\lambda + \dddot{\lambda}\rho\cos\lambda$$

and therefore $\langle \vec{u},\ddot{\vec{u}}\rangle = \dot{\rho}\ddot{\rho} + \rho^2\dot{\varphi}\ddot{\varphi}\cos^2\lambda + \rho^2\dot{\lambda}\dddot{\lambda}$.

Then the radiation terms of (12) take the form

$$\ddot{u}_1^{(2)}\cos\varphi\cos\lambda + \ddot{u}_2^{(2)}\sin\varphi\cos\lambda + \ddot{u}_3^{(2)}\sin\lambda = \ddot{\rho}\ ,$$
$$\ddot{u}_1^{(2)}\sin\varphi - \ddot{u}_2^{(2)}\cos\varphi = -\ddot{\varphi}\rho\cos\lambda\ ,$$
$$\ddot{u}_1^{(2)}\cos\varphi\sin\lambda + \ddot{u}_2^{(2)}\sin\varphi\sin\lambda - \ddot{u}_3^{(2)}\cos\lambda = -\dddot{\lambda}\rho.$$

Considering $\Delta_2 \approx c$ (12) becomes:

$$\begin{aligned}
\ddot{\rho} &= \rho\dot{\varphi}^2\cos^2\lambda + \rho\dot{\lambda}^2 + \frac{e_1e_2}{m_2c^2}\frac{c^2-\dot{\rho}^2}{\rho^2} - \frac{e_2^2}{m_2c^3}\dddot{\rho}\ ,\\
\ddot{\varphi} &= -\frac{2\dot{\rho}\dot{\varphi}}{\rho} + 2\dot{\varphi}\dot{\lambda}tg\lambda - \frac{e_1e_2}{m_2c^2}\frac{\dot{\rho}\dot{\varphi}}{\rho^2} - \frac{e_2^2}{m_2c^3}\dddot{\varphi}\ ,\\
\ddot{\lambda} &= -\frac{-2\dot{\rho}\dot{\lambda}}{\rho} - \frac{\dot{\varphi}^2\sin 2\lambda}{2} + \frac{e_1e_2}{m_2c^2}\frac{\dot{\rho}\dot{\lambda}}{\rho^2} - \frac{e_2^2}{m_2c^3}\dddot{\lambda}\ .
\end{aligned} \tag{13}$$

Remark 2. For $\lambda = 0$ we obtain the two-dimensional case without radiation terms

$$\ddot{\rho} = \rho\dot{\varphi}^2 + \frac{e_1e_2}{m_2c^2}\frac{c^2-\dot{\rho}^2}{\rho^2}\ ,\ \ddot{\varphi} = -\frac{2\dot{\rho}\dot{\varphi}}{\rho} - \frac{e_1e_2}{m_2c^2}\frac{\dot{\rho}\dot{\varphi}}{\rho^2}$$

derived in [6], [7].

2.5 Existence-Uniqueness

Here we again denote by $t = \theta$ and present the above system in an equivalent form setting $r(t) = \dot{\rho}(t), \phi(t) = \dot{\varphi}(t), \eta(t) = \dot{\lambda}(t)$:

$$\dot{r} = \rho\phi^2\cos^2\lambda + \rho\eta^2 - \frac{c^2}{\rho} \equiv G_r,\ \ \dot{\phi} = 2\phi\eta\, tg\lambda - \frac{2r\phi}{\rho} \equiv G_\phi\ ,\ \ \dot{\eta} = -\frac{2r\eta}{\rho} - \frac{\phi^2\sin 2\lambda}{2} \equiv G_\eta\ . \tag{14}$$

Assumption (C) in spherical coordinates has the form:

$$\dot{\rho}^2 + \rho^2\dot{\lambda}^2 + \rho^2\dot{\varphi}^2\cos^2\lambda \le \bar{c}^2 < c^2 \Rightarrow |r| \le \bar{c}, |\rho\eta| \le \bar{c}, |\rho\phi| \le \bar{c} \ll c\ .$$

By $C_T^{\infty}[-T,0]$ we denote the set of all infinitely differentiable functions, satisfying the condition $r(-T)=r(0)=0$. Introduce the following sets:

$$M_r=\left\{r\in C_T^{\infty}[-T,0]:\left|r^{(m)}(t)\right|\le\omega^m R_0 e^{\mu(t+T)};\ r(-T)=r(0)=0;\ \int_{-T}^{0}r(s)ds=0\right\},$$

$$d_{(m)}(r,\bar r)=\sup\left\{\frac{\left|r^{(m)}(t)-\bar r^{(m)}(t)\right|}{\omega^m}e^{-\mu(t+T)}:t\in[-T,0]\right\}\quad(m=0,1,2,...);$$

$$M_\phi=\left\{\phi\in C_T^{\infty}[-T,0]:\left|\phi^{(m)}(t)\right|\le\omega^m\Phi_0 e^{\mu(t+T)},\ \phi(-T)=\phi(0)=\phi_0,\ \int_{-T}^{0}\phi(s)ds=T\phi_0\right\},$$

$$d_{(m)}(\phi,\bar\phi)=\sup\left\{\frac{\left|\phi^{(m)}(t)-\bar\phi^{(m)}(t)\right|}{\omega^m}e^{-\mu(t+T)}:t\in[-T,0]\right\}\ (m=0.1,2,...);$$

$$M_\eta=\left\{\eta\in C_T^{\infty}[-T,0]:\left|\eta^{(m)}(t)\right|\le\omega^m\Lambda_0 e^{\mu(t+T)};\ \eta(-T)=\eta(0)=0;\ \int_{-T}^{0}\eta(s)ds=0;\left|\int_{-T}^{t}\eta(s)ds\right|\le\frac{\pi}{2}-\delta\right\},$$

$$d_{(m)}(\eta,\bar\eta)=\sup\left\{\frac{\left|\eta^{(m)}(t)-\bar\eta^{(m)}(t)\right|}{\omega^m}e^{-\mu(t+T)}:t\in[-T,0]\right\}\quad(m=0,1,2,...),$$

where $0<\delta<\frac{\pi}{2}$, $\mu>\omega>0$ are strictly positive constants, $r^{(m)}$ means the m-derivative of r and $r^{(0)}(t)=r(t)$.

The set $M_r\times M_\phi\times M_\eta$ turns out in a uniform space by introducing a countable family of pseudo-metrics (cf. [14]): $d_{(m)}((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta))=d_{(m)}(r,\bar r)+d_{(m)}(\phi,\bar\phi)+d_{(m)}(\eta,\bar\eta)$. Here the index set is $A=\{0,1,2,...\}$.

We define an operator $B=\left(B_r,B_\phi,B_\eta\right)$ acting on the function space $M_r\times M_\phi\times M_\eta$:

$$B_r(r,\phi,\eta)(t):=\int_{-T}^{t}G_r(r,\phi,\eta)(s)ds-\left(\frac{t+T}{T}-\frac{1}{2}\right)\int_{-T}^{0}G_r(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p}G_r(r,\phi,\eta)(s)dsdp\ ,t\in[-T,0]$$

$$B_\phi(r,\phi,\eta)(t):=\phi_0+\int_{-T}^{t}G_\phi(r,\phi,\eta)(s)ds-\left(\frac{t+T}{T}-\frac{1}{2}\right)\int_{-T}^{0}G_\phi(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p}G_\phi(r,\phi,\eta)(s)dsdp,t\in[-T,0]$$

$$B_\eta(r,\phi,\eta)(t):=\int_{-T}^{t}G_\eta(r,\phi,\eta)(s)ds-\left(\frac{t+T}{T}-\frac{1}{2}\right)\int_{-T}^{0}G_\eta(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p}G_\eta(r,\phi,\eta)(s)dsdp\ ,t\in[-T,0].$$

Lemma 1. For some $n\in N$ the following inequalities are valid for $t\in[-T,0]$:

$$\left|r(t)\right|\le\frac{\omega^n}{\mu^n}R_0e^{\mu(t+T)};\ \left|r^{(1)}(t)\right|\le\frac{\omega^n}{\mu^n}\omega R_0e^{\mu(t+T)};\ \left|r^{(2)}(t)\right|\le\frac{\omega^n}{\mu^n}\omega^2R_0e^{\mu(t+T)},$$

$$\left|\eta(t)\right|\le\frac{\omega^n}{\mu^n}\Lambda_0e^{\mu(t+T)},\ \left|\eta^{(1)}(t)\right|\le\frac{\omega^n}{\mu^n}\omega\Lambda_0e^{\mu(t+T)},\left|\eta^{(2)}(t)\right|\le\frac{\omega^n}{\mu^n}\omega^2\Lambda_0e^{\mu(t+T)},\ ....$$

$$\left|\phi(t)\right|\le\phi_0+\frac{\omega^n}{\mu^n}\Phi_0e^{\mu(t+T)},\ \left|\phi^{(1)}(t)\right|\le\frac{\omega^n}{\mu^n}\omega\Phi_0e^{\mu(t+T)},\left|\phi^{(2)}(t)\right|\le\frac{\omega^n}{\mu^n}\omega^2\Phi_0e^{\mu(t+T)},\ ...,$$

$$d_{(0)}(r,\bar r)\le\frac{\omega^n}{\mu^n}d_{(m)}\left(r,\bar r\right);\ \ d_{(s)}(r^{(s)},\bar r^{(s)})\le\omega^s\frac{\omega^n}{\mu^n}d_{(n+s)}\left(r,\bar r\right),(s=1,2,...),$$

$$d_{(0)}(\phi,\bar\phi)\le\frac{\omega^n}{\mu^n}d_{(n)}\left(\phi,\bar\phi\right);\ \ d_{(s)}(\phi^{(s)},\bar\phi^{(s)})\le\omega^s\frac{\omega^n}{\mu^n}d_{(n+s)}\left(\phi^{(n+s)},\bar\phi^{(n+s)}\right),(s=1,2,...),$$

$$d_{(0)}(\eta,\bar{\eta}) \le \frac{\omega^n}{\mu^n} d_{(n)}(\eta,\bar{\eta})\,;\;\; d_{(s)}(\eta^{(s)},\bar{\eta}^{(s)}) \le \omega^s \frac{\omega^n}{\mu^n} d_{(n+s)}\left(\eta^{(n+s)},\bar{\eta}^{(n+s)}\right), (s=1,2,...)\,.$$

Proof: For the first Bohr orbit $\bar{c} \approx 10^6$ and then for sufficiently large $\mu > 0$ and $\mu T = const. > 0$ we obtain

$$|r(t)| = \left|\int_{-T}^{t} r^{(1)}(t_1)dt_1\right| = \left|\int_{-T}^{t}\left(r_1 + \int_{-T}^{t_1} r^{(2)}(t_2)dt_2\right)dt_1\right| = ... = \left|\int_{-T}^{t}\left(r_1 + \int_{-T}^{t_1}...\int_{-T}^{t_{n-1}} r^{(n)}(t_n)dt_n...dt_1\right)\right| \le$$

$$\le \frac{r_1}{\mu}e^{\mu(t+T)} + ... + \frac{r_{n-1}}{\mu^{n-1}}e^{\mu(t+T)} + \frac{\omega^n}{\mu^n}R_0 e^{\mu(t+T)} \le \frac{\bar{c}}{\mu}e^{\mu T} + ... + \frac{\bar{c}}{\mu^{n-1}}e^{\mu T} + \frac{\omega^n}{\mu^n}R_0 e^{\mu(t+T)} \approx \frac{\omega^n}{\mu^n}R_0 e^{\mu(t+T)}$$

$$|\phi(t)| = \left|\phi_0 + \int_{-T}^{t}\phi^{(1)}(t_1)dt_1\right| = \left|\phi_0 + \int_{-T}^{t}\left(\phi_1 + \int_{-T}^{t_1} r^{(2)}(t_2)dt_2\right)dt_1\right| = ... = \left|\phi_0 + \int_{-T}^{t}\left(\phi_1 + \int_{-T}^{t_1}...\int_{-T}^{t_{n-1}} r^{(n)}(t_n)dt_n...dt_1\right)\right| \le$$

$$\le \phi_0 + \frac{\phi_1}{\mu}e^{\mu(t+T)} + ... + \frac{\phi_{n-1}}{\mu^{n-1}}e^{\mu(t+T)} + \frac{\omega^n}{\mu^n}\Phi_0 e^{\mu(t+T)} \le \phi_0 + \frac{\phi_1}{\mu}e^{\mu T} + ... + \frac{\phi_{n-1}}{\mu^{n-1}}e^{\mu T} + \frac{\omega^n}{\mu^n}\Phi_0 e^{\mu(t+T)} \approx$$

$$\approx \phi_0 + \frac{\omega^n}{\mu^n}\Phi_0 e^{\mu(t+T)}.$$

In a similar way we obtain $|\eta(t)| \le \frac{\omega^n}{\mu^n}\Lambda_0 e^{\mu(t+T)}$, $|\eta^{(1)}(t)| \le \frac{\omega^n}{\mu^n}\omega\Lambda_0 e^{\mu(t+T)}$ … and so on for the higher derivatives. The inequalities for pseudo-metrics are straightforward, which completes the proof of Lemma 1.

Corollary 1.

$$d_{(0)}\left((r,\phi,\eta),(\bar{r},\bar{\phi},\bar{\eta})\right) \le \frac{\omega^n}{\mu^n} d_{(n)}\left((r,\phi,\eta),(\bar{r},\bar{\phi},\bar{\eta})\right), \tag{15}$$

$$d_{(1)}\left((r,\phi,\eta),(\bar{r},\bar{\phi},\bar{\eta})\right) \le \frac{\omega^n}{\mu^n}\omega d_{(n+1)}\left((r,\phi,\eta),(\bar{r},\bar{\phi},\bar{\eta})\right), ...$$

Lemma 2. The distance between particles $\rho(t)$ satisfies the inequalities $0 < \rho_{\min} \le \rho(t) \le \rho_{\max}$.

Proof: Let $\rho_0 = \rho(-T) > 0$ be the distance between particles at $t = -T$. Then for sufficiently large $\mu > 0$ and $\mu T = const. > 0$ we have for $t \in [-T, 0]$:

$$\rho(t) \ge \rho_0 - \int_{-T}^{t}|r(s)|ds \ge \rho_0 - R_0\int_{-T}^{t} e^{\mu(s+T)}ds = \rho_0 - R_0\frac{e^{\mu(t+T)} - 1}{\mu} \ge \rho_0 - R_0\frac{e^{\mu T} - 1}{\mu} \equiv \rho_{\min} > 0\,.$$

In a similar way we infer $\rho(t) \le \rho_0 + R_0\int_{-T}^{t}|r(s)|ds \le \rho_0 + R_0\int_{-T}^{t} e^{\mu(s+T)}ds \le \rho_0 + R_0\frac{e^{\mu T} - 1}{\mu} \equiv \rho_{\max}$.

Finally, we must check that $\rho_{\min} < \rho_{\max} \Leftrightarrow \rho_0 - R_0\frac{e^{\mu T} - 1}{\mu} < \rho_0 + R_0\frac{e^{\mu T} - 1}{\mu} \Leftrightarrow e^{\mu T} - 1 > 0\ (-T < 0)$.

Lemma 2 is thus proved.

Lemma 3 (Main Lemma I) The system (14) has a solution iff the operator

$$B = (B_r(r,\phi,\eta), B_\phi(r,\phi,\eta), B_\eta(r,\phi,\eta)) : M_r \times M_\phi \times M_\eta \to M_r \times M_\phi \times M_\eta$$

has a fixed point.

Proof: After integration of (14) in view of $r(-T)=0$ and $\eta(-T)=0$ we obtain

$$r(t)=\int_{-T}^{t} G_r(r,\phi,\eta)(s),\ \phi(t)=\phi_0+\int_{-T}^{t} G_\phi(r,\phi,\eta)(s)ds,\ \eta(t)=\int_{-T}^{t} G_\eta(r,\phi,\eta)(s)ds\ .$$

Let us put $t=0$. Then

$$0=r(0)=\int_{-T}^{0} G_r(r,\phi,\eta)ds \Rightarrow \int_{-T}^{0} G_r(r,\phi,\eta)ds\ =0\ ,\ \phi(0)=\phi_0+\int_{-T}^{0} G_\phi(r,\phi,\eta)ds \Rightarrow \int_{-T}^{0} G_\phi(r,\phi,\eta)ds\ =0\ ,$$
$$0=\eta(0)=\int_{-T}^{0} G_\eta(r,\phi,\eta)ds \Rightarrow \int_{-T}^{0} G_\eta(r,\phi,\eta)ds\ =0\ .$$

Therefore,

$$r(t)=\int_{-T}^{t} G_r(r,\phi,\eta)ds \Leftrightarrow r(t)=\int_{-T}^{t} G_r(r,\phi,\eta)ds-\left(\frac{t+T}{T}-\frac{1}{2}\right)\int_{-T}^{0} G_r(r,\phi,\eta)(s)ds\ ,$$
$$\phi(t)=\phi_0+\int_{-T}^{t} G_\phi(r,\phi,\eta)ds \Leftrightarrow \phi(t)=\phi_0+\int_{-T}^{t} G_\phi(r,\phi,\eta)ds\ -\left(\frac{t+T}{T}-\frac{1}{2}\right)\int_{-T}^{0} G_\phi(r,\phi,\eta)(s)ds\ , \tag{16}$$
$$\eta(t)=\int_{-T}^{t} G_\eta(r,\phi,\eta)ds \Leftrightarrow \eta(t)=\int_{-T}^{t} G_\eta(r,\phi,\eta)ds-\left(\frac{t+T}{T}-\frac{1}{2}\right)\int_{-T}^{0} G_\eta(r,\phi,\eta)(s)ds\ .$$

Changing the order of integration and considering that $r(0)=r(-T)=0$ and $\int_{-T}^{0} r(s)ds=0$ we have

$$\int_{-T}^{0}\int_{-T}^{p} G_r(r,\phi,\eta)(s)dsdp=\int_{-T}^{0}(0-s)G_r(r,\phi,\eta)(s)ds=-\int_{-T}^{0} s\frac{dr(s)}{ds}ds=$$
$$=-\int_{-T}^{0} s\,dr(s)=-0.r(0)-T.\ r(-T)+\int_{-T}^{0} r(s)ds=0.$$

Consequently (16) can be written in the form:

$$r(t)=\int_{-T}^{t} G_r(r,\phi,\eta)ds-\left(\frac{t+T}{T}-\frac{1}{2}\right)\int_{-T}^{0} G_r(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p} G_r(r,\phi,\eta)(s)dsdp\ ,$$

that is, $r=B_r(r,\phi,\eta)$.

In a similar way one obtains:

$$\phi(t)=\phi_0+\int_{-T}^{t} G_\phi(r,\phi,\eta)ds-\left(\frac{t+T}{T}-\frac{1}{2}\right)\int_{-T}^{0} G_\phi(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p} G_\phi(r,\phi,\eta)(s)dsdp$$
$$\eta(t)=\int_{-T}^{t} G_\eta(r,\phi,\eta)ds-\left(\frac{t+T}{T}-\frac{1}{2}\right)\int_{-T}^{0} G_\eta(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p} G_\eta(r,\phi,\eta)(s)dsdp\ .$$

We have shown that the solution (r,ϕ,η) of (14) is a fixed point of the operator equation:

$$(r,\phi,\eta)=(B_r(r,\phi,\eta),B_\phi(r,\phi,\eta),B_\eta(r,\phi,\eta)) .$$

Conversely, let $(r,\phi,\eta)\in M_r\times M_\phi\times M_\eta$ be a fixed point of B. Then

$$r(t)=\int_{-T}^{t}G_r(r,\phi,\eta)ds-\left(\frac{t+T}{T}-\frac{1}{2}\right)\int_{-T}^{0}G_r(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p}G_r(r,\phi,\eta)(s)dsdp$$

for $t=-T$ gives

$$r(-T)=\int_{-T}^{-T}G_r(r,\phi,\eta)ds-\left(\frac{-T+T}{T}-\frac{1}{2}\right)\int_{-T}^{0}G_r(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{s}G_r(r,\phi,\eta)(\theta)d\theta ds\Rightarrow$$
$$\Rightarrow 0=\frac{1}{2}\int_{-T}^{0}G_r(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{s}G_r(r,\phi,\eta)(\theta)d\theta ds .$$

We show that $\int_{-T}^{0}G_r(r,\phi,\eta)(s)ds=0$ that implies $\int_{-T}^{0}\int_{-T}^{s}G_r(r,\phi,\eta)(\theta)d\theta ds=0$.

Indeed, let us suppose that $\left|\int_{-T}^{0}G_r(r,\phi,\eta)(s)ds\right|=\delta>0$. Then

$$\left|\int_{-T}^{0}G_r(r,\phi,\eta)(s)ds\right|\le\left|\int_{-T}^{0}\left(\rho\phi^2\cos^2\lambda+\rho\eta^2-\frac{c^2}{\rho}\right)dt\right|\le$$
$$\le\left(\bar{c}\Phi_0+\bar{c}\Lambda_0+\frac{c^2}{\rho_{\min}}\right)\int_{-T}^{0}e^{\mu(s+T)}ds=\left(\bar{c}\Phi_0+\bar{c}\Lambda_0+\frac{c^2}{\rho_{\min}}\right)\frac{e^{\mu T}-1}{\mu}.$$

The last term becomes smaller than δ for sufficiently large $\mu>0$ and $\mu T=const$. Consequently $r(t)=\int_{-T}^{t}G_r(r,\phi,\eta)ds\Rightarrow \dot{r}(t)=G_r(r,\phi,\eta)(t)$.

From the second component of B for $t=-T$ we have

$$\phi(-T)=\phi_0+\int_{-T}^{-T}G_\phi(r,\phi,\eta)ds-\left(\frac{-T+T}{T}-\frac{1}{2}\right)\int_{-T}^{0}G_\phi(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{s}G_\phi(r,\phi,\eta)(\theta)d\theta ds$$

and in view of $\phi(-T)=\phi_0$ we obtain $0=\frac{1}{2}\int_{-T}^{0}G_\phi(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{s}G_\phi(r,\phi,\eta)(\theta)d\theta ds$.

In a similar way we show that

$$\int_{-T}^{0}G_\phi(r,\phi,\eta)(s)ds=0\Rightarrow\int_{-T}^{0}\int_{-T}^{s}G_\phi(r,\phi,\eta)(\theta)d\theta ds=0 \text{ and } \int_{-T}^{0}G_\eta(r,\phi,\eta)(s)ds=0\Rightarrow\int_{-T}^{0}\int_{-T}^{s}G_\eta(r,\phi,\eta)(\theta)d\theta ds=0 .$$

Therefore $\dot{\phi}(t)=F_\phi(r,\phi,\eta)$ and $\dot{\eta}(t)=F_\eta(r,\phi,\eta)$, that is, (14) has a periodic solution.

Lemma 3 is thus proved.

Lemma 4. The operator $B=(B_r,B_\phi,B_\eta)$ maps $M_r\times M_\phi\times M_\eta$ into itself provided,

$$\frac{e^{\mu T}}{\mu}\left(\bar{c}\Phi_0+\bar{c}\Lambda_0+\frac{c^2}{\hat{\rho}}\right)\le R_0\,,\ \left[\frac{\phi_0}{\Phi_0}+\left(\Lambda_0 tg\left(\frac{\pi}{2}-\delta\right)+\frac{\bar{c}}{\hat{\rho}}\right)\frac{2e^{\mu T}}{\mu}\right]\Phi_0\le\Phi_0\,,\ \left(\frac{2\bar{c}\Lambda_0}{\hat{\rho}}+\frac{{\Phi_0}^2e^{\mu T}}{2}\right)\frac{e^{\mu T}}{\mu}\le\Lambda_0\,,$$

$$\left(\frac{2\bar{c}\Lambda_0}{\hat{\rho}}+\frac{{\Phi_0}^2e^{\mu T}}{2}\right)\frac{e^{2\mu T}}{\mu^2}\le\frac{\pi}{2}-\delta\,.$$

Proof: For the first component $B_r(r,\phi,\eta)$ one obtains

$$0=r(-T)=B_r(r,\phi,\eta)(-T)=\frac{1}{2}\int_{-T}^{0}G_r(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{s}G_r(r,\phi,\eta)(\theta)d\theta ds\ . \tag{17}$$

Then

$$B_r(r,\phi,\eta)(0)=\int_{-T}^{0}G_r(r,\phi,\eta)(s)ds-\left(\frac{T}{T}-\frac{1}{2}\right)\int_{-T}^{0}G_r(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p}G_r(r,\phi,\eta)(s)dsdp\ =$$

$$=\frac{1}{2}\int_{-T}^{0}G_r(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p}G_r(r,\phi,\eta)(s)dsdp=0$$

and

$$B_r(r,\phi,\eta)(-T)=\int_{-T}^{-T}G_r(r,\phi,\eta)(s)ds-\left(\frac{-T+T}{T}-\frac{1}{2}\right)\int_{-T}^{0}G_r(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p}G_r(r,\phi,\eta)(s)dsdp\ =$$

$$=\frac{1}{2}\int_{-T}^{0}G_r(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p}G_r(r,\phi,\eta)(s)dsdp=0\ .$$

Since $\int_{-T}^{0}\left(\frac{t+T}{T}-\frac{1}{2}\right)dt=0$ we obtain

$$\int_{-T}^{0}B_r(r,\phi,\eta)(t)\,dt=\int_{-T}^{0}\int_{-T}^{t}G_r(r,\phi,\eta)(s)ds\,dt-\int_{-T}^{0}\left(\frac{t+T}{T}-\frac{1}{2}\right)dt\int_{-T}^{0}G_r(r,\phi,\eta)(s)ds-\int_{-T}^{0}\int_{-T}^{\theta}G_r(r,\phi,\eta)(s)dsd\theta\ =0.$$

Besides

$$\left|B_r(r,\phi,\eta)(t)\right|\le\int_{-T}^{t}\left|G_r(r,\phi,\eta)(s)\right|ds+\int_{-T}^{0}\left|G_r(r,\phi,\eta)(s)\right|ds\le$$

$$\le\left|\int_{-T}^{t}\left(\rho\phi^2\cos^2\lambda+\rho\eta^2-\frac{c^2}{\rho}\right)ds\right|+\left|\int_{-T}^{0}\left(\rho\phi^2\cos^2\lambda+\rho\eta^2-\frac{c^2}{\rho}\right)ds\right|\le$$

$$\le\left(\bar{c}\Phi_0+\bar{c}\Lambda_0+\frac{c^2}{\hat{\rho}}\right)\left(\int_{-T}^{t}e^{\mu(s+T)}ds+\int_{-T}^{0}e^{\mu(s+T)}ds\right)\le$$

$$\le e^{\mu(t+T)}\left(\frac{1}{\mu}+\frac{e^{\mu T}-1}{\mu}\right)\left(\bar{c}\Phi_0+\bar{c}\Lambda_0+\frac{c^2}{\hat{\rho}}\right)=e^{\mu(t+T)}\,\frac{e^{\mu T}}{\mu}\left(\bar{c}\Phi_0+\bar{c}\Lambda_0+\frac{c^2}{\hat{\rho}}\right)\le R_0e^{\mu(t+T)}.$$

For the second component of B we have:

$$\phi(0)=B_\phi(r,\phi,\eta)(0)\Leftrightarrow\phi_0=\phi_0+\int_{-T}^{0}G_\phi(r,\phi,\eta)(s)ds-\left(\frac{T}{T}-\frac{1}{2}\right)\int_{-T}^{0}G_\phi(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p}G_\phi(r,\phi,\eta)(s)dsdp\Leftrightarrow$$

$$\Leftrightarrow 0=\frac{1}{2}\int_{-T}^{0}G_\phi(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p}G_\phi(r,\phi,\eta)(s)dsdp$$

and

$$\phi(-T)=B_\phi(r,\phi,\eta)(-T)\Leftrightarrow$$

$$\phi_0=\phi_0+\int_{-T}^{-T}G_\phi(r,\phi,\eta)(s)ds-\left(\frac{-T+T}{T}-\frac{1}{2}\right)\int_{-T}^{0}G_\phi(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p}G_\phi(r,\phi,\eta)(s)dsdp\Leftrightarrow$$

$$0=\frac{1}{2}\int_{-T}^{0}G_\phi(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p}G_\phi(r,\phi,\eta)(s)dsdp.$$

Therefore

$$B_\phi(r,\phi,\eta)(-T)=\phi_0+\frac{1}{2}\int_{-T}^{0}G_\phi(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{-T}^{0}\int_{-T}^{p}G_\phi(r,\phi,\eta)(s)dsdp=\phi_0.$$

Besides

$$\int_{-T}^{0}B_\phi(r,\phi,\eta)(t)\,dt=\phi_0+\int_{-T}^{0}\int_{-T}^{t}G_\phi(r,\phi,\eta)(s)ds\,dt-\int_{-T}^{0}\left(\frac{t+T}{T}-\frac{1}{2}\right)dt\int_{-T}^{0}G_\phi(r,\phi,\eta)(s)ds-\int_{-T}^{0}\int_{-T}^{\theta}G_\phi(r,\phi,\eta)(s)dsd\theta\ =\phi_0T$$

and

$$\left|B_\phi(r,\phi,\eta)(t)\right|\le\phi_0+\int_{-T}^{t}\left|G_\phi(r,\phi,\eta)(s)\right|ds+\int_{-T}^{0}\left|G_\phi(r,\phi,\eta)(s)\right|ds\le$$

$$\le\phi_0+\int_{-T}^{t}\left|2\phi\eta\tan\lambda-\frac{2r\phi}{\rho}\right|ds+\int_{-T}^{0}\left|2\phi\eta\tan\lambda-\frac{2r\phi}{\rho}\right|ds\le$$

$$\le\phi_0+\left(2\Phi_0\Lambda_0e^{\mu T}\tan\left(\frac{\pi}{2}-\delta\right)+\frac{2\overline{c}\Phi_0}{\widehat{\rho}}\right)\left(\int_{-T}^{t}e^{\mu(s+T)}ds+\int_{-T}^{0}e^{\mu(s+T)}ds\right)\le$$

$$\le\left[\frac{\phi_0}{\Phi_0}+\left(\Lambda_0\tan\left(\frac{\pi}{2}-\delta\right)+\frac{\overline{c}}{\widehat{\rho}}\right)\frac{2e^{\mu T}}{\mu}\right]\Phi_0e^{\mu(t+T)}\le\Phi_0e^{\mu(t+T)}.$$

For the third component of B we get

$$\left|B_\eta(r,\phi,\eta)(t)\right|\le\int_{-T}^{t}\left|G_\eta(r,\phi,\eta)(s)\right|ds+\int_{-T}^{0}\left|G_\eta(r,\phi,\eta)(s)\right|ds\le\int_{-T}^{t}\left|-\frac{2r\eta}{\rho}-\frac{\phi^2\sin2\lambda}{2}\right|ds+\int_{-T}^{0}\left|-\frac{2r\eta}{\rho}-\frac{\phi^2\sin2\lambda}{2}\right|ds\le$$

$$\le\left(\frac{2\overline{c}\Lambda_0}{\widehat{\rho}}+\frac{\Phi_0{}^2e^{\mu T}}{2}\right)\left(\int_{-T}^{t}e^{\mu(s+T)}ds+\int_{-T}^{0}e^{\mu(s+T)}ds\right)\le e^{\mu(t+T)}\left(\frac{2\overline{c}\Lambda_0}{\widehat{\rho}}+\frac{\Phi_0{}^2e^{\mu T}}{2}\right)\frac{e^{\mu T}}{\mu}\le\Lambda_0e^{\mu(t+T)};$$

$$\int_{-T}^{t}\left|B_{\eta}(r,\phi,\eta)ds\right|dt \le \int_{-T}^{t}\int_{-T}^{s}\left|G_{\eta}(r,\phi,\eta)ds\right|dt + \int_{-T}^{t}\int_{-T}^{0}\left|G_{\eta}(r,\phi,\eta)ds\right|dt \le$$
$$\le \left(\frac{2\overline{c}\Lambda_0}{\widehat{\rho}} + \frac{\Phi_0{}^2 e^{\mu T}}{2}\right)\left(\int_{-T}^{t}\int_{-T}^{s} e^{\mu(p+T)}dpds + \int_{-T}^{t}\int_{-T}^{0} e^{\mu(s+T)}dsdt\right) \le \left(\frac{2\overline{c}\Lambda_0}{\widehat{\rho}} + \frac{\Phi_0{}^2 e^{\mu T}}{2}\right)\left(\int_{-T}^{t}\frac{e^{\mu(s+T)}-1}{\mu}ds + \int_{-T}^{t}\frac{e^{\mu T}-1}{\mu}ds\right) \le$$
$$\le \left(\frac{2\overline{c}\Lambda_0}{\widehat{\rho}} + \frac{\Phi_0{}^2 e^{\mu T}}{2}\right)\left(\frac{e^{\mu(t+T)}-1}{\mu^2} + \frac{(e^{\mu T}-1)\left(e^{\mu(t+T)}-1\right)}{\mu^2}\right) \le \left(\frac{2\overline{c}\Lambda_0}{\widehat{\rho}} + \frac{\Phi_0{}^2 e^{\mu T}}{2}\right)\frac{e^{2\mu T}}{\mu^2} \le \frac{\pi}{2} - \delta.$$

Lemma 4 is thus proved.

Introduce on the Cartesian product $M_r \times M_{\phi} \times M_{\eta}$ a family of pseudo-metrics:

$$d_{(m)}((r,\phi,\eta),(\overline{r},\overline{\phi},\overline{\eta})) = d_{(m)}(r,\overline{r}) + d_{(m)}(\phi,\overline{\phi}) + d_{(m)}(\eta,\overline{\eta}), (m = 0,1,2,...).$$

Theorem 1. The system (14) possesses a unique solution belonging to $M_r \times M_{\phi} \times M_{\eta}$.

Proof: We show that the operator $B = \left(B_r(.), B_{\phi}(.), B_{\eta}(.)\right): M_r \times M_{\phi} \times M_{\eta} \to M_r \times M_{\phi} \times M_{\eta}$ possesses a unique fixed point in $M_r \times M_{\phi} \times M_{\eta}$. Indeed, in view of Lemma 3 the operator B maps $M_r \times M_{\phi} \times M_{\eta}$ into itself. It remains to show that B is a $(\Phi, j)-$contractive operator. Recall $\Delta_{12}^2 \approx c^2$ and we obtain:

$$\left|\frac{\partial}{\partial\rho}\left(\rho\phi^2\cos^2\lambda + \rho\eta^2 - \frac{c^2}{\rho}\right)\right| = \left|\phi^2\cos^2\lambda + \eta^2 + \frac{c^2}{\rho^2}\right| \le \left(\Phi_0{}^2 + \Lambda_0{}^2 e^{2\mu(t+T)} + \frac{c^2}{\widehat{\rho}^2}\right)e^{2\mu(t+T)};$$
$$\left|\frac{\partial}{\partial r}\left(\rho\phi^2\cos^2\lambda + \rho\eta^2 - \frac{c^2}{\rho}\right)\right| = 0;\quad \left|\frac{\partial}{\partial\phi}\left(\rho\phi^2\cos^2\lambda + \rho\eta^2 - \frac{c^2}{\rho}\right)\right| = \left|2\rho\phi\cos^2\lambda\right| \le 2\overline{c};$$
$$\left|\frac{\partial}{\partial\lambda}\left(\rho\phi^2\cos^2\lambda + \rho\eta^2 - \frac{c^2}{\rho}\right)\right| \le 2\overline{c}\Phi_0 e^{\mu(t+T)};\quad \left|\frac{\partial}{\partial\eta}\left(\rho\phi^2\cos^2\lambda + \rho\eta^2 - \frac{c^2}{\rho}\right)\right| = \left|2\rho\eta\right| \le 2\overline{c};$$
$$\left|\frac{\partial}{\partial\rho}\left(2\phi\eta\tan\lambda - \frac{2r\phi}{\rho}\right)\right| \le \frac{2\overline{c}\Phi_0}{\widehat{\rho}^2}e^{\mu(t+T)};\quad \left|\frac{\partial}{\partial r}\left(2\phi\eta\tan\lambda - \frac{2r\phi}{\rho}\right)\right| = \left|\frac{2\phi}{\rho}\right| \le \frac{2\Phi_0}{\widehat{\rho}}e^{\mu(t+T)};$$
$$\left|\frac{\partial}{\partial\phi}\left(2\phi\eta\tan\lambda - \frac{2r\phi}{\rho}\right)\right| = \left|2\eta\tan\lambda - \frac{2r}{\rho}\right|\left|\dot{\varphi}\right| \le \left(2\Lambda_0\tan\left(\pi/2-\delta\right) + \frac{2\overline{c}}{\widehat{\rho}}\right)e^{\mu(t+T)};$$
$$\left|\frac{\partial}{\partial\lambda}\left(2\phi\eta\tan\lambda - \frac{2r\phi}{\rho}\right)\right| \le \frac{2\Phi_0\Lambda_0}{\cos^2\left(\pi/2-\delta\right)}e^{2\mu(t+T)};\left|\frac{\partial}{\partial\eta}\left(2\phi\eta\tan\lambda - \frac{2r\phi}{\rho}\right)\right| \le 2\Phi_0\tan\left(\pi/2-\delta\right)e^{\mu(t+T)};$$
$$\left|\frac{\partial}{\partial\rho}\left(-\frac{2r\eta}{\rho} - \frac{\phi^2\sin 2\lambda}{2}\right)\right| \le \frac{2\overline{c}\Lambda_0 e^{\mu(t+T)}}{\widehat{\rho}^2};\qquad \left|\frac{\partial}{\partial r}\left(-\frac{2r\eta}{\rho} - \frac{\phi^2\sin 2\lambda}{2}\right)\right| \le \frac{2\Lambda_0 e^{\mu(t+T)}}{\widehat{\rho}};$$
$$\left|\frac{\partial}{\partial\phi}\left(-\frac{2r\eta}{\rho} - \frac{\phi^2\sin 2\lambda}{2}\right)\right| \le \Phi_0 e^{\mu(t+T)};\ \left|\frac{\partial}{\partial\lambda}\left(-\frac{2r\eta}{\rho} - \frac{\phi^2\sin 2\lambda}{2}\right)\right| \le \Phi_0{}^2 e^{2\mu(t+T)};\ \left|\frac{\partial}{\partial\eta}\left(-\frac{2r\eta}{\rho} - \frac{\phi^2\sin 2\lambda}{2}\right)\right| \le \frac{2\overline{c}}{\widehat{\rho}}.$$

The last estimates imply

$$\left|B_r(r,\phi,\eta)(t) - B_r(\overline{r},\overline{\phi},\overline{\eta})(t)\right| \le \int_{-T}^{t}\left|G_r(r,\phi,\eta)(s) - G_r(\overline{r},\overline{\phi},\overline{\eta})(s)\right|ds + \int_{-T}^{0}\left|G_r(r,\phi,\eta)(s) - G_r(\overline{r},\overline{\phi},\overline{\eta})(s)\right|ds \le$$
$$\le \left(\Phi_0{}^2 + \Lambda_0{}^2 + \frac{c^2}{\widehat{\rho}^2}\right)\left(\int_{-T}^{t} e^{2\mu(s+T)}\int_{-T}^{s}\left|r(p) - \overline{r}(p)\right|dpds + \int_{-T}^{0} e^{2\mu(s+T)}\int_{-T}^{s}\left|r(p) - \overline{r}(p)\right|dpds\right) +$$

$$+2\overline{c}\left(\int_{-T}^{t} e^{3\mu(s+T)}ds+\int_{-T}^{0} e^{3\mu(s+T)}ds\right)d(\phi,\overline{\phi})++2\overline{c}\Phi_0\left(\int_{-T}^{t} e^{\mu(t+T)}\left|\int_{-T}^{s}\eta(p)-\overline{\eta}(p)dp\right|ds+\int_{-T}^{0} e^{\mu(t+T)}\left|\int_{-T}^{s}\eta(p)-\overline{\eta}(p)dp\right|ds\right)+$$

$$+2\Phi_0\tan\left(\frac{\pi}{2}-\delta\right)\left(\int_{-T}^{t} e^{\mu(s+T)}\left|\eta(s)-\overline{\eta}(s)\right|ds+\int_{-T}^{0} e^{\mu(s+T)}\left|\eta(s)-\overline{\eta}(s)\right|ds\right)\le$$

$$\le\left({\Phi_0}^2+{\Lambda_0}^2+\frac{c^2}{\widehat{\rho}^2}\right)\left(\frac{e^{3\mu(t+T)}-1}{3\mu^2}+\frac{e^{3\mu T}-1}{3\mu^2}\right)d(r,\overline{r})+2\overline{c}\left(\frac{e^{3\mu(t+T)}-1}{3\mu}+\frac{e^{3\mu T}-1}{3\mu}\right)d(\phi,\overline{\phi})+$$

$$+2\overline{c}\Phi_0\left(\frac{e^{3\mu(t+T)}-1}{3\mu^2}+\frac{e^{3\mu T}-1}{3\mu^2}\right)d(\eta,\overline{\eta})+2\Phi_0\tan\left(\frac{\pi}{2}-\delta\right)\left(\frac{e^{3\mu(t+T)}-1}{3\mu}+\frac{e^{3\mu T}-1}{3\mu}\right)d(\eta,\overline{\eta})\le$$

$$\le\left[{\Phi_0}^2+{\Lambda_0}^2+\frac{c^2}{\widehat{\rho}^2}+2\overline{c}\Phi_0+2\Phi_0\tan\left(\frac{\pi}{2}-\delta\right)\right]\frac{2(e^{3\mu T}-1)}{3\mu}d_{(m)}\left((r,\phi,\eta),(\overline{r},\overline{\phi},\overline{\eta})\right)\Rightarrow$$

$$d_{(m)}\left(B_r(r,\phi,\eta),B_r(\overline{r},\overline{\phi},\overline{\eta})\right)\le\left[{\Phi_0}^2+{\Lambda_0}^2+\frac{c^2}{\widehat{\rho}^2}+2\overline{c}\Phi_0+2\Phi_0\tan\left(\frac{\pi}{2}-\delta\right)\right]\frac{2(e^{3\mu T}-1)}{3\mu}d_{(m)}\left((r,\phi,\eta),(\overline{r},\overline{\phi},\overline{\eta})\right).$$

For the second component we obtain

$$\left|B_\phi(r,\phi,\eta)(t)-B_\phi(\overline{r},\overline{\phi},\overline{\eta})(t)\right|\le\int_{-T}^{t}\left|G_\phi(r,\phi,\eta)(s)-G_\phi(\overline{r},\overline{\phi},\overline{\eta})(s)\right|ds+\int_{-T}^{0}\left|G_\phi(r,\phi,\eta)(s)-G_\phi(\overline{r},\overline{\phi},\overline{\eta})(s)\right|ds\le$$

$$\le\frac{2\overline{c}\Phi_0}{\widehat{\rho}^2}\left(\frac{e^{2\mu(t+T)}-1}{2\mu^2}+\frac{e^{2\mu T}-1}{2\mu^2}\right)d(r,\overline{r})+\frac{2\Phi_0}{\widehat{\rho}}\left(\frac{e^{2\mu(t+T)}-1}{2\mu}+\frac{e^{2\mu T}-1}{2\mu}\right)d(r,\overline{r})+$$

$$+\left(2\Lambda_0\,tg\left(\pi/2-\delta\right)+\frac{2\overline{c}}{\widehat{\rho}}\right)\left(\frac{e^{2\mu(t+T)}-1}{2\mu}+\frac{e^{2\mu T}-1}{2\mu}\right)d(\phi,\overline{\phi})+$$

$$+\frac{2\Phi_0\Lambda_0}{\cos^2\left(\pi/2-\delta\right)}\left(\frac{e^{3\mu(t+T)}-1}{3\mu^2}+\frac{e^{3\mu T}-1}{3\mu^2}\right)d(\eta,\overline{\eta})+2\Phi_0\tan\left(\pi/2-\delta\right)\left(\frac{e^{2\mu(t+T)}-1}{2\mu}+\frac{e^{2\mu T}-1}{2\mu}\right)d(\eta,\overline{\eta})\le$$

$$\le\Phi_0\left[\left(\frac{\overline{c}}{\widehat{\rho}^2}+\frac{\Lambda_0}{\cos^2\left((\pi/2)-\delta\right)}\right)\frac{e^{3\mu T}-1}{\mu^2}+\left(\frac{2}{\widehat{\rho}}+\Lambda_0\tan\left(\frac{\pi}{2}-\delta\right)+\frac{\overline{c}}{\widehat{\rho}}+\tan\left(\frac{\pi}{2}-\delta\right)\right)\frac{e^{2\mu T}-1}{\mu}\right]e^{\mu(t+T)}d_{(m)}\left((r,\phi,\eta),(\overline{r},\overline{\phi},\overline{\eta})\right).$$

Therefore

$$d_{(m)}(B_\phi(r,\phi,\eta),B_\phi(\overline{r},\overline{\phi},\overline{\eta}))\le$$

$$\le\left[\left(\frac{\overline{c}\Phi_0}{\widehat{\rho}^2}+\frac{\Phi_0\Lambda_0}{\cos^2\left((\pi/2)-\delta\right)}\right)\frac{e^{3\mu T}-1}{\mu^2}+\left(\frac{2\Phi_0+\overline{c}}{\widehat{\rho}}+(\Lambda_0+\Phi_0)\tan\left(\frac{\pi}{2}-\delta\right)\right)\frac{e^{2\mu T}-1}{\mu}\right]d_{(m)}\left((r,\phi,\eta),(\overline{r},\overline{\phi},\overline{\eta})\right).$$

For the third component we obtain

$$\left|B_\eta(r,\phi,\eta)(t)-B_\eta(\overline{r},\overline{\phi},\overline{\eta})(t)\right|\le\int_{-T}^{t}\left|G_\eta(r,\phi,\eta)(s)-G_\eta(\overline{r},\overline{\phi},\overline{\eta})(s)\right|ds+\int_{-T}^{0}\left|G_\eta(r,\phi,\eta)(s)-G_\eta(\overline{r},\overline{\phi},\overline{\eta})(s)\right|ds\le$$

$$\le\frac{2\overline{c}\Lambda_0}{\widehat{\rho}^2}\left(\frac{e^{2\mu(t+T)}-1}{2\mu^2}+\frac{e^{2\mu T}-1}{2\mu^2}\right)d(r,\overline{r})+\frac{2\Lambda_0}{\widehat{\rho}}\left(\frac{e^{2\mu(t+T)}-1}{2\mu}+\frac{e^{2\mu T}-1}{2\mu}\right)d(r,\overline{r})+$$

$$+\Phi_0\left(\frac{e^{2\mu(t+T)}-1}{2\mu}+\frac{e^{2\mu T}-1}{2\mu}\right)d(\phi,\overline{\phi})+{\Phi_0}^2\left(\frac{e^{3\mu(t+T)}-1}{3\mu^2}+\frac{e^{3\mu T}-1}{3\mu^2}\right)d(\eta,\overline{\eta})+\frac{2\overline{c}}{\widehat{\rho}}\left(\frac{e^{\mu(t+T)}-1}{\mu}+\frac{e^{\mu T}-1}{\mu}\right)d(\eta,\overline{\eta})\le$$

$$\le\left(\frac{2\overline{c}\Lambda_0}{\widehat{\rho}^2}\frac{e^{2\mu T}-1}{\mu^2}+\frac{2\Lambda_0}{\widehat{\rho}}\frac{e^{2\mu T}-1}{\mu}+\Phi_0\frac{e^{2\mu T}-1}{\mu}+{\Phi_0}^2\frac{e^{2\mu T}-1}{\mu^2}+\frac{4\overline{c}}{\widehat{\rho}}\frac{e^{\mu T}-1}{\mu}\right)e^{\mu(t+T)}d_{(m)}((r,\phi,\eta),(\overline{r},\overline{\phi},\overline{\eta}))$$

and therefore

$$d_{(m)}\left(B_\eta(r,\phi,\eta),B_\eta(\overline{r},\overline{\phi},\overline{\eta})\right)\le$$
$$\le\left(\frac{2\overline{c}\Lambda_0}{\widehat{\rho}^2}\frac{e^{2\mu T}-1}{\mu^2}+\frac{2\Lambda_0}{\widehat{\rho}}\frac{e^{2\mu T}-1}{\mu}+\Phi_0\frac{e^{2\mu T}-1}{\mu}+\Phi_0{}^2\frac{e^{2\mu T}-1}{\mu^2}+\frac{4\overline{c}}{\widehat{\rho}}\frac{e^{\mu T}-1}{\mu}\right)d_{(m)}((r,\phi,\eta),(\overline{r},\overline{\phi},\overline{\eta})).$$

The above inequalities imply

$$d_{(m)}\left((B_r,B_\phi,B_\eta),(\overline{B}_r,\overline{B}_\phi,\overline{B}_\eta)\right)\le(K_r+K_\phi+K_\eta)d_{(m)}\left((r,\phi,\eta),(\overline{r},\overline{\phi},\overline{\eta})\right).$$

For sufficiently large $\mu>0$ and small $\delta>0$ $K_r+K_\phi+K_\eta<1$. If the last inequality is not satisfied we can apply Corollary 1 (cf. (15)) and obtain $\frac{\omega^n}{\mu^n}\left(K_r+K_\phi+K_\eta\right)<1$. Therefore, the contraction inequality becomes

$$d_{(m)}\left((B_r,B_\phi,B_\eta),(\overline{B}_r,\overline{B}_\phi,\overline{B}_\eta)\right)\le\frac{\omega^n}{\mu^n}\left(K_r+K_\phi+K_\eta\right)d_{(m+n)}\left((r,\phi,\eta),(\overline{r},\overline{\phi},\overline{\eta})\right)$$

which shows that the following map arises: $j:A\to A, j(m)=m+n$. It is easy to verify that the space $M_r\times M_\phi\times M_\eta$ is *j*-bounded and *B* is a $(\Phi,j)-$contractive operator in the sense of [14]. The fixed point of *B* is a solution of (14).

Theorem 1 is thus proved.

2.6 Formulation of the Periodic Problem

Let us set $\dot{\rho}=r$, $\dot{\varphi}=\phi$, $\dot{\lambda}=\eta$. Then $\rho(t)=\rho_0+\int_0^t r(s)ds$; $\varphi(t)=\varphi_0+\int_0^t \phi(s)ds$, $\lambda(t)=\int_0^t \eta(s)ds$. Recall $\beta^2\approx 0$, $\Delta_2\approx c$ and reduce system (13) to the following one:

$$\begin{aligned}
&\dot{r}=\rho\phi^2\cos^2\lambda+\rho\eta^2+\frac{e_1e_2}{m_2}\frac{1}{\rho^2}-\frac{e_2^2}{m_2c^3}\ddot{r}\equiv F_r(r,\phi,\eta),\\
&\dot{\phi}=\frac{-2r\phi}{\rho}+2\phi\eta\, tg\lambda-\frac{e_1e_2}{m_2c^2}\frac{r\phi}{\rho^2}-\frac{e_2^2}{m_2c^3}\ddot{\phi}\equiv F_\phi(r,\phi,\eta),\\
&\dot{\eta}=-\frac{-2r\eta}{\rho}-\frac{\phi^2\sin 2\lambda}{2}+\frac{e_1e_2}{m_2c^2}\frac{r\eta}{\rho^2}-\frac{e_2^2}{m_2c^3}\ddot{\eta}\equiv F_\eta(r,\phi,\eta).
\end{aligned}\tag{18}$$

We look for a solution $(r,\phi,\eta)\in M_r\times M_\phi\times M_\eta$ of (18) such that $\rho(t)$ and $\lambda(t)$ to be *T*-periodic ones and $\varphi(t)\to\infty$ as $t\to\infty$. To simplify the next calculations, we recall denotations $T_k=kT(k=0,1,2,...)$. Obviously $T_{k+1}-T_k=T$.

As above denote by $r^{(m)}(t),\ (m=0,1,2,...)$ the *m*-th derivative of the function $r(t)$ and $r^{(0)}(t)=r(t)$. By $C_T{}^\infty[0,\infty)$ denote the set of all infinitely many differentiable *T*-periodic functions. Introduce the following sets of functions, where $T_k=kT\,(k=0,1,2,...)$ and $0<\delta<\frac{\pi}{2}$:

$$M_r = \left\{ r \in C_T^{\infty}[0,\infty) : \left| r^{(m)}(t) \right| \le \omega^m R_0 e^{\mu(t-T_k)};\ r(T_k) = 0;\ \int_{T_k}^{T_{k+1}} r(s)ds = 0 \right\}, t \in [T_k, T_{k+1}] ;$$

$$d_{(k,m)}(r,\bar{r}) = \sup\left\{ \frac{\left| r^{(m)}(t) - \bar{r}^{(m)}(t) \right|}{\omega^m} e^{-\mu(t-T_k)} : t \in [T_k, T_{k+1}] \right\};\ (k = 0,1,2,\dots),(m = 0,1,2,\dots);$$

$$M_\phi = \left\{ \phi \in C_T^{\infty}[0,\infty) : \left| \phi^{(m)}(t) \right| \le \omega^m \Phi_0 e^{\mu(t-T_k)},\ \phi(T_k) = \phi_0,\ \int_{T_k}^{T_{k+1}} \phi(s)ds = T\phi_0 \right\}, t \in [T_k, T_{k+1}] ,$$

$$d_{(k,m)}(\phi,\bar{\phi}) = \sup\left\{ \frac{\left| \phi^{(m)}(t) - \bar{\phi}^{(m)}(t) \right|}{\omega^m} e^{-\mu(t-T_k)} : t \in [T_k, T_{k+1}] \right\},\ (k = 0,1,2,\dots),(m = 0.1,2,\dots),$$

$$M_\eta = \left\{ \eta \in C_T^{\infty}[0,\infty) : \left| \eta^{(m)}(t) \right| \le \omega^m \Upsilon_0 e^{\mu(t-T_k)};\ \eta(T_k) = 0;\ \int_{T_k}^{T_{k+1}} \eta(s)ds = 0; \left| \int_{T_k}^{t} \eta(s)ds \right| \le \frac{\pi}{2} - \delta \right\},$$

$$d_{(k,m)}(\eta,\bar{\eta}) = \sup\left\{ \frac{\left| \eta^{(m)}(t) - \bar{\eta}^{(m)}(t) \right|}{\omega^m} e^{-\mu(t-T_k)} : t \in [T_k, T_{k+1}] \right\};\ (k = 0,1,2,\dots),\ (m = 0,1,2,\dots).$$

Here $\mu > \omega$ are strictly positive constants.

The set $M_r \times M_\phi \times M_\eta$ turns out in a uniform space by introducing a countable family of pseudo-metrics (cf. [11]):

$$d_{(k,m)}((r,\phi,\eta),(\bar{r},\bar{\phi},\bar{\eta})) = d_{(k,m)}(r,\bar{r}) + d_{(k,m)}(\phi,\bar{\phi}) + d_{(k,m)}(\eta,\bar{\eta}).$$

Here the index set $A = \{(k,m)\}$ consists of the ordered pairs of numbers ($k,m = 0,1,2,\dots$).

Lemma 5. If $(r,\varphi,\eta) \in M_r \times M_\phi \times M_\eta$, then $\rho(t) = \rho_0 + \int_0^t r(s)ds$ and $\lambda(t) = \int_0^t \eta(s)ds$ are *T*-periodic functions and $\lim_{t\to\infty}\left(\varphi_0 + \int_0^t \phi(s)ds \right) = \infty (\rho_0 , \phi_0 > 0)$.

Proof: Let us set $s = p + T$. Since $\int_T^{t+T} r(s)ds = \int_0^t r(p+T)dp = \int_0^t r(p)dp$ and $\int_0^T r(s)ds = 0$ it follows

$$\rho(t+T) = \rho_0 + \int_0^{t+T} r(s)ds = \rho_0 + \int_0^t r(s)ds + \int_t^0 r(s)ds + \int_0^T r(s)ds + \int_T^{t+T} r(s)ds = \rho_0 + \int_0^t r(s)ds + \int_t^0 r(s)ds + \int_0^t r(\theta)d\theta = \rho(t).$$

In a similar way we prove that $\lambda(t+T) = \lambda(t)$. We notice that for every $t > T$ there is $m \in N$ such that $[0,t] = [0,T_{m+1}] \cup [T_{m+1},t]$. If $t \to \infty$ then $m \to \infty$ which implies

$$\varphi(t) = \varphi_0 + \int_0^t \phi(s)ds = \varphi_0 + \sum_{k=0}^{m} \int_{T_k}^{T_{k+1}} \phi(s)ds + \int_{T_{k+1}}^{t} \phi(s)ds \ge \varphi_0 + mT\phi_0 - \int_{T_{k+1}}^{t} |\phi(s)|ds \ge$$
$$\ge \varphi_0 + mT\phi_0 - T\max\left\{ |\phi(s)| : s \in [T_{k+1},T_{k+2}] \right\} \underset{m\to\infty}{\to} \infty.$$

Lemma 5 is thus proved.

Lemma 6. The distance between particles $\rho(t)$ is a bounded function.

Proof: $\rho_{\min} = \rho_0 - R_0 \frac{e^{\mu T} - 1}{\mu} \le \rho(t) = \rho_0 + \int_0^t r(s)ds \le \rho_0 + R_0 \frac{e^{\mu T} - 1}{\mu} = \rho_{\max}$.

Recall that μ is sufficiently large.

Lemma 7. The sets M_r, M_ϕ and M_η are closed.

The proof is straightforward.

Lemma 8. If $(r,\phi,\eta)\in M_r\times M_\phi\times M_\eta$, then $F_r(r,\phi,\eta)(t)$, $F_\phi(r,\phi,\eta)(t)$ and $F_\eta(r,\phi,\eta)(t)$ are *T*-periodic functions.

The proof is straightforward.

Lemma 9. ([15]) If $F_r(r,\phi,\eta)(t)$, $F_\phi(r,\phi,\eta)(t)$, $F_\eta(r,\phi,\eta)(t)$ are T -periodic function, then

$$\int_{T_k}^{T_{k+1}}\int_{T_k}^{s}F_r(r,\phi,\eta)(p)dpds=\int_{T_{k+1}}^{T_{k+2}}\int_{T_{k+1}}^{s}F_r(r,\phi,\eta)(p)dpds\,;\ \int_{T_k}^{T_{k+1}}\int_{T_k}^{s}F_\phi(r,\phi,\eta)(p)dpds=\int_{T_{k+1}}^{T_{k+2}}\int_{T_{k+1}}^{s}F_\phi(r,\phi,\eta)(p)dpds\,;$$

$$\int_{T_k}^{T_{k+1}}\int_{T_k}^{s}F_\eta(r,\phi,\eta)(p)dpds=\int_{T_{k+1}}^{T_{k+2}}\int_{T_{k+1}}^{s}F_\eta(r,\phi,\eta)(p)dpds\,.$$

We assign to the periodic problem (18) the operator $B=(B_r,B_\phi,B_\eta):M_r\times M_\phi\times M_\eta\to M_r\times M_\phi\times M_\eta$ defined by the formulas:

$$B_r(r,\phi,\eta)(t):=\int_{T_k}^{t}F_r(r,\phi,\eta)(s)ds-\left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int_{T_k}^{T_{k+1}}F_r(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta}F_r(r,\phi,\eta)(s)dsd\theta\qquad t\in[T_k,T_{k+1}]\,,$$

$$B_\phi(r,\phi,\eta)(t):=\phi_0+\int_{T_k}^{t}F_\phi(r,\phi,\eta)(s)ds-\left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int_{T_k}^{T_{k+1}}F_\phi(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta}F_\phi(r,\phi,\eta)(s)dsd\theta\,,\quad t\in[T_k,T_{k+1}]$$

$$B_\eta(r,\phi,\eta)(t):=\int_{T_k}^{t}F_\eta(r,\phi,\eta)(s)ds-\left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int_{T_k}^{T_{k+1}}F_\eta(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta}F_\eta(r,\phi,\eta)(s)dsd\theta,\quad t\in[T_k,T_{k+1}]\,.\ k=0,1,2,\ldots\,.$$

Remark 3. Assumption (C) implies $r^2+\rho^2\eta^2+\rho^2\phi^2\cos\lambda\le R_0{}^2e^{2\mu T}+\rho^2\left(\Upsilon_0{}^2+\Phi_0{}^2\right)e^{2\mu T}\le\bar{c}^2<c^2$ and $\Delta_2{}^2+r^2=c^2-\rho^2\eta^2-\rho^2\varphi^2\cos\lambda\le\bar{c}^2$, $\Delta_2{}^2+\varphi^2\rho^2\cos^2\lambda\le\bar{c}^2$, $\Delta_2{}^2+\rho^2\eta^2\le\bar{c}^2$.

Recall $|e_1e_2|/m_2=e_2{}^2/m_2$ and $0<\beta=\bar{c}/c\ll 1$.

Lemma 10. The following estimates of the right-hand sides of (18) are valid:

$$\left|F_r(r,\phi,\eta)\right|=\left|\rho\phi^2\cos^2\lambda+\rho\eta^2+\frac{e_1e_2}{m_2}\frac{1}{\rho^2}-\frac{e_2^2}{m_2c^3}\ddot{r}\right|\le$$

$$\le\left|\bar{c}\Phi_0e^{\mu(t-T_k)}+\bar{c}\Lambda_0e^{\mu(t-T_k)}+\frac{e_2{}^2}{m_2}\left(\frac{1}{\rho_{\min}{}^2}+\frac{\omega^2R_0e^{\mu(t-T_k)}}{c^3}\right)\right|\le$$

$$\le e^{\mu(t-T_k)}\left[\bar{c}\Phi_0+\bar{c}\Lambda_0+\frac{e_2{}^2}{m_2}\left(\frac{1}{\rho_{\min}{}^2}+\frac{\omega^2R_0}{c^3}\right)\right];$$

$$\left|F_\phi(r,\phi,\eta)\right|=\left|\frac{-2r\phi}{\rho}+2\phi\eta\tan\lambda-\frac{e_1e_2}{m_2c^2}\frac{r\phi}{\rho^2}-\frac{e_2^2}{m_2c^3}\ddot{\phi}\right|\le$$

$$\le\left|\frac{2\bar{c}\Phi_0e^{\mu(t-T_k)}}{\rho_{\min}}+2\Phi_0\Lambda_0e^{2\mu(t-T_k)}\tan\left(\frac{\pi}{2}-\delta\right)+\frac{e_2^2}{m_2}\left(\frac{\bar{c}\Phi_0e^{\mu(t-T_k)}}{c^2\rho_{\min}{}^2}+\frac{\omega^2\Phi_0e^{\mu(t-T_k)}}{c^3}\right)\right|\le$$

$$\le e^{\mu(t-T_k)}\left[\frac{2\bar{c}}{\rho_{\min}}+2\Lambda_0e^{\mu T}\tan\left(\frac{\pi}{2}-\delta\right)+\frac{e_2^2}{m_2}\left(\frac{\bar{c}}{c^2\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)\right]\Phi_0\le\Phi_0e^{\mu(t-T_k)};$$

$$\left|F_\eta(r,\phi,\eta)\right| = \left|\frac{2r\eta}{\rho} - \frac{\phi^2 \sin 2\lambda}{2} + \frac{e_1 e_2}{m_2 c^2}\frac{r\eta}{\rho^2} - \frac{e_2^2}{m_2 c^3}\ddot{\eta}\right| \le$$
$$\le \left(\frac{2\bar{c}\Lambda_0 e^{\mu(t-T_k)}}{\rho_{\min}} + \frac{1}{2}\Phi_0{}^2 e^{2\mu(t-T_k)} + \frac{e_2^2}{m_2}\left(\frac{\bar{c}\Lambda_0 e^{\mu(t-T_k)}}{c^2 \rho_{\min}{}^2} + \frac{\omega^2 \Lambda_0 e^{\mu(t-T_k)}}{c^3}\right)\right) \le$$
$$\le e^{\mu(t-T_k)}\left(\frac{2\bar{c}\Lambda_0}{\rho_{\min}} + \frac{\Phi_0{}^2 e^{\mu T}}{2} + \frac{e_2^2}{m_2}\Lambda_0\left(\frac{\bar{c}}{c^2\rho_{\min}{}^2} + \frac{\omega^2}{c^3}\right)\right).$$

Lemma 11. (Main Lemma B) The problem (18) has a solution $(r,\phi,\eta) \in M_r \times M_\phi \times M_\eta$ iff the operator B has a fixed point belonging to $M_r \times M_\phi \times M_\eta$.

The proof is similar to the one of Lemma 3.

2.7 Existence-Uniqueness Theorem

The main result is:

Theorem 2. Let the following inequalities be fulfilled:

$$\left[\bar{c}\Phi_0 + \bar{c}\Lambda_0 + \frac{e_2{}^2}{m_2}\left(\frac{1}{\rho_{\min}{}^2} + \frac{\omega^2 R_0}{c^3}\right)\right]\frac{e^{\mu T}}{\mu} \le R_0 \ ; \ \phi_0 + \left[\frac{2\bar{c}}{\rho_{\min}} + 2\Lambda_0 e^{\mu T}\, tg\left(\frac{\pi}{2} - \delta\right) + \frac{e_2^2}{m_2}\left(\frac{\bar{c}}{c^2\rho_{\min}{}^2} + \frac{\omega^2}{c^3}\right)\right]\frac{e^{\mu T}}{\mu}\Phi_0 \le \Phi_0 \ ;$$

$$\left(\frac{2\bar{c}\Lambda_0}{\rho_{\min}} + \frac{\Phi_0{}^2 e^{\mu T}}{2} + \frac{e_2^2}{m_2}\Lambda_0\left(\frac{\bar{c}}{c^2\rho_{\min}{}^2} + \frac{\omega^2}{c^3}\right)\right)\frac{e^{\mu T}}{\mu} \le \Lambda_0 \ ; \ \left(\frac{2\bar{c}\Lambda_0}{\rho_{\min}} + \frac{\Phi_0{}^2 e^{\mu T}}{2} + \frac{e_2^2}{m_2}\Lambda_0\left(\frac{\bar{c}}{c^2\rho_{\min}{}^2} + \frac{\omega^2}{c^3}\right)\right)\frac{e^{2\mu T}}{\mu^2} \le \frac{\pi}{2} - \delta \ ,$$

$$\left[\bar{c}\Phi_0 + \bar{c}\Lambda_0 + \frac{e_2{}^2}{m_2}\left(\frac{1}{\rho_{\min}{}^2} + \frac{\omega^2 R_0}{c^3}\right)\right]\left(1 + \frac{e^{\mu T} - 1}{\mu T}\right) \le \omega R_0 \ ;$$

$$\left[\frac{2\bar{c}}{\rho_{\min}} + 2\Lambda_0 e^{\mu T}\, tg\left(\frac{\pi}{2} - \delta\right) + \frac{e_2^2}{m_2}\left(\frac{\bar{c}}{c^2\rho_{\min}{}^2} + \frac{\omega^2}{c^3}\right)\right]\Phi_0\left(1 + \frac{e^{\mu T} - 1}{\mu T}\right) \le \omega\Phi_0 \ ;$$

$$\left[\frac{2\bar{c}\Lambda_0}{\rho_{\min}} + \frac{\Phi_0{}^2 e^{\mu T}}{2} + \frac{e_2^2}{m_2}\Lambda_0\left(\frac{\bar{c}}{c^2\rho_{\min}{}^2} + \frac{\omega^2}{c^3}\right)\right]\left(1 + \frac{e^{\mu T} - 1}{\mu T}\right) \le \omega\Lambda_0$$ and so on.

It follows infinite number inequalities for higher derivatives.

Then there exists a unique solution of (18) belonging to $M_r \times M_\phi \times M_\eta$.

Proof: First we show that B maps the set $M_r \times M_\phi \times M_\eta$ into itself. Indeed, since $\int_{T_k}^{T_{k+1}}\left(\frac{t - T_k}{T} - \frac{1}{2}\right)dt = 0$ we obtain

$$\int_{T_k}^{T_{k+1}} B_r(r,\phi,\eta)(t)\,dt = \int_{T_k}^{T_{k+1}}\int_{T_k}^{t} F_r(r,\phi,\eta)(s)ds\,dt - \int_{T_k}^{T_{k+1}}\left(\frac{t - T_k}{T} - \frac{1}{2}\right)dt\int_{T_k}^{T_{k+1}} F_r(r,\phi,\eta)(s)ds - \int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta} F_r(r,\phi,\eta)(s)dsd\theta = 0$$

and

$$B_r(r,\phi,\eta)(T_k) = \int_{T_k}^{T_k} F_r(r,\phi,\eta)(s)ds - \left(\frac{T_k - T_k}{T} - \frac{1}{2}\right)\int_{T_k}^{T_{k+1}} F_r(r,\phi,\eta)(s)ds - \frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta} F_r(r,\phi,\eta)(s)dsd\theta =$$

$$=\frac{1}{2}\int_{T_k}^{T_{k+1}}F_r(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta}F_r(r,\phi,\eta)(s)dsd\theta=0$$

(cf. the proof of the Main Lemma I). In view of the first inequality from Theorem 2 we obtain

$$\left|B_r(r,\phi,\eta)(t)\right|\le\int_{T_k}^{t}\left|F_r(r,\phi,\eta)(s)\right|ds+\frac{1}{2}\int_{T_k}^{T_{k+1}}\left|F_r(r,\phi,\eta)\right|ds+\frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta}\left|F_r(r,\phi,\eta)\right|dsd\theta=\int_{T_k}^{t}\left|F_r(r,\phi,\eta)\right|ds+\int_{T_k}^{T_{k+1}}\left|F_r(r,\phi,\eta)\right|ds\le$$
$$\le e^{\mu(t-T_k)}\left[\bar{c}\Phi_0+\bar{c}\Lambda_0+\frac{e_2^{\,2}}{m_2}\left(\frac{1}{\rho_{\min}^{\,2}}+\frac{\omega^2R_0}{c^3}\right)\right]\left(\frac{1}{\mu}+\frac{e^{\mu T}-1}{\mu}\right)\le e^{\mu(t-T_k)}\left[\bar{c}\Phi_0+\bar{c}\Lambda_0+\frac{e_2^{\,2}}{m_2}\left(\frac{1}{\rho_{\min}^{\,2}}+\frac{\omega^2R_0}{c^3}\right)\right]\frac{e^{\mu T}}{\mu}\le R_0e^{\mu(t-T_k)}.$$

Therefore $B_r(r,\phi,\eta)\in M_r$.

For the second component we have $B_\phi(r,\phi,\eta)(T_k)=\phi_0$ and

$$\int_{T_k}^{T_{k+1}}B_\phi(r,\phi,\eta)dt=\phi_0T+\int_{T_k}^{T_{k+1}}\int_{T_k}^{t}F_\phi(r,\phi,\eta)ds\,dt-\int_{T_k}^{T_{k+1}}\left(\frac{t-T_k}{T}-\frac{1}{2}\right)dt\int_{T_k}^{T_{k+1}}F_\phi(r,\phi,\eta)(s)ds-\int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta}F_\phi(r,\phi,\eta)(s)dsd\theta=\phi_0T$$
$$\left|B_\phi(r,\phi,\eta)(t)\right|\le\phi_0e^{\mu(t-T_k)}+e^{\mu(t-T_k)}\left[\frac{2\bar{c}}{\rho_{\min}}+2\Lambda_0e^{\mu T}\tan\left(\frac{\pi}{2}-\delta\right)+\frac{e_2^2}{m_2}\left(\frac{\bar{c}}{c^2\rho_{\min}^{\,2}}+\frac{\omega^2}{c^3}\right)\right]\frac{e^{\mu T}}{\mu}\Phi_0\le\Phi_0e^{\mu(t-T_k)}.$$

Consequently $B_\phi(r,\phi,\eta)\in M_\phi$.

For the third component we obtain $\int_{T_k}^{T_{k+1}}B_\eta(r,\phi,\eta)(t)\,dt=0$, $B_\eta(r,\phi,\eta)(T_k)=0$ and

$$\left|B_\eta(r,\phi,\eta)(t)\right|\le\left(\frac{2\bar{c}\Lambda_0}{\rho_{\min}}+\frac{\Phi_0^{\,2}e^{\mu T}}{2}+\frac{e_2^2}{m_2}\Lambda_0\left(\frac{\bar{c}}{c^2\rho_{\min}^{\,2}}+\frac{\omega^2}{c^3}\right)\right)\frac{e^{\mu T}}{\mu}e^{\mu(t-T_k)}\le\Lambda_0e^{\mu(t-T_k)},$$
$$\left|\int_{T_k}^{t}B_\eta(r,\phi,\eta)(s)ds\right|\le\left(\frac{2\bar{c}\Lambda_0}{\rho_{\min}}+\frac{\Phi_0^{\,2}e^{\mu T}}{2}+\frac{e_2^2}{m_2}\Lambda_0\left(\frac{\bar{c}}{c^2\rho_{\min}^{\,2}}+\frac{\omega^2}{c^3}\right)\right)\frac{e^{\mu T}}{\mu}\int_{T_k}^{t}e^{\mu(s-T_k)}ds\le$$
$$\le\left(\frac{2\bar{c}\Lambda_0}{\rho_{\min}}+\frac{\Phi_0^{\,2}e^{\mu T}}{2}+\frac{e_2^2}{m_2}\Lambda_0\left(\frac{\bar{c}}{c^2\rho_{\min}^{\,2}}+\frac{\omega^2}{c^3}\right)\right)\frac{e^{\mu T}}{\mu}\frac{e^{\mu T}-1}{\mu}\le\frac{\pi}{2}-\delta.$$

For the first derivatives we have respectively

$$\left|\frac{dB_r(r,\phi,\eta)(t)}{dt}\right|\le\left|F_r(r,\phi,\eta)(t)\right|+\frac{1}{T}\int_{T_k}^{T_{k+1}}\left|F_r(r,\phi,\eta)(s)\right|ds\le\left[\bar{c}\Phi_0+\bar{c}\Lambda_0+\frac{e_2^{\,2}}{m_2}\left(\frac{1}{\rho_{\min}^{\,2}}+\frac{\omega^2R_0}{c^3}\right)\right]\left(1+\frac{e^{\mu T}-1}{\mu T}\right)\le\omega R_0e^{\mu(t-T_k)},$$
$$\left|\frac{dB_\phi(r,\phi,\eta)(t)}{dt}\right|\le\left|F_\phi(r,\phi,\eta)(t)\right|+\frac{1}{T}\int_{T_k}^{T_{k+1}}\left|F_\phi(r,\phi,\eta)(s)\right|ds\le\left[\frac{2\bar{c}}{\rho_{\min}}+2\Lambda_0e^{\mu T}\tan\left(\frac{\pi}{2}-\delta\right)+\frac{e_2^2}{m_2}\left(\frac{\bar{c}}{c^2\rho_{\min}^{\,2}}+\frac{\omega^2}{c^3}\right)\right]\Phi_0\left(1+\frac{e^{\mu T}-1}{\mu T}\right)\le\omega\Phi_0e^{\mu(t-T_k)},$$
$$\left|\frac{dB_\eta(r,\phi,\eta)(t)}{dt}\right|\le\left|F_\eta(r,\phi,\eta)(t)\right|+\frac{1}{T}\int_{T_k}^{T_{k+1}}\left|F_\eta(r,\phi,\eta)(s)\right|ds\le\left[\frac{2\bar{c}\Lambda_0}{\rho_{\min}}+\frac{\Phi_0^{\,2}e^{\mu T}}{2}+\frac{e_2^2}{m_2}\Lambda_0\left(\frac{\bar{c}}{c^2\rho_{\min}^{\,2}}+\frac{\omega^2}{c^3}\right)\right]\left(1+\frac{e^{\mu T}-1}{\mu T}\right)\le\omega\Lambda_0e^{\mu(t-T_k)}.$$

We notice that in the right-hand sides of the above inequalities the power of the frequency increases which implies a satisfying of the inequalities for the higher derivatives.

To prove that the operator B is contractive one we use the inequalities:

$d_{(k,0)}((r,\phi,\eta),(\bar{r},\bar{\phi},\bar{\eta}))\le\frac{\omega^n}{\mu^n}R_0\,d_{(k,n)}((r,\phi,\eta),(\bar{r},\bar{\phi},\bar{\eta}))$ and $\beta^3\approx0$

and obtain

$$\begin{aligned}
&\left|F_r(r,\phi,\eta)(s)-F_r(\bar r,\bar\phi,\bar\eta)(s)\right|\le\\
&\le\left|\frac{\partial\bar F_r}{\partial\rho}\right|\left|\rho(s)-\bar\rho(s)\right|+\left|\frac{\partial\bar F_r}{\partial r}\right|\left|r(s)-\bar r(s)\right|+\left|\frac{\partial\bar F_r}{\partial\varphi}\right|\left|\phi(s)-\bar\phi(s)\right|+\left|\frac{\partial\bar F_r}{\partial\lambda}\right|\left|\lambda(s)-\bar\lambda(s)\right|+\left|\frac{\partial\bar F_r}{\partial\eta}\right|\left|\eta(s)-\bar\eta(s)\right|\le\\
&\le\left|\frac{\partial\bar F_r}{\partial\rho}\right|\frac{e^{\mu(t-T_k)}}{\mu}d_{(k,0)}(r,\bar r)+\left|\frac{\partial\bar F_r}{\partial r}\right|e^{\mu(t-T)}d_{(k,0)}(r,\bar r)+\left|\frac{\partial\bar F_r}{\partial\phi}\right|e^{\mu(t-T_k)}d_{(k,0)}(\phi,\bar\phi)+\\
&+\left|\frac{\partial\bar F_r}{\partial\lambda}\right|\frac{e^{\mu(t-T_k)}}{\mu}d_{(k,0)}(\eta,\bar\eta)+\left|\frac{\partial\bar F_r}{\partial\eta}\right|e^{\mu(t-T_k)}d_{(k,0)}(\eta,\bar\eta)\le\\
&\le e^{\mu(t-T_k)}\left(\left|\frac{\partial\bar F_r}{\partial\rho}\right|\frac{1}{\mu}+\left|\frac{\partial\bar F_r}{\partial r}\right|+\left|\frac{\partial\bar F_r}{\partial\phi}\right|+\left|\frac{\partial\bar F_r}{\partial\lambda}\right|\frac{1}{\mu}+\left|\frac{\partial\bar F_r}{\partial\eta}\right|\right)\left(d_{(k,0)}(r,\bar r)+d_{(k,0)}(\phi,\bar\phi)+d_{(k,0)}(\eta,\bar\eta)\right),
\end{aligned}$$

where $\left|\frac{\partial\bar F_r}{\partial\rho}\right|,\ldots,\left|\frac{\partial\bar F_r}{\partial\eta}\right|$ are supremums of the partial derivatives.

Then in view of the definition of $B_r(r,\phi,\eta)$ we have

$$\begin{aligned}
&\left|B_r(r,\phi,\eta)(t)-B_r(\bar r,\bar\phi,\bar\eta)(t)\right|\le\int_{T_k}^{t}\left|F_r(r,\phi,\eta)(s)-F_r(\bar r,\bar\phi,\bar\eta)(s)\right|ds+\int_{T_k}^{T_{k+1}}\left|F_r(r,\phi,\eta)(s)-F_r(\bar r,\bar\phi,\bar\eta)(s)\right|ds\le\\
&\le\left(\frac{e^{\mu(t-kT)}}{\mu}+\frac{e^{\mu T}-1}{\mu}\right)\left(\left|\frac{\partial\bar F_r}{\partial\rho}\right|\frac{1}{\mu}+\left|\frac{\partial\bar F_r}{\partial r}\right|+\left|\frac{\partial\bar F_r}{\partial\phi}\right|+\left|\frac{\partial\bar F_r}{\partial\lambda}\right|\frac{1}{\mu}+\left|\frac{\partial\bar F_r}{\partial\eta}\right|\right)\left(d_{(k,0)}(r,\bar r)+d_{(k,0)}(\phi,\bar\phi)+d_{(k,0)}(\eta,\bar\eta)\right)\le\\
&\le e^{\mu(t-kT)}\frac{e^{\mu T}}{\mu}\left(\left|\frac{\partial\bar F_r}{\partial\rho}\right|\frac{1}{\mu}+\left|\frac{\partial\bar F_r}{\partial r}\right|+\left|\frac{\partial\bar F_r}{\partial\phi}\right|+\left|\frac{\partial\bar F_r}{\partial\lambda}\right|\frac{1}{\mu}+\left|\frac{\partial\bar F_r}{\partial\eta}\right|\right)d_{(k,0)}((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta)).
\end{aligned}$$

It follows $d_{(k,0)}(B_r(r,\phi,\eta),B_r(\bar r,\bar\phi,\bar\eta))\le K_r d_{(k,0)}((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta))$, where

$$K_r=\frac{e^{\mu T}}{\mu}\left(\left|\frac{\partial\bar F_r}{\partial\rho}\right|\frac{1}{\mu}+\left|\frac{\partial\bar F_r}{\partial r}\right|+\left|\frac{\partial\bar F_r}{\partial\phi}\right|+\left|\frac{\partial\bar F_r}{\partial\lambda}\right|\frac{1}{\mu}+\left|\frac{\partial\bar F_r}{\partial\eta}\right|\right).$$

In a similar way we obtain $d_{(k,0)}(B_\phi(r,\phi,\eta),B_\phi(\bar r,\bar\phi,\bar\eta))\le K_\phi d_{(k,0)}((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta))$, where

$$K_\phi=\frac{e^{\mu T}}{\mu}\left(\left|\frac{\partial\bar F_\phi}{\partial\rho}\right|\frac{1}{\mu}+\left|\frac{\partial\bar F_\phi}{\partial r}\right|+\left|\frac{\partial\bar F_\phi}{\partial\phi}\right|+\left|\frac{\partial\bar F_\phi}{\partial\lambda}\right|\frac{1}{\mu}+\left|\frac{\partial\bar F_\phi}{\partial\eta}\right|\right)$$

and $d_{(k,0)}(B_\eta(r,\phi,\eta),B_\eta(\bar r,\bar\phi,\bar\eta))\le K_\eta d_{(k,0)}((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta))$, where

$$K_\eta=\frac{e^{\mu T}}{\mu}\left(\left|\frac{\partial\bar F_\eta}{\partial\rho}\right|\frac{1}{\mu}+\left|\frac{\partial\bar F_\eta}{\partial r}\right|+\left|\frac{\partial\bar F_\eta}{\partial\phi}\right|+\left|\frac{\partial\bar F_\eta}{\partial\lambda}\right|\frac{1}{\mu}+\left|\frac{\partial\bar F_\eta}{\partial\eta}\right|\right).$$

The above inequalities imply:

$$d_{(k,0)}\left(\left(B_r(r,\phi,\eta),B_\phi(r,\phi,\eta),B_\eta(r,\phi,\eta)\right),\left(B_r(\bar r,\bar\phi,\bar\eta),B_\phi(\bar r,\bar\phi,\bar\eta),B_\eta(\bar r,\bar\phi,\bar\eta)\right)\right)\le\left(K_r+K_\phi+K_\eta\right)d_{(k,0)}\left((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta)\right).$$

If $K_r+K_\phi+K_\eta$ is not smaller than 1, we choose m_0 in such a way that $\left(\frac{\omega}{\mu}\right)^{m_0}\left(K_r+K_\phi+K_\eta\right)<1$. Then we have

$$d_{(k,0)}\left(\left(B_r(r,\phi,\eta),B_\phi(r,\phi,\eta),B_\eta(r,\phi,\eta)\right),\left(B_r(\bar r,\bar\phi,\bar\eta),B_\phi(\bar r,\bar\phi,\bar\eta),B_\eta(\bar r,\bar\phi,\bar\eta)\right)\right)\le\left(\frac{\omega}{\mu}\right)^{m_0}\left(K_r+K_\phi+K_\eta\right)d_{(k,m_0)}\left((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta)\right).$$

For the derivative $\frac{dB_r r,\phi,\eta)(t)}{dt} = F_r(r,\phi,\eta)(t) - \frac{1}{T}\int_{T_k}^{T_{k+1}} F_r(r,\phi,\eta)(s)ds,\ t\in[T_k,T_{k+1}]$ we obtain

$$\left|\frac{dB_r(r,\phi,\eta)(t)}{dt} - \frac{dB_r(\bar r,\bar\phi,\bar\eta)(t)}{dt}\right| \le \left|F_r(r,\phi,\eta)(t) - F_r(\bar r,\bar\phi,\bar\eta)(t)\right| + \frac{1}{T}\int_{T_k}^{T_{k+1}}\left|F_r(r,\phi,\eta)(s) - F_r(\bar r,\bar\phi,\bar\eta)(s)\right|ds \le$$
$$\le \left|\frac{\partial \bar F_r}{\partial\rho}\right|\left|\rho(t)-\bar\rho(t)\right| + \left|\frac{\partial \bar F_r}{\partial r}\right|\left|r(t)-\bar r(t)\right| + \left|\frac{\partial \bar F_r}{\partial\phi}\right|\left|\phi(t)-\bar\phi(t)\right| + \left|\frac{\partial \bar F_r}{\partial\lambda}\right|\left|\lambda(t)-\bar\lambda(t)\right| + \left|\frac{\partial \bar F_r}{\partial\eta}\right|\left|\eta(t)-\bar\eta(t)\right| +$$
$$+\frac{1}{T}\int_{T_k}^{T_{k+1}}\left(\left|\frac{\partial \bar F_r}{\partial\rho}\right|\left|\rho(s)-\bar\rho(s)\right| + \left|\frac{\partial \bar F_r}{\partial r}\right|\left|r(s)-\bar r(s)\right| + \left|\frac{\partial \bar F_r}{\partial\varphi}\right|\left|\phi(s)-\bar\phi(s)\right| + \left|\frac{\partial \bar F_r}{\partial\lambda}\right|\left|\lambda(s)-\bar\lambda(s)\right| + \left|\frac{\partial \bar F_r}{\partial\eta}\right|\left|\eta(s)-\bar\eta(s)\right|\right)ds \le$$
$$\le K_r^{\,1} e^{\mu(t-T_k)} d_{(k,0)}((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta))\,.$$

It follows

$$d_{(k,1)}\left(B_r(r,\phi,\eta), B_r(\bar r,\bar\phi,\bar\eta)\right) \le K_r^{\,1}\frac{\omega}{\mu} d_{(k,1)}((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta))\,,$$
$$d_{(k,1)}\left(B_\phi(r,\phi,\eta), B_\phi(\bar r,\bar\phi,\bar\eta)\right) \le K_\phi^{\,1}\frac{\omega}{\mu} d_{(k,1)}((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta))\,,$$
$$d_{(k,1)}\left(B_\eta(r,\phi,\eta), B_\eta(\bar r,\bar\phi,\bar\eta)\right) \le K_\eta^{\,1}\frac{\omega}{\mu} d_{(k,1)}((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta))$$

and hence

$$d_{(k,1)}\left(\left(B_r(r,\phi,\eta), B_\phi(r,\phi,\eta), B_\eta(r,\phi,\eta)\right),\left(B_r(\bar r,\bar\phi,\bar\eta), B_\phi(\bar r,\bar\phi,\bar\eta), B_\eta(\bar r,\bar\phi,\bar\eta)\right)\right) \le \left(K_r^{\,1} + K_\phi^{\,1} + K_\eta^{\,1}\right)\frac{\omega^{m+1}}{\mu^{m+1}} d_{(k,m)}\left((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta)\right).$$

For the second derivative $\frac{d^2B_r(r,\phi,\eta)(t)}{dt^2} = \frac{dF_r(r,\phi,\eta)(t)}{dt}$, $t\in[T_k,T_{k+1}]$ we obtain $d_{(k,2)}\left(\left(B_r(r,\phi,\eta), B_\phi(r,\phi,\eta), B_\eta(r,\phi,\eta)\right),\left(B_r(\bar r,\bar\phi,\bar\eta), B_\phi(\bar r,\bar\phi,\bar\eta), B_\eta(\bar r,\bar\phi,\bar\eta)\right)\right) \le$
$\le \left(K_r^{\,2} + K_\phi^{\,2} + K_\eta^{\,2}\right)\frac{\omega^{m+2}}{\mu^{m+2}} d_{(k,m)}\left((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta)\right)$ and so on.

It is easy to see that $\frac{dF_r(r,\phi,\eta)}{dt}, \frac{dF_\phi(r,\phi,\eta)}{dt}, \frac{dF_\eta(r,\phi,\eta)}{dt}$ contain the third derivatives of the unknown functions r,ϕ,η and therefore their estimates contain ω^3. But the estimates for the second derivatives contains ω^2. Consequently, the power of ω does not increase.

Define the map $j: A\to A$ in the following way:

$$j(k,0)=(k,m),\quad j^2(k,0)=(k,2m),...;\ j(k,1)=(k,m_1),\quad j^2(k,1)=(k,2m_1),\dots\ .$$

It is easy to verify that the space $M_r\times M_\phi\times M_\eta$ is *j*-bounded in the sense of [15]. Indeed, $\sup\left\{d_{(k,p+m_s)}((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta)) : p\right\} =$
$= \sup\left\{d_{(k,p+m_s)}(r,\bar r) : m_s\right\} + \sup\left\{d_{(k,p+m_s)}(\varphi,\bar\varphi) : m_s\right\} + \sup\left\{d_{(k,p+m_s)}(\eta,\bar\eta) : m_s\right\} \le 2(R_0+\Phi_0+\Lambda_0)e^{\mu T} < \infty\quad (s=1,2,3,...)$.

Therefore, operator B possesses unique fixed point which is a solution of the problem.

Theorem 2 is thus proved.

2.8 Continuous "Jump"

To describe the transitions of the moving (second) particle from the ground state to a higher energy excited state we apply a radial force $\tilde{\rho}(t)$ to (18). From the mathematical point of view, we add this force to the first equation and assume that $\tilde{\rho}(t)$ possesses some suitable properties:

$$\begin{aligned}
\ddot{\rho} &= \rho\dot{\varphi}^2\cos^2\lambda + \rho\dot{\lambda}^2 - \frac{|e_1e_2|}{m_2c^2}\frac{c^2}{\rho^2} + \frac{e_2^2}{m_2c^3}\dddot{\rho} + \tilde{\rho}(t)\,;\\
\ddot{\varphi} &= \frac{-2\dot{\rho}\dot{\varphi}}{\rho} + 2\dot{\varphi}\dot{\lambda}\,tg\lambda + \frac{|e_1e_2|}{m_2c^2}\frac{\dot{\rho}\dot{\varphi}}{\rho^2} - \frac{e_2^2}{m_2c^3}\dddot{\varphi}\,;\\
\ddot{\lambda} &= \frac{-2\dot{\rho}\dot{\lambda}}{\rho} - \frac{\dot{\varphi}^2\sin 2\lambda}{2} + \frac{|e_1e_2|}{m_2c^2}\frac{\dot{\rho}\dot{\lambda}}{\rho^2} - \frac{e_2^2}{m_2c^3}\dddot{\lambda}\,.
\end{aligned} \tag{19}$$

We would like to point out, however, that this transition is performed along a certain continuous trajectory. We show the existence of such trajectories.

We recall that we have already solved the Initial system on $[-T,0]$:

$$\dot{r} = \rho\phi^2\cos^2\lambda + \rho\eta^2 - \frac{c^2}{\rho}\,;\; \dot{\phi} = 2\phi\eta tg\lambda - \frac{2r\phi}{\rho}\,;\; \dot{\eta} = -\frac{2r\eta}{\rho} - \frac{\phi^2\sin 2\lambda}{2}$$

and we take the values of the solution at the right end $t=0$ to be initial conditions for (19).

We modify the function spaces. Denote again by $C_T^{\infty}[0,\infty)$ the set of all infinitely differentiable *T*-periodic functions such that $r(T_k)=0,(k=0,1,...)$. Let $k_0,k_1\in N,\ 0<k_0<k_1$. We redefine every function $r(.)\in C_T^{\infty}[0,\infty)$ (denoting again by $r(.)$) in the following way:

$$r(.) = \begin{cases} r(t), & t\in[0,T_{k_0}),\\ r(t)+\tilde{r}(t), & t\in[T_{k_0},T_{k_1}],\\ r(t), & t\in(T_{k_1},\infty) \end{cases}$$

where $\tilde{r}(.)\in C^{\infty}[0,\infty)$ and $\tilde{r}(t)\ge 0$, where $\tilde{r}(t)=\dot{\tilde{\rho}}(t)$.

The function $\tilde{r}(t)$ is defined on $[T_{k_0},T_{k_1}]$. It is *T*-periodic and $\tilde{r}(T_{k_0})=...=\tilde{r}(T_{k_1})=0$ and $\int_{T_k}^{T_{k+1}}\tilde{r}(s)ds = T\,\tilde{r}_0,(k=k_0,...,k_1-1),\tilde{r}_0 = const.>0$.

Then we rewrite the above system for $t\in[0,\infty)$:

$$\dot{r}(t) = \tilde{F}_r(r,\phi,\eta) = \begin{cases} \rho(t)\phi^2(t)\cos^2\lambda(t) + \rho(t)\eta^2(t) - \frac{|e_1e_2|}{m_2}\frac{1}{\rho^2(t)} + \frac{e_2^2}{m_2c^3}\ddot{r}(t) + \tilde{r}(t),\ t\in[T_{k_0},T_{k_1}]\\ \rho(t)\phi^2(t)\cos^2\lambda(t) + \rho(t)\eta^2(t) - \frac{|e_1e_2|}{m_2}\frac{1}{\rho^{2(t)}} + \frac{e_2^2}{m_2c^3}\ddot{r}(t), t\in[0,\infty)\setminus[T_{k_0},T_{k_1}] \end{cases},$$

$$\dot{\phi}(t)=\frac{-2r(t)\phi(t)}{\rho(t)}+2\phi(t)\eta(t)\,tg\,\lambda(t)+\frac{|e_1e_2|}{m_2c^2}\frac{r(t)\phi(t)}{\rho^2(t)}-\frac{e_2^2}{m_2c^3}\ddot{\phi}(t)\equiv F_\phi(r,\phi,\eta),\;\; t\in[0,\infty) \tag{20}$$
$$\dot{\eta}(t)=\frac{-2r(t)\eta(t)}{\rho(t)}-\frac{\phi^2(t)\sin 2\lambda(t)}{2}+\frac{|e_1e_2|}{m_2c^2}\frac{r(t)\eta(t)}{\rho^2(t)}-\frac{e_2^2}{m_2c^3}\ddot{\eta}(t)\equiv F_\eta(r,\phi,\eta)\;, t\in[0,\infty)$$

and because of $\langle\vec{u},\ddot{u}\rangle=\dot{\rho}\dddot{\rho}+\rho^2\dot{\varphi}\dddot{\varphi}\cos^2\lambda+\rho^2\dot{\lambda}\dddot{\lambda}$ we obtain for the energy of the second particle

$$\frac{dW^{(2)}}{dt}=-e_2{}^2\left(\frac{r}{\rho^2}+\frac{r\ddot{r}+\rho^2\phi\ddot{\phi}\cos^2\lambda+\rho^2\eta\ddot{\eta}}{c^2}\right)\Rightarrow$$
$$W^{(2)}(t)=-e_2{}^2\left(-\frac{1}{\rho(t)}+\frac{1}{\rho(0)}+\frac{1}{c^2}\int_0^t\left(r\ddot{r}+\rho^2\phi\ddot{\phi}\cos^2\lambda+\rho^2\eta\ddot{\eta}\right)ds\right).$$

For $t=T$ one obtains:

$$W_T=-e_2{}^2\left(-\frac{1}{\rho(T)}+\frac{1}{\rho(0)}+\frac{1}{c^2}\int_0^T\left(r\ddot{r}+\rho^2\phi\ddot{\phi}\cos^2\lambda+\rho^2\eta\ddot{\eta}\right)ds\right)=\frac{-e_2{}^2}{c^2}\int_0^T\left(r\ddot{r}+\rho^2\phi\ddot{\phi}\cos^2\lambda+\rho^2\eta\ddot{\eta}\right)ds\;.$$

We notice $\int_0^T r(s)\ddot{r}(s)ds=\int_0^T r(s)d\dot{r}(s)=r(T)\dot{r}(T)-r(0)\dot{r}(0)-\int_0^T\dot{r}(s)dr(s)=-\int_0^T\dot{r}^2(s)ds;$

$$\int_0^T\rho^2\phi\ddot{\phi}\cos^2\lambda ds=\rho^2(\xi_T)\cos^2\lambda(\xi_T)\int_0^T\phi(s)\ddot{\phi}(s)ds=\rho^2(\xi_T)\cos^2\lambda(\xi_T)\left[\int_0^T\phi(s)d\dot{\phi}(s)\right]=$$
$$=\rho^2(\xi_T)\cos^2\lambda(\xi_T)\left[\phi(T)\dot{\phi}(T)-\phi(0)\dot{\phi}(0)-\int_0^T\dot{\phi}^2(s)ds\right]=-\rho^2(\xi_T)\cos^2\lambda(\xi_T)\int_0^T\dot{\phi}^2(s)ds;$$
$$\int_0^T\rho^2\eta\ddot{\eta}ds=\rho^2(\varsigma_T)\int_0^T\eta(s)d\dot{\eta}(s)=\rho^2(\varsigma_T)\left[\eta(T)\dot{\eta}(T)-\eta(0)\dot{\eta}(0)-\int_0^T\dot{\eta}^2(s)ds\right]=-\rho^2(\varsigma_T)\int_0^T\dot{\eta}^2(s)ds\;.$$

But $\langle\vec{u},\vec{u}\rangle=\dot{\rho}^2+\rho^2\dot{\lambda}^2+\rho^2\dot{\varphi}^2\cos^2\lambda=r^2+\rho^2\phi^2\cos^2\lambda+\rho^2\eta^2$. Since T is small number, we assume $\rho\approx const.,\lambda\approx const.$ and then

$$W_T=-e_2{}^2\left(\frac{1}{c^2}\int_0^T\left(r\ddot{r}+\rho^2\phi\ddot{\phi}\cos^2\lambda+\rho^2\eta\ddot{\eta}\right)ds\right)=\frac{-e_2{}^2}{c^2}\left(\int_0^T r\ddot{r}ds+\int_0^T\rho^2\phi\ddot{\phi}\cos^2\lambda ds+\int_0^T\rho^2\eta\ddot{\eta}ds\right)=$$
$$=\frac{e_2{}^2}{c^2}\left(\int_0^T\dot{r}^2(s)ds+\rho^2(\xi_T)\cos^2\lambda(\xi_T)\int_0^T\dot{\phi}^2(s)ds+\rho^2(\varsigma_T)\int_0^T\dot{\eta}^2(s)ds\right). \tag{21}$$

To obtain transition (“forbidden” in Quantum mechanics) trajectories we introduce the following function spaces:

$$\tilde{M}_r=\left\{r(.)\in C_T{}^\infty[0,\infty):\left|\frac{d^mr(t)}{dt^m}\right|\le\omega^mR_0e^{\mu(t-T_k)};\int_{T_k}^{T_{k+1}}r(s)ds=0,(k=0,1,...,k_0-1,\;k_1,k_1.+1,...);\right.$$
$$\left.\int_{T_k}^{T_{k+1}}r(s)ds=T\,\tilde{r}_0,;\left|\frac{d^m\left(r(t)+\tilde{r}(t)\right)}{dt^m}\right|\le\omega^m(R_0+\tilde{R}_0)e^{\mu(t-T_k)}(k=k_0,k_0+1,...,k_1-1)\right\}$$
$$M_\phi=\left\{\phi\in C_T{}^\infty[0,\infty):\left|\frac{d^m\phi(t)}{dt^m}\right|\le\omega^m\Phi_0e^{\mu(t-T_k)};\int_{T_k}^{T_{k+1}}\phi(s)ds=T\phi_0,(k=0,1,...)\right\}, \tag{22}$$

$$M_\eta = \left\{ \eta \in C_T^{\ \infty}[0,\infty) : \left| \frac{d^m \eta(t)}{dt^m} \right| \le \omega^m \Lambda_0 e^{\mu(t-T_k)} ; \int_{T_k}^{T_{k+1}} \eta(t)dt = 0, \left| \int_{T_k}^{t} \eta(s)ds \right| \le \frac{\pi}{2} - \delta, (k = 0,1,...) \right\},$$

$(k, m = 0,1,2,...)$.

Lemma 12. The solution (r,ϕ,η) of (20) belonging to $\tilde{M}_r \times M_\phi \times M_\eta$ generates transition trajectories.

Proof: Indeed, in the interval $[0,T_{k_0}]$ the distance between particles $\rho(t)$ is *T*-periodic function and $\rho_{\min} = \rho_0 - R_0\left(e^{\mu T} - 1\right)/\mu \le \rho(t) \le \rho_0 + R_0\left(e^{\mu T} - 1\right)/\mu = \rho_{\max}$.

Let us calculate

$$\rho(T_{k_1}) = \rho_0 + \int_0^{T_{k_1}} r(s)ds =$$

$$= \rho_0 + \int_0^{T} r(s)ds + \int_T^{T_2} r(s)ds + ... + \int_{T_{k_0-1}}^{T_{k_0}} r(s)ds + \int_{T_{k_0}}^{T_{k_0+1}} r(s)ds + \int_{T_{k_0+1}}^{T_{k_0+2}} r(s)ds + ... + \int_{T_{k_1-2}}^{T_{k_1-1}} r(s)ds + \int_{T_{k_1-1}}^{T_{k_1}} r(s)ds =$$

$$= \rho_0 + \int_{T_{k_0}}^{T_{k_0+1}} r(s)ds + \int_{T_{k_0+1}}^{T_{k_0+2}} r(s)ds + ... + \int_{T_{k_1-2}}^{T_{k_1-1}} r(s)ds + \int_{T_{k_1-1}}^{T_{k_1}} r(s)ds = \rho_0 + (k_1 - k_0)T\,\tilde{r}_0.$$

Then since $\int_{T_k}^{T_{k+1}} r(s)ds = 0$, $(k = k_1, k_1 + 1, ...)$ we infer that the distance function $\rho(t)$ increases in the interval: $[T_{k_0}, T_{k_1}]$ from $\rho_0 + \int_0^t r(p)dp$, $t \in [0, T_{k_0}]$ to $\rho_0 + T_{k_1-k_0}\tilde{r}_0 + \int_{T_{k_1}}^{t} r(p)dp$, $t \in [T_{k_1}, \infty)$ and further on it remains periodic one $\rho(t) = \rho_1 + \int_{T_{k_1}}^{t} r(p)dp$, where $\rho_1 = \rho_0 + T_{k_1-k_0}\tilde{r}$.

Lemma 12 is thus proved.

Principal Remark 4. Substituting $r(t)$ by $r(t) + \tilde{r}(t)$ we notice an increasing of the energy. Indeed,

$$\tilde{W}_T = \frac{e_2^{\ 2}}{c^2}\left(\int_0^T \dot{r}^2(s)ds + \left(\rho_0 + \int_0^{\xi_T} (r(p) + \tilde{r}(p))dp \right)^2 \cos^2\lambda(\xi_T) \int_0^T \dot{\phi}^2(s)ds + \left(\rho_0 + \int_0^{\xi_T} (r(p) + \tilde{r}(p))dp \right)^2 \int_0^T \dot{\eta}^2(s)ds \right) > W_T .$$

In such a way we proved an existence of transition (jump) trajectories of the second particle (electron) from the ground state to a higher energy excited state after absorbing energy caused by the radial periodic force $\tilde{r}(t)$ **.**

In what follows we prove an existence of functions $(r,\phi,\eta) \in M_r \times M_\phi \times M_\eta$ which guarantee an existence of such trajectories. We endow $\tilde{M}_r \times M_\phi \times M_\eta$ with a saturated family of pseudo-metrics

$$d_{(m,k)}((r,\phi,\eta),(\overline{r},\overline{\phi},\overline{\eta})) = d_{(m,k)}(r,\overline{r}) + d_{(m,k)}(\phi,\overline{\phi}) + d_{(m,k)}(\eta,\overline{\eta}) ,$$

where

$$d_{(m,k)}(r,\overline{r}) = \sup\left\{ \frac{1}{\omega^m} \left| \frac{d^m r(t)}{dt^m} - \frac{d^m \overline{r}(t)}{dt^m} \right| e^{-\mu(t-T_k)} : \ t \in [T_k; T_{k+1}] \right\},$$

$$d_{(m,k)}(\phi,\bar{\phi}) = \sup\left\{\frac{1}{\omega^m}\left|\frac{d^m\phi(t)}{dt^m} - \frac{d^m\bar{\phi}(t)}{dt^m}\right| e^{-\mu(t-T_k)} : t\in[T_k;T_{k+1}]\right\},$$

$$d_{(m,k)}(\eta,\bar{\eta}) = \sup\left\{\frac{1}{\omega^m}\left|\frac{d^m\eta(t)}{dt^m} - \frac{d^m\bar{\eta}(t)}{dt^m}\right| e^{-\mu(t-T_k)} : t\in[T_k;T_{k+1}]\right\} \ (k,m=0,1,2,...).$$

The set $\tilde{M}_r \times M_\phi \times M_\eta$ turns out in a uniform space with countable family of pseudo-metrics (cf. [14]). Here the index set consists of the ordered pairs $A=\{(k,m)\}$ of numbers $k,m=0,1,2,....$

Lemma 13. The sets $\tilde{M}_r$, M_ϕ , M_η and $\tilde{M}_r \times M_\phi \times M_\eta$ are closed.

The proof is given in the Appendix.

Lemma 14. ([15]) If $(r,\phi,\eta)\in M_r\times M_\phi \times M_\eta$, then $F_r(r,\phi,\eta)(t)$, $F_\phi(r,\phi,\eta)(t)$ and $F_\eta(r,\phi,\eta)(t)$ are *T*-periodic functions.

We assign to the problem (20) the operator $B=(B_r,B_\phi,B_\eta)$ defined on $M_r\times M_\phi\times M_\eta$ for $t\in[T_k,T_{k+1}]$ by the formulas:

$$B_r(r,\phi,\eta)(t) := \int_{T_k}^{t}\tilde{F}_r(r,\phi,\eta)(s)ds - \left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int_{T_k}^{T_{k+1}}\tilde{F}_r(r,\phi,\eta)(s)ds - \frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta}\tilde{F}_r(r,\phi,\eta)(s)dsd\theta,$$

$$B_\phi(r,\phi,\eta)(t) := \phi_0 + \int_{T_k}^{t}F_\phi(r,\phi,\eta)(s)ds - \left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int_{T_k}^{T_{k+1}}F_\phi(r,\phi,\eta)(s)ds - \frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta}F_\phi(r,\phi,\eta)(s)dsd\theta,$$

$$B_\eta(r,\phi,\eta)(t) := \int_{T_k}^{t}F_\eta(r,\phi,\eta)(s)ds - \left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int_{T_k}^{T_{k+1}}F_\eta(r,\phi,\eta)(s)ds - \frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta}F_\eta(r,\phi,\eta)(s)dsd\theta$$

$$t\in[T_k,T_{k+1}] \ (k=0,1,2,3,\ldots).$$

Recall that **Assumption (C)** implies $R_0{}^2e^{2\mu T} + \rho^2\left(\Lambda_0{}^2+\Phi_0{}^2\right)e^{2\mu T} \le \bar{c}^2 < c^2$, and $0<\beta=\bar{c}/c<1$.

Lemma 15. The following inequalities are valid:

$$\left|\tilde{F}_r(r,\phi,\eta)\right| \le \left[\bar{c}(\Phi_0+\Lambda_0) + \frac{|e_1e_2|}{m_2}\left(\frac{1}{\rho_{\min}{}^2}+\frac{\omega^2R_0}{c^3}\right)+\tilde{r}_{\max}\right]e^{\mu(t-T_k)};$$

$$\left|F_\phi(r,\phi,\eta)\right| \le \left[\frac{2R_0e^{\mu T}}{\rho_{\min}} + 2\Lambda_0\tan\left(\frac{\pi}{2}-\delta\right) + \frac{|e_1e_2|}{m_2}\left(\frac{R_0e^{\mu T}}{c^2\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)\right]\Phi_0e^{\mu(t-T_k)};$$

$$\left|F_\eta(r,\phi,\eta)\right| \le \left[\frac{2R_0\Lambda_0e^{\mu T}}{\rho_{\min}} + \frac{\Phi_0{}^2e^{\mu T}}{2} + \frac{|e_1e_2|}{m_2}\left(\frac{R_0e^{\mu T}}{c^2\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)\Lambda_0\right]e^{\mu(t-T_k)}.$$

The proof is straightforward.

Lemma 16 (Main Lemma) The problem (20) has a solution $(r,\phi,\eta)\in\tilde{M}_r\times M_\phi\times M_\eta$ iff the operator B has a fixed point belonging to $\tilde{M}_r\times M_\phi\times M_\eta$.

Proof: We consider only the case $t\in[T_{k_0},T_{k_1}]$ because for $t\in[0,T_{k_0})\cup(T_{k_1},\infty)$ the proof repeats the one from the previous subsection. Let $(r,\phi,\eta)\in\tilde{M}_r\times M_\phi\times M_\eta$ be a T -periodic solution of the system:

$$\dot{r}(t)=\tilde{F}_r(r,\phi,\eta)(t),\ \dot{\phi}(t)=F_\phi(r,\phi,\eta)(t),\ \dot{\eta}(t)=F_\eta(r,\phi,\eta)(t).$$

Then after integration (in view of $\tilde{r}(T_k)=0 \Rightarrow r(T_k)=0$) we obtain:

$$r(t)=\int_{T_k}^{t}\tilde{F}_r(r,\phi,\eta)(s)ds\ ,\ \ \phi(t)=\phi_0+\int_{T_k}^{t}F_\varphi(r,\phi,\eta)(s)ds\ \ ,\ \ \eta(t)=\int_{T_k}^{t}F_\eta(r,\phi,\eta)(s)ds$$

and put $t=T_{k+1}$ which implies $0=r(T_{k+1})=\int_{T_k}^{T_{k+1}}\tilde{F}_r(r,\phi,\eta)(S)ds \Rightarrow \int_{T_k}^{T_{k+1}}\tilde{F}_r(r,\phi,\eta)(s)ds=0$.

Therefore $r(t)=\int_{T_k}^{t}\tilde{F}_r(r,\phi,\eta)ds \Rightarrow r(t)=\int_{T_k}^{t}\tilde{F}_r(r,\phi,\eta)ds-\left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int_{T_k}^{T_{k+1}}\tilde{F}_r(r,\phi,\eta)(s)ds$.

Changing the order of integration and considering that $\dot{r}(t)=\tilde{F}_r(r,\phi,\eta)(t)$ and $\int_{T_k}^{T_{k+1}}\tilde{r}(s)ds=T\,\tilde{r}_0$ we have

$$\int_{T_k}^{T_{k+1}}\int_{T_k}^{p}\tilde{F}_r(r,\phi,\eta)(s)dsdp=\int_{T_k}^{T_{k+1}}(T_{k+1}-s)\tilde{F}_r(r,\phi,\eta)(s)ds=\int_{T_k}^{T_{k+1}}(T_{k+1}-s)\dot{r}(s)ds=T_{k+1}\int_{T_k}^{T_{k+1}}\dot{r}(s)ds-\int_{T_k}^{T_{k+1}}s\dot{r}(s)ds=$$

$$=T_{k+1}\left(r(T_{k+1})-r(T_k)\right)-\int_{T_k}^{T_{k+1}}sdr(s)=T_{k+1}\left(r(T_{k+1})-r(T_k)\right)-\left(T_{k+1}r(T_{k+1})-T_kr(T_k)\right)+\int_{T_k}^{T_{k+1}}r(s)ds=0.$$

Consequently $r(t)=\int_{T_k}^{t}\tilde{F}_r(r,\phi,\eta)ds-\left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int_{T_k}^{T_{k+1}}\tilde{F}_r(r,\phi,\eta)ds$ can be rewritten in the form:

$$r(t)=\int_{T_k}^{t}\tilde{F}_r(r,\phi,\eta)ds-\left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int_{T_k}^{T_{k+1}}\tilde{F}_r(r,\phi,\eta)ds-\frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{p}\tilde{F}_r(r,\phi,\eta)dsdp\ .$$

The last equality means $r=B_r(r,\phi,\eta)$.

Repeating the above reasoning we conclude that $\phi=B_\phi(r,\phi,\eta)$, $\eta=B_\eta(r,\phi,\eta)$. In other words $(r,\phi,\eta)=\left(B_r(r,\phi,\eta),B_\phi(r,\phi,\eta),B_\eta(r,\phi,\eta)\right)$, $\eta=B_\eta(r,\phi,\eta)$,

that is, the operator B has a fixed point.

Conversely, let $(r,\phi,\eta)\in\tilde{M}_r\times M_\phi\times M_\eta$ be a fixed point of B. Then

$$r(t)=\int_{T_k}^{t}\tilde{F}_r(r,\phi,\eta)ds-\left(\frac{t-T_k}{T}-\frac{1}{2}\right)\int_{T_k}^{T_{k+1}}\tilde{F}_r(r,\phi,\eta)ds-\frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{p}\tilde{F}_r(r,\phi,\eta)dsdp$$

which for $t=T_k$ implies

$$0=r(T_k)=\int_{T_k}^{T_k}\tilde{F}_r(r,\phi,\eta)ds-\left(\frac{T_k-T_k}{T}-\frac{1}{2}\right)\int_{T_k}^{T_{k+1}}F_r(r,\phi,\eta)ds-\frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{s}F_r(r,\phi,\eta)d\theta ds \Rightarrow$$

$$\Rightarrow 0=\frac{1}{2}\int_{T_k}^{T_{k+1}}F_r(r,\phi,\eta)ds-\frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{s}F_r(r,\phi,\eta)d\theta ds\ .$$

We show that $\int_{T_k}^{T_{k+1}}F_r(r,\phi,\eta)(s)ds=0$. Indeed, let us suppose that $\left|\int_{T_k}^{T_{k+1}}F_r(r,\phi,\eta)ds\right|=\xi>0$.

Then in view of Lemma 15:

$$\left|\int_{T_k}^{T_{k+1}} F_r(r,\phi,\eta)(s)ds\right| \le \left(\bar{c}(\Phi_0+\Lambda_0)+\frac{|e_1e_2|}{m_2}\left(\frac{1}{\rho_{\min}{}^2}+\frac{R_0\omega^2}{c^3}\right)+\tilde{r}_{\max}\right)\frac{e^{\mu T}-1}{\mu}$$

and the last term becomes less than ξ for sufficiently large $\mu>0$ and $\mu T = const$. Consequently, $r(t)=\int_{T_k}^{t}\tilde{F}_r(r,\phi,\eta)(s)ds \Rightarrow \dot{r}(t)=\tilde{F}_r(r,\phi,\eta)(t)$.

In a similar way for the second and third components of B we obtain $\dot{\phi}(t) = F_\phi(r,\phi,\eta)(t)$ and $\dot{\eta}(t) = F_\eta(r,\phi,\eta)(t)$ which completes the proof.

2.9 Existence-Uniqueness Basic System

The main result is the following theorem:

Theorem 3. Let the following inequalities be fulfilled:

$$\left[\bar{c}(\Phi_0+\Lambda_0)+\frac{|e_1e_2|}{m_2}\left(\frac{1}{\rho_{\min}{}^2}+\frac{\omega^2R_0}{c^3}\right)+\tilde{r}_{\max}\right]\frac{1}{\mu}\le R_0+\tilde{R}_0;$$

$$\phi_0+\left[\frac{2R_0e^{\mu T}}{\rho_{\min}}+2\Lambda_0\tan\left(\frac{\pi}{2}-\delta\right)+\frac{|e_1e_2|}{m_2}\left(\frac{R_0e^{\mu T}}{c^2\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)\right]\frac{\Phi_0}{\mu}\le\Phi_0;$$

$$\left[\frac{2R_0\Lambda_0e^{\mu T}}{\rho_{\min}}+\frac{\Phi_0{}^2e^{\mu T}}{2}+\frac{|e_1e_2|}{m_2}\left(\frac{R_0e^{\mu T}}{c^2\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)\Lambda_0\right]\frac{1}{\mu}\le\Lambda_0;$$

$$\left[\frac{2R_0\Lambda_0e^{\mu T}}{\rho_{\min}}+\frac{\Phi_0{}^2e^{\mu T}}{2}+\frac{|e_1e_2|}{m_2}\left(\frac{R_0e^{\mu T}}{c^2\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)\Lambda_0\right]\frac{e^{\mu T}-1}{\mu^2}\left(1+\frac{1}{T}+\frac{e^{\mu T}-1}{2}\right)\le\frac{\pi}{2}-\delta;$$

$$\left(1+\frac{e^{\mu T}-1}{\mu T}\right)\left[\bar{c}(\Phi_0+\Lambda_0)+\frac{|e_1e_2|}{m_2}\left(\frac{1}{\rho_{\min}{}^2}+\frac{\omega^2R_0}{c^3}\right)+\tilde{r}_{\max}\right]e^{\mu(t-T_k)}\le\omega(R_0+\tilde{R}_0);$$

$$\left(1+\frac{e^{\mu T}-1}{\mu T}\right)\left(\frac{\phi_0}{\Phi_0}+\frac{2R_0e^{\mu T}}{\rho_{\min}}+2\Lambda_0\tan\left(\frac{\pi}{2}-\delta\right)+\frac{|e_1e_2|}{m_2}\left(\frac{R_0e^{\mu T}}{c^2\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)\right)\Phi_0\le\omega\Phi_0;$$

$$\left(1+\frac{e^{\mu T}-1}{\mu T}\right)\left[\frac{2R_0\Lambda_0e^{\mu T}}{\rho_{\min}}+\frac{\Phi_0{}^2e^{\mu T}}{2}+\frac{|e_1e_2|}{m_2}\left(\frac{R_0e^{\mu T}}{c^2\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)\Lambda_0\right]\le\omega\Lambda_0 \text{ and so on.}$$

(For higher derivatives see **Appendix**)

Then there exists a unique solution of (20) belonging to $\tilde{M}_r\times M_\phi\times M_\eta$.

Proof: First we show that B maps the set $M_r\times M_\phi\times M_\eta$ into itself. Indeed, since $\int_{T_k}^{T_{k+1}}\left(\frac{t-T_k}{T}-\frac{1}{2}\right)dt=0$ we obtain

$$\int_{T_k}^{T_{k+1}} B_r(r,\phi,\eta)(t)\,dt = \int_{T_k}^{T_{k+1}}\int_{T_k}^{t} F_r(r,\phi,\eta)(s)ds\,dt - \int_{T_k}^{T_{k+1}}\left(\frac{t-T_k}{T}-\frac{1}{2}\right)dt\int_{T_k}^{T_{k+1}} F_r(r,\phi,\eta)(s)ds -$$
$$-T\frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta} F_r(r,\phi,\eta)(s)dsd\theta+\int_{T_k}^{T_{k+1}}\tilde{r}(s)ds = T\,\tilde{r}_0$$

and

$$B_r(r,\phi,\eta)(T_k)=\int_{T_k}^{T_k}\tilde{F}_r(r,\phi,\eta)(s)ds+\tilde{r}(T_k)-\left(\frac{T_k-T_k}{T}-\frac{1}{2}\right)\int_{T_k}^{T_{k+1}}\tilde{F}_r(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta}\tilde{F}_r(r,\phi,\eta)(s)dsd\theta=$$

$$=\frac{1}{2}\int_{T_k}^{T_{k+1}}\tilde{F}_r(r,\phi,\eta)(s)ds-\frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta}\tilde{F}_r(r,\phi,\eta)(s)dsd\theta=0$$

(cf. the proof of the Main Lemma).

We must show the inequalities for $t\in[T_k,T_{k+1}],k=k_0,k_0+1,..,k_1-1$, where $k_0<k_1$ are natural numbers. For the first component we have

$$\left|B_r(r,\phi,\eta)(t)\right|\le\left|\int_{T_k}^{t}\left[F_r(r,\phi,\eta)(s)+\tilde{r}(s)\right]ds\right|\le\left[\overline{c}(\Phi_0+\Lambda_0)+\frac{|e_1e_2|}{m_2}\left(\frac{1}{\rho_{\min}{}^2}+\frac{\omega^2R_0}{c^3}\right)+\tilde{r}_{\max}\right]\int_{T_k}^{t}e^{\mu(s-T_k)}ds\le$$

$$\le\left[\overline{c}(\Phi_0+\Lambda_0)+\frac{|e_1e_2|}{m_2}\left(\frac{1}{\rho_{\min}{}^2}+\frac{\omega^2R_0}{c^3}\right)+\tilde{r}_{\max}\right]\frac{e^{\mu(t-T_k)}-1}{\mu}\le(R_0+\tilde{R}_0)e^{\mu(t-T_k)}.$$

For the second component we have

$$\int_{T_k}^{T_{k+1}}B_\phi(r,\phi,\eta)(t)\,dt=\phi_0T+\int_{T_k}^{T_{k+1}}\int_{T_k}^{t}F_\phi(r,\phi,\eta)ds\,dt-\int_{T_k}^{T_{k+1}}\left(\frac{t-T_k}{T}-\frac{1}{2}\right)dt\int_{T_k}^{T_{k+1}}F_\phi(r,\phi,\eta)ds-\int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta}F_\phi(r,\phi,\eta)dsd\theta\ =\phi_0T$$

and

$$\left|B_\phi(r,\phi,\eta)(t)\right|\le\phi_0+\left|\int_{T_k}^{t}F_\phi(r,\phi,\eta)(s)ds\right|\le\phi_0+\left[\frac{2R_0e^{\mu T}}{\rho_{\min}}+2\Lambda_0\tan\left(\frac{\pi}{2}-\delta\right)+\frac{|e_1e_2|}{m_2}\left(\frac{R_0e^{\mu T}}{c^2\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)\right]\Phi_0\int_{T_k}^{t}e^{\mu(s-T_k)}ds\le$$

$$\le\left\{\phi_0+\left[\frac{2R_0e^{\mu T}}{\rho_{\min}}+2\Lambda_0tg\left(\frac{\pi}{2}-\delta\right)+\frac{|e_1e_2|}{m_2}\left(\frac{R_0e^{\mu T}}{c^2\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)\right]\frac{\Phi_0}{\mu}\right\}e^{\mu(t-T_k)}\le\Phi_0e^{\mu(t-T_k)}$$

and finally for the third component we obtain:

$$\int_{T_k}^{T_{k+1}}B_\eta(r,\phi,\eta)(t)\,dt=0\ ,\ B_\eta(r,\phi,\eta)(T_k)=0\ ,$$

$$\left|B_\eta(r,\phi,\eta)(t)\right|\le\left[\frac{2R_0\Lambda_0e^{\mu T}}{\rho_{\min}}+\frac{\Phi_0{}^2e^{\mu T}}{\mu^2}+\frac{|e_1e_2|}{m_2}\left(\frac{R_0e^{\mu T}}{c^2\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)\Lambda_0\right]\frac{e^{\mu(t-T_k)}}{\mu}\le e^{\mu(t-T_k)}\Lambda_0$$

and

$$\left|\int_{T_k}^{t}B_\eta(r,\phi,\eta)(s)ds\right|\le\left|\int_{T_k}^{t}\int_{T_k}^{s}F_\eta(r,\phi,\eta)dpds\right|+\left|\int_{kT}^{t}\left(\frac{s-T_k}{T}-\frac{1}{2}\right)ds\right|\left|\int_{T_k}^{T_{k+1}}F_\eta(r,\phi,\eta)ds\right|+\left|\frac{1}{T}\int_{T_k}^{T_{k+1}}\int_{T_k}^{\theta}F_\eta(r,\phi,\eta)dsd\theta\right|\int_{T_k}^{t}ds\le$$

$$\le\left[\frac{2R_0\Lambda_0e^{\mu T}}{\rho_{\min}}+\frac{\Phi_0{}^2e^{\mu T}}{2}+\frac{|e_1e_2|}{m_2}\left(\frac{R_0e^{\mu T}}{c^2\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)\Lambda_0\right]\frac{e^{\mu(t-T_k)}-1}{\mu^2}\left[1+\frac{1}{T}+\frac{e^{\mu T}-1}{2}\right]\le$$

$$\le\left[\frac{2R_0\Lambda_0e^{\mu T}}{\rho_{\min}}+\frac{\Phi_0{}^2e^{\mu T}}{2}+\frac{|e_1e_2|}{m_2}\left(\frac{R_0e^{\mu T}}{c^2\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)\Lambda_0\right]\frac{e^{\mu T}-1}{\mu^2}\left(1+\frac{1}{T}+\frac{e^{\mu T}-1}{2}\right)\le\frac{\pi}{2}-\delta.$$

Therefore $B_r(r,\phi,\eta)\in\tilde{M}_r$, $B_\phi(r,\phi,\eta)\in M_\phi$, $B_\eta(r,\phi,\eta)\in M_\eta$.

For the first derivatives we have respectively

$$\left|\frac{dB_r(r,\phi,\eta)(t)}{dt}\right| \le \left|\tilde{F}_r(r,\phi,\eta)(t)\right| + \frac{1}{T}\int_{T_k}^{T_{k+1}}\left|\tilde{F}_r(r,\phi,\eta)(s)\right|ds \le$$

$$\le \left(1+\frac{e^{\mu T}-1}{\mu T}\right)\left[\bar{c}(\Phi_0+\Lambda_0)+\frac{|e_1e_2|}{m_2}\left(\frac{1}{\rho_{\min}{}^2}+\frac{\omega^2 R_0}{c^3}\right)+\tilde{r}_{\max}\right]e^{\mu(t-T_k)} \le \omega(R_0+\tilde{R}_0)e^{\mu(t-T_k)};$$

$$\left|\frac{dB_\phi(r,\phi,\eta)(t)}{dt}\right| \le \left|F_\phi(r,\phi,\eta)(t)\right| + \frac{1}{T}\int_{T_k}^{T_{k+1}}\left|F_\phi(r,\phi,\eta)(s)\right|ds \le$$

$$\le \left(1+\frac{e^{\mu T}-1}{\mu T}\right)\left\{\phi_0+\left[\frac{2R_0e^{\mu T}}{\rho_{\min}}+2\Lambda_0\tan\left(\frac{\pi}{2}-\delta\right)+\frac{|e_1e_2|}{m_2}\left(\frac{R_0e^{\mu T}}{c^2\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)\right]\Phi_0\right\}e^{\mu(t-T_k)} \le \omega\Phi_0 e^{\mu(t-T_k)};$$

$$\left|\frac{dB_\eta(r,\phi,\eta)(t)}{dt}\right| \le \left|F_\eta(r,\phi,\eta)(t)\right| + \frac{1}{T}\int_{T_k}^{T_{k+1}}\left|F_\eta(r,\phi,\eta)(s)\right|ds \le$$

$$\le \left(1+\frac{e^{\mu T}-1}{\mu T}\right)\left[\frac{2R_0\Lambda_0e^{\mu T}}{\rho_{\min}}+\frac{\Phi_0{}^2e^{\mu T}}{2}+\frac{|e_1e_2|}{m_2}\left(\frac{R_0e^{\mu T}}{c^2\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)\Lambda_0\right]e^{\mu(t-T_k)} \le \omega\Lambda_0 e^{\mu(t-T_k)}.$$

For the higher order derivatives see **Appendix**.

We show that the operator *B* is (Φ, j) – contractive one in the sense of [14], [16].
In view of **Appendix**

$$d_{(k,0)}\left(F_r(r,\phi,\eta),F_r(\bar{r},\bar{\phi},\bar{\eta})\right) \le K_r d_{(k,0)}\left((r,\phi,\eta),(\bar{r},\bar{\phi},,\bar{\eta})\right),$$

where

$$K_r = \left[\left(\Phi_0{}^2+\Lambda_0{}^2+\frac{2|e_1e_2|}{m_2\rho_{\min}{}^3}+2\rho_{\max}\Phi_0{}^2\right)\frac{e^{\mu T}}{\mu}+\frac{e_2^2}{m_2}\left(\frac{2}{c^2}\frac{R_0{}^2e^{\mu T}}{\rho_{\min}{}^2}+\frac{\omega^2}{c^3}\right)+\Phi_0+2\rho_{\max}\Lambda_0\right]e^{\mu T}$$

and $d_{(k,0)}\left(F_\phi(r,\phi,\eta),F_\phi(\bar{r},\bar{\phi},\bar{\eta})\right) \le K_\phi d_{(k,0)}\left((r,\phi,\eta),(\bar{r},\bar{\phi},,\bar{\eta})\right)$,

where

$$K_\phi = \left(\frac{R_0\Phi_0e^{2\mu T}}{\rho_{\min}{}^2}+\frac{2|e_1e_2|}{m_2c^2}\frac{R_0\Phi_0e^{2\mu T}}{\rho_{\min}{}^3}+\frac{2\Phi_0\Lambda_0e^{2\mu T}}{\cos^2\left(\frac{\pi}{2}-\delta\right)}\right)\frac{1}{\mu}+\frac{\Phi_0e^{\mu T}}{\rho_{\min}}+\frac{|e_1e_2|}{m_2c^2}\frac{\Phi_0e^{\mu T}}{\rho_{\min}{}^2}+$$

$$+\frac{2R_0e^{\mu T}}{\rho_{\min}}+2\Lambda_0e^{\mu T}\tan\left(\frac{\pi}{2}-\delta\right)+\frac{|e_1e_2|}{m_2c^2}\frac{R_0e^{\mu T}}{\rho_{\min}{}^2}+2\Phi_0e^{\mu T}\tan\left(\frac{\pi}{2}-\delta\right)+\frac{e_2^2\omega^2}{m_2c^3}$$

and

$$d_{(k,0)}\left(F_\eta(r,\phi,\eta),F_\eta(\bar{r},\bar{\phi},\bar{\eta})\right) \le K_\eta d_{(k,0)}\left((r,\phi,\eta),(\bar{r},\bar{\phi},\bar{\eta})\right),$$

where

$$K_\eta = \left(\frac{2R_0\Lambda_0}{\rho_{\min}{}^2}+\frac{2|e_1e_2|}{m_2c^2}\frac{R_0\Lambda_0}{\rho_{\min}{}^3}+\Phi_0{}^2\right)\frac{e^{2\mu T}}{\mu}+\frac{2\Lambda_0e^{\mu T}}{\rho_{\min}}+\frac{|e_1e_2|}{m_2c^2}\frac{\Lambda_0e^{\mu T}}{\rho_{\min}{}^2}+\Phi_0e^{\mu T}+\frac{2R_0e^{\mu T}}{\rho_{\min}}+\frac{|e_1e_2|}{m_2c^2}\frac{R_0e^{\mu T}}{\rho_{\min}{}^2}+\frac{e_2^2\omega^2}{m_2c^3}.$$

The above inequalities imply

$$d_{(k,0)}\left(\left(F_r(r,\phi,\eta),F_\phi(r,\phi,\eta),F_\eta(r,\phi,\eta)\right),\left(F_r(\bar r,\bar\phi,\bar\eta),F_\phi(\bar r,\bar\phi,\bar\eta),F_\eta(\bar r,\bar\phi,\bar\eta)\right)\right)\le$$
$$\le\left(K_r+K_\phi+K_\eta\right)d_{(k,0)}\left((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta)\right).$$

If $K_r+K_\phi+K_\eta$ is not smaller than 1 we take *m* sufficiently large such that $\left(K_r+K_\phi+K_\eta\right)\frac{\omega^m}{\mu^m}<1$.

Then

$$d_{(k,0)}\left(\left(F_r(r,\phi,\eta),F_\phi(r,\phi,\eta),F_\eta(r,\phi,\eta)\right),\left(F_r(\bar r,\bar\phi,\bar\eta),F_\phi(\bar r,\bar\phi,\bar\eta),F_\eta(\bar r,\bar\phi,\bar\eta)\right)\right)\le$$
$$\le\left(K_r+K_\phi+K_\eta\right)\frac{\omega^m}{\mu^m}d_{(k,m)}\left((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta)\right).$$

For the first derivative we get

$$d_{(k,1)}\left(\left(\frac{dB_r(r,\phi,\eta)}{dt},\frac{dB_\phi(r,\phi,\eta)}{dt},\frac{dB_\eta(r,\phi,\eta)}{dt}\right),\left(\frac{dB_r(\bar r,\bar\phi,\bar\eta)}{dt},\frac{dB_\phi(\bar r,\bar\phi,\bar\eta)}{dt},\frac{dB_\eta(\bar r,\bar\phi,\bar\eta)}{dt}\right)\right)\le$$
$$\le\left(K_r^{\,1}+K_\varphi^{\,1}+K_\eta^{\,1}\right)\frac{\omega^{m_1}}{\mu^{m_1}}d_{(k,m_1)}\left((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta)\right).$$

For the second derivative $\frac{d^2B_r(k)(r,\phi,\eta)(t)}{dt^2}=\frac{dF_r(r,\phi,\eta)(t)}{dt}$, $t\in[T_k,T_{k+1}]$ we obtain

$$d_{(k,2)}\left(\left(\frac{d^2B_r(r,\phi,\eta)}{dt^2},\frac{d^2B_\phi(r,\phi,\eta)}{dt^2},\frac{d^2B_\eta(r,\phi,\eta)}{dt^2}\right),\left(\frac{d^2B_r(\bar r,\bar\phi,\bar\eta)}{dt^2},\frac{d^2B_\varphi(\bar r,\bar\phi,\bar\eta)}{dt^2},\frac{d^2B_\eta(\bar r,\bar\phi,\bar\eta)}{dt^2}\right)\right)\le$$
$$\le3\left(K_r^{\,2}+K_\phi^{\,2}+K_\eta^{\,2}\right)\frac{\omega^{m_2}}{\mu^{m_2}}d_{(k,m_2)}\left((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta)\right).$$

It is easy to see that $\frac{dF_r(r,\phi,\eta)}{dt},\frac{dF_\phi(r,\phi,\eta)}{dt},\frac{dF_\eta(r,\phi,\eta)}{dt}$ contain the third derivatives of the unknown functions r,ϕ,η and therefore their estimates contain the multiplier ω^3. But the estimates for the second derivatives contain ω^2 . Consequently, the power of ω does not increase.

Define the map $j:A\to A$ in the following way: $j(k,0)=(k,m),\quad j^2(k,0)=(k,2m),...$; $j(k,1)=(k,m_1)$, $j^2(k,1)=(k,2m_1),\dots$.

It is easy to verify that the space $M_r\times M_\phi\times M_\eta$ is *j*-bounded in the sense of [14]. Indeed,

$$\sup\left\{d_{(k,p+m_s)}((r,\phi,\eta),(\bar r,\bar\phi,\bar\eta)):p\right\}=$$
$$=\sup\left\{d_{(k,p+m_s)}(r,\bar r):m_s\right\}+\sup\left\{d_{(k,p+m_s)}(\phi,\bar\phi):m_s\right\}+\sup\left\{d_{(k,p+m_s)}(\eta,\bar\eta):m_s\right\}\le2(R_0+\tilde R_0+\Phi_0+\Upsilon_0)e^{\mu T}<\infty$$

$(s=1,2,3,...)$. Consequently, B has a fixed point which is a solution of the problem formulated- an existence of transition trajectories.

Theorem 3 is thus proved.

To show an existence of transition trajectories from exited to the ground state we consider the system

$$\dot{r} = \tilde{\tilde{F}}_r(r,\phi,\eta) = \begin{cases} \rho\phi^2\cos^2\lambda + \rho\eta^2 - \dfrac{|e_1e_2|}{m_2}\dfrac{1}{\rho^2} + \dfrac{e_2^2}{m_2c^3}\ddot{r} - \tilde{\tilde{r}},\ t\in[T_{k_0},T_{k_1}] \\ \rho\phi^2\cos^2\lambda + \rho\eta^2 - \dfrac{|e_1e_2|}{m_2}\dfrac{1}{\rho^2} + \dfrac{e_2^2}{m_2c^3}\ddot{r},\ t\in[0,\infty)\setminus[T_{k_0},T_{k_1}] \end{cases}$$

$$\dot{\phi} = \frac{-2r\phi}{\rho} + 2\phi\eta\tan\lambda + \frac{|e_1e_2|}{m_2c^2}\frac{r\phi}{\rho^2} - \frac{e_2^2}{m_2c^3}\ddot{\phi} \equiv F_\phi(r,\phi,\eta),\ \ t\in[0,\infty) \tag{23}$$

$$\dot{\eta} = \frac{-2r\eta}{\rho} - \frac{\phi^2\sin 2\lambda}{2} + \frac{|e_1e_2|}{m_2c^2}\frac{r\eta}{\rho^2} - \frac{e_2^2}{m_2c^3}\ddot{\eta} \equiv F_\eta(r,\phi,\eta)\,,\ t\in[0,\infty)\,.$$

We introduce the following function spaces:

$$\tilde{M}_r = \Bigg\{ r(.)\in C_T^{\ \infty}[0,\infty):\ \left|\frac{d^m r(t)}{dt^m}\right| \le \omega^m R_0 e^{\mu(t-T_k)};\ \int_{T_k}^{T_{k+1}} r(s)ds = 0, (k=0,1,...,k_0-1,\ k_1,k_1.+1,...);$$

$$\int_{T_k}^{T_{k+1}} r(s)ds = T\,\tilde{r}_0,;\left|\frac{d^m\left(r(t)-\tilde{r}(t)\right)}{dt^m}\right| \le \omega^m(R_0-\tilde{R}_0)e^{\mu(t-T_k)}, R_0 > \tilde{R}_0 > 0, (k=k_0,k_0+1,...,k_1-1)\Bigg\}$$

$$M_\phi = \left\{ \phi\in C_T^{\ \infty}[0,\infty):\left|\frac{d^m\phi(t)}{dt^m}\right| \le \omega^m\Phi_0 e^{\mu(t-T_k)};\ \int_{T_k}^{T_{k+1}}\phi(s)ds = T\phi_0, (k=0,1,...)\right\},$$

$$M_\eta = \left\{ \eta\in C_T^{\ \infty}[0,\infty):\left|\frac{d^m\eta(t)}{dt^m}\right| \le \omega^m\Lambda_0 e^{\mu(t-T_k)};\ \int_{T_k}^{T_{k+1}}\eta(t)dt = 0,\ \left|\int_{T_k}^{t}\eta(s)ds\right| \le \frac{\pi}{2}-\delta, (k=0,1,...)\right\},$$

$$(k,m = 0,1,2,...)\,.$$

Then the energy decreases from W_T to $\tilde{\tilde{W}}_T$:

$$\tilde{\tilde{W}}_T = \frac{e_2^{\ 2}}{c^2}\left(\int_0^T \dot{r}^2(s)ds + \left(\rho_0 + \int_0^{\xi_T}(r(p)-\tilde{\tilde{r}}(p))dp\right)^2\cos^2\lambda(\xi_T)\int_0^T\dot{\phi}^2(s)ds + \left(\rho_0 + \int_0^{\xi_T}(r(p)-\tilde{\tilde{r}}(p))dp\right)^2\int_0^T\dot{\eta}^2(s)ds\right) < W_T\,.$$

The proof repeats the above reasoning.

Lemma 17. The solution (r,ϕ,η) of system (23) belonging to $\tilde{M}_r\times M_\phi\times M_\eta$ generates transition trajectories from excited state to the ground state.

Proof: Indeed, in the interval $[0,T_{k_0}]$ the distance between particles $\rho(t)$ is *T*-periodic function and $\rho_{\min} = \rho_1 - R_0\left(e^{\mu T}-1\right)/\mu \le \rho(t) \le \rho_1 + R_0\left(e^{\mu T}-1\right)/\mu = \rho_{\max}$.

Let us calculate

$$\rho(T_{k_1}) = \rho_1 + \int_0^{T_{k_1}} r(s)ds =$$

$$= \rho_1 + \int_0^{T} r(s)ds + \int_T^{T_2} r(s)ds + ... + \int_{T_{k_0-1}}^{T_{k_0}} r(s)ds + \int_{T_{k_0}}^{T_{k_0+1}} r(s)ds + \int_{T_{k_0+1}}^{T_{k_0+2}} r(s)ds + ... + \int_{T_{k_1-2}}^{T_{k_1-1}} r(s)ds + \int_{T_{k_1-1}}^{T_{k_1}} r(s)ds =$$

$$= \rho_1 + \int_{T_{k_0}}^{T_{k_0+1}} r(s)ds - \int_{T_{k_0+1}}^{T_{k_0+2}} \tilde{r}(s)ds + ... + \int_{T_{k_1-2}}^{T_{k_1-1}} \tilde{r}(s)ds + \int_{T_{k_1-1}}^{T_{k_1}} \tilde{r}(s)ds = \rho_1 - (k_1 - k_0)T\,\tilde{r}_0.$$

Then since $\int_{T_k}^{T_{k+1}} r(s)ds = 0$, $(k = k_1, k_1+1, ...)$ we infer that the distance function $\rho(t)$ decreases in the interval $[T_{k_0}, T_{k_1}]$ from $\rho_1 + \int_0^t r(p)dp$, $t \in [0, T_{k_0}]$ to $\rho_1 - T_{k_1-k_0}\,\tilde{r}_0 + \int_{T_{k_1}}^t r(p)dp$, $t \in [T_{k_1}, \infty)$ and it is again periodic one $\rho(t) = \rho_1 - (k_1 - k_0)T\,\tilde{r} + \int_{T_{k_1}}^t r(p)dp$.

Lemma 17 is thus proved.

Numerical verification of the results obtained

The inequalities from Theorem 3 can be checked using the results from [17], [18].

For the first Bohr orbit we have $\rho_0 = 5{,}3.10^{-11} m$, $\rho_{\min} = 5.10^{-11}, \rho_{\max} \approx 5{,}4.10^{-11}$ while for the second one $\rho_1 = 2^2.5{,}4.10^{-11} m = 21{,}6.10^{-11} m$ (cf. [11]). In order to describe the transition, we use $\rho_0 + (k_1 - k_0)T = \rho_1$ and obtain $5{,}3.10^{-11} + (k_1 - k_0)T = 21{,}6.10^{-11} \Rightarrow k_1 - k_0 \cong 21{,}6.10^{-11}/T$. The last estimate shows that the particle (electron) must perform $16.10^{-11}/T$ rounds to pass from the first stationary state to the second one. Since $1 - \beta^2 \approx 1$, $\nu = 6{,}55.10^{15}, \omega = 2\pi.\nu \approx 4{,}1.10^{16}$, then $T = 1/\nu = 1/(6{,}55.10^{15}) \approx 1{,}53.10^{-16}$, $k_1 - k_0 \cong 21{,}6.10^{-11}/1{,}53.10^{-16} \approx 14.10^5$.

Considering the values (cf. [12]-[14]) $e_2^2/m_2 \approx 10^{-8}$, $c \approx 3.10^8$, and choosing $\mu = 10^{16} \Rightarrow e^{\mu T} = e^{1{,}53} \approx 4{,}62$ and $\dfrac{\phi_0}{\Phi_0} < 1$ sufficiently small we verify the inequalities from Theorem 3:

$$\left[5{,}4.10^{-11}\left(\Phi_0{}^2 + \Upsilon_0{}^2\right)4{,}62 + 10^{-8}\left(\frac{1}{(5.10^{-11})^2} + \frac{(4{,}1.10^{16})^2}{137.9.10^{16}}\right)\right]\frac{1}{10^{16}} \le R_0 + \tilde{R}_0\,;$$

$$\frac{\phi_0}{\Phi_0} + 4{,}62\left[\frac{2R_0}{5.10^{-11}} + 2\Lambda_0\, tg\left(\frac{\pi}{2} - \delta\right) + 10^{-8}\left(\frac{1}{3.10^8(5.10^{-11})^2.137} + \frac{(4{,}1.10^{16})^2}{27.10^{24}}\right)\right]\frac{1}{10^{16}} \le 1\,;$$

$$4{,}62\left[\frac{2R_0}{5.10^{-11}} + \frac{5{,}4.10^{-11}\Phi_0^2.4{,}62}{2} + 10^{-8}\left(\frac{R_0}{9.10^{16}(5.10^{-11})^2} + \frac{(4{,}1.10^{16})^2.4{,}62}{27.10^{24}}\right)\right]\frac{1}{10^{16}} \le 1\,.$$

Passing to the next stationary state can be achieved in a similar way.

2.10 Existence-Uniqueness of Escape Trajectories

To obtain escape trajectories we again must change the function spaces. Namely, we apply the radial force on the interval $[T_{k_0}, \infty)$ and introduce the spaces:

$$M_r = \left\{ r(.) \in C_T^{\infty}[0,\infty) : \left| \frac{d^m r(t)}{dt^m} \right| \le \omega^m R_0 e^{\mu(t-T_k)}; \int_{T_k}^{T_{k+1}} r(s)ds = 0, (k = 0,1,...,k_0 - 1); \left| \frac{d^m r(t)}{dt^m} \right| \le \omega^m R_0 e^{\mu(t-T_k)}, \right.$$

$$\left. \int_{T_k}^{T_{k+1}} r(s)ds = T\tilde{r}_0; \left| \frac{d^m \left(r(t) + \tilde{r}(t) \right)}{dt^m} \right| \le \omega^m (R_0 + \tilde{R}_0) e^{\mu(t-T_k)} (k = k_0, k_0 + 1, ...) \right\},$$

$$M_\phi = \left\{ \phi \in C_T^{\infty}[0,\infty) : \left| \frac{d^m \phi(t)}{dt^m} \right| \le \omega^m \Phi_0 e^{\mu(t-T_k)}; \int_{T_k}^{T_{k+1}} \phi(s)ds = T\phi_0, (k = 0,1,...) \right\},$$

$$M_\eta = \left\{ \eta \in C_T^{\infty}[0,\infty) : \left| \frac{d^m \eta(t)}{dt^m} \right| \le \omega^m \Lambda_0 e^{\mu(t-T_k)}; \int_{T_k}^{T_{k+1}} \eta(t)dt = 0, \left| \int_{T_k}^{t} \eta(s)ds \right| \le \frac{\pi}{2} - \delta, (k = 0,1,...) \right\},$$

$(k, m = 0,1,2,...)$.

One can prove as in Lemma 17 that $\rho(t) \to \infty$ as $t \to \infty$ for $t > T_{k_0}$. For the angle function $\varphi(t)$ there are various cases. If we choose the last space we obtain escape trajectories with $\varphi(t) \to \infty$. But another possible case is the function $\varphi(t)$ to be bounded one and yet $\rho(t) \to \infty$. This case is realized provided:

$$M_\phi = \left\{ \phi \in C_T^{\infty}[0,\infty) : \left| \frac{d^m \phi(t)}{dt^m} \right| \le \omega^m \Phi_0 e^{\mu(t-T_k)}; \int_{T_k}^{T_{k+1}} \phi(s)ds = 0, (k = 0,1,...) \right\}.$$

3. CONCLUSION

In the present paper we have formulated conditions for the existence of periodic orbit of particle moving around nuclei, transition from on energy level to another one and escape motion. This is performed in the 3D-Kepler formulation of the 2-body problem. In this manner we confirm N. Bohr hypothesis from 1923. We propose the main goals of the Bohr’s hypothesis from [18]. The starting point for the Bohr’s theory of the atom is the recognition of the fact that the Rutherford’s model of the atom cannot, within the framework of classical physics, explain that stability which one knows from everyday life experience to be a property of the atom. According to classical electrodynamics (at the time - author's note), a system composed of a positive nucleus with electron orbiting about it would inevitably radiate light, so that the orbits of the electrons would shrink and finally collapse into the nucleus.

In fact, we confirm the famous Bohr’s hypothesis in the framework of the present classical electrodynamics on the base of our generalized Synge’s model.

We note some papers studying similar problems with different approaches [19-24].

COMPETING INTERESTS

Author has declared that no competing interests exist.

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APPENDIX

A1. Solution Sets Are Closed

$$M_r = \left\{ r \in C_T^{\infty}[0,\infty) : \left| r^{(m)}(t) \right| \le \omega^m R_0 e^{\mu(t-kT)};\ r^{(0)}(kT) = 0;\quad \int_{kT}^{(k+1)T} r(s)ds = 0 \right\}, t \in [kT,(k+1)T];$$

$$M_\phi = \left\{ \phi \in C_T^{\infty}[0,\infty) : \left| \phi^{(m)}(t) \right| \le \omega^m \Phi_0 e^{\mu(t-kT)},\ \phi^{(0)}(kT) = \phi_0,\ \int_{kT}^{(k+1)T} \phi(s)ds = T\phi_0 \right\}, t \in [kT,(k+1)T];$$

$$M_\eta = \left\{ \eta \in C_T^{\infty}[0,\infty) : \left| \eta^{(m)}(t) \right| \le \omega^m \Upsilon_0 e^{\mu(t-kT)};\ \eta^{(0)}(kT) = 0;\quad \int_{kT}^{(k+1)T} \eta(s)ds = 0; \left| \int_{kT}^{t} \eta(s)ds \right| \le \frac{\pi}{2} - \delta \right\}$$

$$(k = 0,1,2,\ldots),(m = 0,1,2,\ldots).$$

Indeed, let $\{r_s\}_{s=1}^{\infty} \in M_r, \{\phi_s\}_{s=1}^{\infty} \in M_\phi, \{\eta_s\}_{s=1}^{\infty} \in M_\eta$ be convergent sequences. We show that their limits belong to corresponding sets, that is, $r_0 \in M_r, \phi_0 \in M_\phi, \eta_0 \in M_\eta$. We have

$$\frac{\left| r_0^{(m)}(t) \right|}{\omega^m} e^{-\mu(t-kT)} - \frac{\left| r_s^{(m)}(t) \right|}{\omega^m} e^{-\mu(t-kT)} \le \frac{\left| r_s^{(m)}(t) - r_0^{(m)}(t) \right|}{\omega^m} e^{-\mu(t-kT)} \le d_{(k,m)}(r_s, r_0) < \varepsilon.$$

It follows

$$\left| r_0^{(m)}(t) \right| < \left| r_s^{(m)}(t) \right| + \varepsilon\, \omega^m e^{\mu T} \le \omega^m R_0 e^{\mu(t-kT)} + \varepsilon\, \omega^m e^{\mu T} \Rightarrow \left| r_0^{(m)}(t) \right| \le \omega^m R_0 e^{\mu(t-kT)}$$

and then

$$\int_{kT}^{(k+1)T} r_0^{(m)}(t)dt < \int_{kT}^{(k+1)T} r_s^{(m)}(t)dt + \varepsilon\, \omega^m \frac{e^{\mu T} - 1}{\mu} \Rightarrow \int_{kT}^{(k+1)T} r_0^{(m)}(t)dt = 0;$$

$$\int_{kT}^{(k+1)T} \phi_0^{(m)}(t)dt < \int_{kT}^{(k+1)T} \phi_s^{(m)}(t)dt + \varepsilon\, \omega^m \frac{e^{\mu T} - 1}{\mu} \Rightarrow \int_{kT}^{(k+1)T} \phi_0^{(m)}(t)dt = \phi_0 T;$$

$$\left| \int_{kT}^{t} \eta_0^{(m)}(p)dp \right| < \left| \int_{kT}^{t} \eta_s^{(m)}(p)dp \right| + \varepsilon\, \omega^m \frac{e^{\mu T} - 1}{\mu} \Rightarrow \left| \int_{kT}^{t} \eta_0^{(m)}(p)dp \right| \le \frac{\pi}{2} - \delta.$$

A2. Lipschitz Estimates for the Right-Hand Sides

$$F_r(r,\varphi,\eta) = \rho\varphi^2 \cos^2\lambda + \rho\eta^2 + \frac{e_1 e_2}{c^2 m_2} \frac{(c^2 - r^2)}{\rho^2} - \frac{e_2^2}{m_2 c^2} \frac{(c^2 - \rho^2\eta^2 - \rho^2\varphi^2\cos\lambda)\ddot{r} + \varphi\rho^2 r\cos^2\lambda\,\ddot{\varphi} + \eta\rho^2 r\,\ddot{\eta}}{\Delta_2^{\,3}}$$

$$F_\varphi(r,\varphi,\eta) = \frac{-2r\varphi}{\rho} + 2\varphi\eta\, tg\lambda - \frac{e_1 e_2}{c^2 m_2} \frac{r\varphi}{\rho^2} - \frac{e_2^2}{m_2 c^2} \frac{r\varphi\ddot{r} + (\Delta_2^{\,2} + \varphi^2\rho^2\cos^2\lambda)\ddot{\varphi} + \varphi\eta\rho^2\ddot{\eta}}{\Delta_2^{\,3}};$$

$$F_\eta(r,\varphi,\eta) = \frac{-2r\eta}{\rho} - \rho\varphi^2 \sin\lambda\cos\lambda - \frac{e_1 e_2}{c^2 m_2} \frac{r\eta}{\rho} - \frac{e_2^2}{m_2 c^2} \frac{r\eta\ddot{r} + \varphi\eta\rho^2\cos^2\lambda\,\ddot{\varphi} + (\Delta_2^{\,2} + \rho^2\eta^2)\ddot{\eta}}{\Delta_2^{\,3}}.$$

Then in view of $\left| e_1 e_2 \right| / m_2 = e_2^2 / m_2$ we get

$$\frac{\partial F_r}{\partial \rho} = \varphi^2 \cos^2 \lambda + \eta^2 - 2\frac{e_1 e_2 (c^2 - r^2)}{c^2 m_2}\frac{1}{\rho^3} + \frac{e_1 e_2 (c^2 - r^2)}{c^3 m_2}\frac{\eta^2 + \varphi^2 \cos\lambda}{\rho {\Delta_2}^3} -$$

$$-\frac{2\rho e_2^2}{m_2 c^2}\frac{\left(-\eta^2 - \varphi^2 \cos\lambda\right)\ddot{r} + \ddot{\varphi}\varphi r \cos^2 \lambda + \ddot{\eta}\eta r}{c^3} -$$

$$-\frac{e_2^2}{m_2 c^2} 3 \frac{\left(c^2 - \rho^2\eta^2 - \rho^2\varphi^2 \cos\lambda\right)\ddot{r} + \ddot{\varphi}\varphi r \rho^2 \cos^2\lambda + \ddot{\eta}\eta r \rho^2}{c^5}\left(\rho\eta^2 + \rho\varphi^2 \cos\lambda\right);$$

$$\left|\frac{\partial F_r}{\partial \rho}\right| \le \frac{2e_2^2}{m_2 \hat{\rho}^3} + \left({\Phi_0}^2 + {\Upsilon_0}^2\right) e^{2\mu T}\left(1 + \frac{e_2^2}{m_2}\left(\frac{c + 4\omega^2 R_0 \hat{\rho}\breve{\rho} e^{\mu T}}{c^5 \hat{\rho}(1-\beta^2)^{5/2}} + \omega^2 \frac{3e^{\mu T}(c^2 R_0 + {\Phi_0}^2 R_0 e^{2\mu T}\breve{\rho}^2 + {\Upsilon_0}^2 R_0 e^{2\mu T}\breve{\rho}^2)}{c^7 (1-\beta^2)^{5/2}}\right)\right) \equiv \frac{\partial \overline{F}_r}{\partial \rho}.$$

In a similar way we obtain estimates for

$$\frac{\partial \overline{F}_r}{\partial r}, \frac{\partial \overline{F}_r}{\partial \phi}, \frac{\partial \overline{F}_r}{\partial \lambda}, \frac{\partial \overline{F}_r}{\partial \eta}, \frac{\partial \overline{F}_\phi}{\partial \rho}, \frac{\partial \overline{F}_\phi}{\partial r}, \frac{\partial \overline{F}_\phi}{\partial \phi}, \frac{\partial \overline{F}_\phi}{\partial \lambda}, \frac{\partial \overline{F}_\phi}{\partial \eta}, \frac{\partial \overline{F}_\eta}{\partial \rho}, \frac{\partial \overline{F}_\eta}{\partial r}, \frac{\partial \overline{F}_\eta}{\partial \phi}, \frac{\partial \overline{F}_\eta}{\partial \lambda}, \frac{\partial \overline{F}_\eta}{\partial \eta}.$$

It is easy to verify that the upper bounds of all partial derivatives contain a multiplier ω^2 because the radiation terms possess second derivatives of the unknown functions.

A3. Estimates of Time-Derivatives of the Right-Hand Sides of Basic System

In view of the Assumption **(C),**

$$r^2 + \rho^2\eta^2 + \rho^2\phi^2 \cos\lambda \le {R_0}^2 e^{2\mu T} + \rho^2 {\Upsilon_0}^2 e^{2\mu T} + \rho^2 {\Phi_0}^2 e^{2\mu T} \le \overline{c}^2 \text{ , and}$$

$$F_r(r,\phi,\eta) = \rho\phi^2 \cos^2\lambda + \rho\eta^2 + \frac{e_1 e_2}{m_2 \rho^2} - \frac{e_2^2}{m_2}\frac{c^2\ddot{r} - \rho^2\eta^2\ddot{r} - \rho^2\phi^2 \cos\lambda \ddot{r} + \phi\ddot{\phi}\rho^2 r \cos^2\lambda + \ddot{\eta}\eta\rho^2 r}{c^5} \approx$$

$$\approx \rho\phi^2 \cos^2\lambda + \rho\eta^2 + \frac{e_1 e_2}{m_2 \rho^2} - \frac{e_2^2}{m_2}\frac{c^2\ddot{r} + \phi\ddot{\phi}\rho^2 r \cos^2\lambda + \ddot{\eta}\eta\rho^2 r}{c^5}$$

we obtain

$$\left|\frac{d^2 B_r(r,\phi,\eta)(t)}{dt^2}\right| == \left|\frac{dF_r(r,\phi,\eta)(t)}{dt}\right| = \left| r\phi^2 \cos^2\lambda + 2\rho\phi\dot{\phi}\cos^2\lambda - \rho\phi^2\eta \sin 2\lambda + r\eta^2 + 2\rho\eta\dot{\eta}\right.$$

$$-\frac{2e_1 e_2}{m_2 \rho^3} r - \frac{e_2^2}{m_2}\frac{c^2\dddot{r} + \ddot{\dot{\phi}}\phi\rho^2 r \cos^2\lambda + \phi\dddot{\phi}\rho^2 r \cos^2\lambda + 2\phi\ddot{\phi}\rho r^2 \cos^2\lambda + \phi\ddot{\phi}\rho^2 \dot{r} \cos^2\lambda - 2\phi\ddot{\phi}\rho^2 r\eta \sin\lambda}{c^5} +$$

$$\left. + \frac{e_2^2}{m_2}\frac{\dddot{\eta}\eta\rho^2 r + \ddot{\eta}\dot{\eta}\rho^2 r + 2\ddot{\eta}\eta\rho r^2 + \ddot{\eta}\eta\rho^2 r^2}{c^5}\right|.$$

Therefore

$$\left|\frac{d^2 B_r(r,\phi,\eta)(t)}{dt^2}\right| = \left|\frac{dF_r(r,\varphi,\eta)(t)}{dt}\right| \le$$

$$\le e^{\mu(t-kT_k)}\overline{c}\Phi_0 + 2\overline{c}\omega\Phi_0 e^{\mu(t-kT_k)} + \overline{c}\Phi_0\Lambda_0 e^{2\mu(t-kT_k)} + \overline{c}e^{2\mu(t-kT_k)}{\Lambda_0}^2 + 2\overline{c}\omega\Lambda_0 e^{\mu(t-kT_k)} +$$

$$+ \frac{2|e_1 e_2|\overline{c}}{m_2 {\rho_{\min}}^3} + \frac{e_2^2}{m_2} e^{\mu(t-T_k)} \frac{c^2\omega^3 R_0 + 4\omega^3\overline{c}^3 + \overline{c}^3 + 2\overline{c}^3\omega^2\Phi_0 + 2\omega^2\overline{c}^3\Lambda_0 + 2\omega^2\overline{c}^3 + \omega^2\overline{c}^4}{c^5} \le \omega^2 R_0 e^{\mu(t-T_k)}.$$

It is easy to see that the inequality is always satisfied:

$$\frac{\bar{c}\Phi_0 + 2\bar{c}\omega\Phi_0 + \bar{c}\Phi_0\Lambda_0 e^{\mu T} + \bar{c}e^{\mu T}\Lambda_0{}^2 + 2\bar{c}\omega\Lambda_0}{\omega^2} + \frac{2|e_1 e_2|\bar{c}}{m_2\omega^2\rho_{\min}{}^3} + \\ + \frac{e_2^2}{m_2}\frac{c^2\omega R_0 + 4\omega\bar{c}^3 + 2\bar{c}^3\Phi_0 + 2\bar{c}^3\Lambda_0 + 2\bar{c}^3 + \bar{c}^4}{c^5} \le R_0 .$$

But in view of the values $\hat{\rho} \approx 5{,}3.10^{-11}$, $\omega \approx 2\pi.6{,}55.10^{15} \approx 4{,}1.10^{16}$, $\frac{e_2^2}{m_2} \approx 10^{-8}$, $c \approx 3.10^8$ (cf. [12] - [14]) we conclude that the last inequality might be satisfied for suitable values of the rest constants.

For the third derivative $\left|\frac{d^3 B_r(k)(r,\phi,\eta)(t)}{dt^3}\right| = \left|\frac{d^2 F_r(r,\phi,\eta)(t)}{dt^2}\right|$ and then the term $\frac{2}{\omega^2}\frac{R_0}{\hat{\rho}^3}$ becomes $\frac{2}{\omega^3}\frac{R_0}{\hat{\rho}^4} \approx \frac{1}{10^{48}}\frac{R_0}{10^{-44}}$ and so on.

For the derivatives of

$$F_\varphi(r,\varphi,\eta) = \frac{-2r\varphi}{\rho} + 2\varphi\eta\tan\lambda - \frac{e_1 e_2}{m_2}\left(\frac{r\varphi}{c^2\rho^2} + \frac{r\varphi\ddot{r} + (c^2 + \varphi^2\rho^2\cos^2\lambda)\ddot{\varphi} + \varphi\eta\rho^2\ddot{\eta}}{c^2 c^3}\right);$$

$$F_\eta(r,\varphi,\eta) = \frac{-2r\eta}{\rho} - \rho\varphi^2\sin\lambda\cos\lambda - \frac{e_1 e_2}{m_2}\left(\frac{r\eta}{c^2\rho^2} + \frac{r\eta\ddot{r} + \varphi\eta\rho^2\cos^2\lambda\ .\ddot{\varphi} + (\Delta_2{}^2 + \rho^2\eta^2)\ddot{\eta}}{c^2 c^3}\right)$$

one can make similar estimates.

We notice that the estimate of $\frac{dF_r(r,\phi,\eta)(t)}{dt}$ contains ω^3 and therefore we obtain

$$\left|\frac{d^2 B_r(k)(r,\phi,\eta)(t)}{dt^2}\right| = \left|\frac{dF_r(r,\phi,\eta)(t)}{dt}\right| \le \omega^2(\ldots)R_0 e^{\mu(t-kT)}$$

(cf. the definition of $M_r \times M_\phi \times M_\eta$).

It is obvious that in the brackets remains ω and the same is valid for the Lipschitz estimate in view of the definition of metric between the second derivatives.

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