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Hermite Positive Definite Solution of the Quaternion Matrix Equation $X^m + B^*XB = C$

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Abstract

This paper discusses the necessary and sufficient conditions for the existence of Hermite positive definite solutions of the quaternion matrix equation $X^m + B^*XB = C\,(m > 0)$ and its iterative solution method. According to the characteristics of the coefficient matrix, a corresponding algebraic equation system is ingeniously constructed, and by discussing the equation system's solvability, the matrix equation's existence interval is obtained. Based on the characteristics of the coefficient matrix, some necessary and sufficient conditions for the existence of Hermitian positive definite solutions of the matrix equation are derived. Then, the upper and lower bounds of the positive actual solutions are estimated by using matrix inequalities. Four iteration formats are constructed according to the given conditions and existence intervals, and their convergence is proven. The selection method for the initial matrix is also provided. Finally, using the complexification operator of quaternion matrices, an equivalent iteration on the complex field is established to solve the equation in the Matlab environment. Two numerical examples are used to test the effectiveness and feasibility of the given method.

Keywords

Quaternion, Matrix Equation, Hermite Positive Definite Solution, Matrix Inequality, Iterative, Convergence

1. Introduction

System theory, stochastic control, and differential methods for solving elliptic partial differential equations frequently involve the class of integer-order nonlinear equations known as algebraic Riccati matrix equations [1] [2] [3] [4].

In recent years, many scholars have extended Riccati matrix equations to matrix equations of non-integer order or symmetric structure type, and have achieved rich results. For example, [5]-[13] studied various iterative methods for Hermite positive definite solutions of nonlinear matrix equations of the form $X^s \pm A^* X^{-q} A = Q$ or $X + A^* X^{-s} A \pm B^* X^{-q} B = Q$ for different values of the unknown matrix indices q, s; in [14] [15] [16] [17], the existence and perturbation analysis of Hermite positive definite solutions of matrix equation $X \pm A^* X^{-q} A = Q$ are given. [18] [19] discussed the extremal solutions of matrix equation $X^s + A^* X^{-q} A = I(Q)$ and the upper and lower bounds of solutions. However, the equations mentioned above are instead discussed in the real number field or the complex number field. Research on the solutions of quaternion nonlinear matrix equations is uncommon [20], particularly for some structural solutions of non-integer order quaternion matrix equations, there is currently no relevant research report.

This paper focuses on the quaternion matrix equation

$$X^m + B^* X B = C \tag{1}$$

study its Hermite positive definite solution, where $m > 0$ is a positive number, $B, C \in \mathbf{Q}^{n\times n}$ and $C > \mathbf{0}$ (Hermite positive definite, hereafter referred to as positive definite) is the known quaternion matrix, $X \in \mathbf{Q}^{n\times n}$ is an unknown matrix. To discuss convenience, let $\overline{A}$ and A^* note the conjugate, conjugate transpose and respectively of the quaternion matrix A. The matrix A is classified as Hermite or self-conjugate if $A = A^*$. All n-order Hermite matrices are classified as $\mathbf{SC}_n(\mathbf{Q})$. The $\lambda_{\max}(A), \lambda_{\min}(A)$ are used to represent the maximum and minimum eigenvalues of the Hermite matrix A, $\| A \| = [\mathrm{tr}(A^* A)]^{1/2}$ respectively denotes the Frobenius norm of the quaternion matrix A. For n-order positive definite matrices A, B, the mean by $A > B$ that $A - B$ is positive definite. For quaternion matrix

$$A = A_1 + A_2 \mathrm{j} \in \mathbf{Q}^{m\times n}$$

We call

$$A^\sigma = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix} \in \mathbf{C}^{2m\times 2n}$$

is the complexization operator of A [21]. The complexification operator concerning a quaternionic matrix A has the following operational properties [22]

$$(A \pm B)^\sigma = A^\sigma \pm B^\sigma, (AB)^\sigma = A^\sigma B^\sigma, (A^{-1})^\sigma = (A^\sigma)^{-1}, \| A^\sigma \| = \sqrt{2} \| A \|.$$

2. Existence of Positive Definite Solutions

This section first discusses some necessary and sufficient conditions for the existence of Hermite positive definite solutions to the quaternion matrix Equation (1). For the coefficient matrix in (1), denote

$$\kappa_u = \lambda_{\max}(B^* B) \ge 0,\quad \kappa_l = \lambda_{\min}(B^* B) \ge 0,\quad \lambda_u = \lambda_{\max}(C),\quad \lambda_l = \lambda_{\min}(C) \tag{2}$$

then, there are the following conclusions.

Theorem 1. Let $m > 0$, $B \in \mathbf{Q}^{n\times n}$ is a nonsingular matrix, $C \in \mathbf{Q}^{n\times n}$ is a positive definite matrix. The real numbers $\kappa_u, \kappa_l, \lambda_u, \lambda_l$ are given by (2), if the following system of algebraic equations

$$\begin{cases} \alpha^m + \kappa_u \beta = \lambda_l \\ \beta^m + \kappa_l \alpha = \lambda_u \end{cases} \tag{3}$$

for α, β has positive real pairs of solutions $\alpha, \beta \in \mathrm{R}^+$, then there must exist a Hermite positive definite solution to the matrix Equation (1).

Proof. Let $\alpha, \beta > 0$ be a positive real pair solution to the system of algebraic Equation (3). The existence of Hermite positive definite solutions of matrix Equation (1) is discussed below in three cases.

Case 1: $\alpha = \beta$. It can be obtained from (3)

$$\alpha(\kappa_u - \kappa_l) = \lambda_l - \lambda_u \tag{4}$$

Because of $\lambda_l \le \lambda_u, \kappa_u \ge \kappa_l$, therefore, Equation (4) holds if and only if

$$\lambda_l = \lambda_u, \kappa_l = \kappa_u \tag{5}$$

According to (5) and the properties of self-conjugate quaternion matrices, it follows that

$$\boldsymbol{B}^* \boldsymbol{B} = \kappa_l \boldsymbol{I}, \quad \boldsymbol{C} = \lambda_l \boldsymbol{I} \tag{6}$$

Then, $\boldsymbol{X} = \alpha \boldsymbol{I}$ is a Hermite positive definite solution of (1).

Case 2: $\alpha < \beta$. At this point, write that $\Upsilon = [\alpha I, \beta I]$, then Υ is a nonempty bounded closed convex set. $\forall \boldsymbol{X} \in \Upsilon$, we obtain

$$\alpha \boldsymbol{I} \le \boldsymbol{X} \le \beta \boldsymbol{I}$$

$$\Rightarrow \alpha \boldsymbol{B}^* \boldsymbol{B} \le \boldsymbol{B}^* \boldsymbol{X} \boldsymbol{B} \le \beta \boldsymbol{B}^* \boldsymbol{B} \Rightarrow \kappa_l \alpha \boldsymbol{I} \le \boldsymbol{B}^* \boldsymbol{X} \boldsymbol{B} \le \kappa_u \beta \boldsymbol{I} \tag{7}$$

Thus, from (7) and in connection with (3), we have

$$\begin{cases} \boldsymbol{C} - \boldsymbol{B}^* \boldsymbol{X} \boldsymbol{B} \ge \boldsymbol{C} - \kappa_u \beta \boldsymbol{I} \ge \lambda_l \boldsymbol{I} - \kappa_u \beta \boldsymbol{I} = (\lambda_l - \kappa_u \beta) \boldsymbol{I} = \alpha^m \boldsymbol{I} \\ \boldsymbol{C} - \boldsymbol{B}^* \boldsymbol{X} \boldsymbol{B} \le \boldsymbol{C} - \kappa_l \alpha \boldsymbol{I} \le \lambda_u \boldsymbol{I} - \kappa_l \alpha \boldsymbol{I} = (\lambda_u - \kappa_l \alpha) \boldsymbol{I} = \beta^m \boldsymbol{I} \end{cases} \tag{8}$$

Therefore, a matrix function can be constructed on Υ

$$F(\boldsymbol{X}) = \sqrt[m]{\boldsymbol{C} - \boldsymbol{B}^* \boldsymbol{X} \boldsymbol{B}} \tag{9}$$

This $F(\boldsymbol{X})$ is continuous Υ, and by (8) we get

$$\alpha \boldsymbol{I} \le F(\boldsymbol{X}) = \sqrt[m]{\boldsymbol{C} - \boldsymbol{B}^* \boldsymbol{X} \boldsymbol{B}} \le \beta \boldsymbol{I} \tag{10}$$

Thus $F(\boldsymbol{X}) \subseteq \Upsilon$, according to the Brouwer fixed point theorem, $F(\boldsymbol{X})$ must have a fixed point on Υ, that is, the matrix Equation (1) has a Hermite positive definite solution.

Case 3: $\alpha > \beta$. At this point, note that $\Omega = [\beta I, \alpha I]$, then Ω is a nonempty bounded closed convex set. $\forall \boldsymbol{X} \in \Omega$, we obtain

$$\beta \boldsymbol{I} \le \boldsymbol{X} \le \alpha \boldsymbol{I}$$

$$\Rightarrow \beta^m \boldsymbol{I} \le \boldsymbol{X}^{\boldsymbol{m}} \le \alpha^m \boldsymbol{I} \Rightarrow \boldsymbol{C} - \alpha^m \boldsymbol{I} \le \boldsymbol{C} - \boldsymbol{X}^{\boldsymbol{m}} \le \boldsymbol{C} - \beta^m \boldsymbol{I} \tag{11}$$

Thus from (11) and in connection with (3) we gain

$$\begin{cases} \boldsymbol{C} - \boldsymbol{X}^{\boldsymbol{m}} \le \boldsymbol{C} - \beta^m \boldsymbol{I} \le \lambda_u \boldsymbol{I} - \beta^m \boldsymbol{I} = (\lambda_u - \beta^m) \boldsymbol{I} = \kappa_l \alpha \boldsymbol{I} \\ \boldsymbol{C} - \boldsymbol{X}^{\boldsymbol{m}} \ge \boldsymbol{C} - \alpha^m \boldsymbol{I} \ge \lambda_l \boldsymbol{I} - \alpha^m \boldsymbol{I} = (\lambda_l - \alpha^m) \boldsymbol{I} = \kappa_u \beta \boldsymbol{I} \end{cases} \tag{12}$$

Again by $\boldsymbol{B}$ nonsingular and (12)

$$\begin{cases}(\boldsymbol{B}^{-1})^*(\boldsymbol{C}-\boldsymbol{X}^m)\boldsymbol{B}^{-1}\le \kappa_l\alpha(\boldsymbol{B}^{-1})^*\boldsymbol{B}^{-1}\le \kappa_l\alpha\kappa_l^{-1}\boldsymbol{I}=\alpha\boldsymbol{I}\\(\boldsymbol{B}^{-1})^*(\boldsymbol{C}-\boldsymbol{X}^m)\boldsymbol{B}^{-1}\ge \kappa_u\beta(\boldsymbol{B}^{-1})^*\boldsymbol{B}^{-1}\ge \kappa_u\beta\kappa_u^{-1}\boldsymbol{I}=\beta\boldsymbol{I}\end{cases}\tag{13}$$

Therefore, a matrix function

$$G(\boldsymbol{X})=(\boldsymbol{B}^{-1})^*(\boldsymbol{C}-\boldsymbol{X}^m)\boldsymbol{B}^{-1}\tag{14}$$

can be constructed on Ω.

This $G(\boldsymbol{X})$ is continuous Ω, and by (13) we get

$$\beta\boldsymbol{I}\le G(\boldsymbol{X})=(\boldsymbol{B}^{-1})^*(\boldsymbol{C}-\boldsymbol{X}^m)\boldsymbol{B}^{-1}\le\alpha\boldsymbol{I}$$

Thus $G(\boldsymbol{X})\subseteq\Omega$, by the Brouwer fixed point theorem, $G(\boldsymbol{X})$ must have a fixed point on Ω, that is, the matrix Equation (1) has a Hermite positive definite solution.

The following discussion further explores the conditions under which the real number $\kappa_u,\kappa_l,\lambda_u,\lambda_l$ in Theorem 1 satisfies, leading to the existence of a solution α,β for positive real number pairs in an algebraic equation system (3). In this regard, the following results are given.

Corollary 1. If the real number $\kappa_u,\kappa_l,\lambda_u,\lambda_l$ in (2) satisfies one of the following conditions

$$\text{(a)}\ \ \kappa_u<\frac{\lambda_l}{\sqrt[m]{\lambda_u}};\ \ \text{(b)}\ \ \kappa_l>\frac{\lambda_u}{\sqrt[m]{\lambda_l}};$$

then, the system of algebraic Equation (3) has positive real pair solutions.

Proof. (a) It is easy to know that the system of Equation (3) is equivalent to the following system of equations.

$$\begin{cases}\beta=\kappa_u^{-1}(\lambda_l-\alpha^m)\\\kappa_u^{-m}(\lambda_l-\alpha^m)^m+\kappa_l\alpha-\lambda_u=0\end{cases}\tag{15}$$

we write

$$f(\alpha)=\kappa_u^{-m}(\lambda_l-\alpha^m)^m+\kappa_l\alpha-\lambda_u\tag{16}$$

It is obvious that $f(\alpha)$ is continuous in $[0,\sqrt[m]{\lambda_l}]$. When the condition (a) is satisfied, we can obtain

$$\lambda_l^m-\kappa_u^m\lambda_u>0\Rightarrow\kappa_u\sqrt[m]{\lambda_u}<\lambda_l.$$

So we get

$$\begin{cases}f(0)=\kappa_u^{-m}\lambda_l^m-\lambda_u=\kappa_u^{-m}(\lambda_l^m-\kappa_u^m\lambda_u)>0\\f(\sqrt[m]{\lambda_l})=\kappa_l\sqrt[m]{\lambda_l}-\lambda_u\le\kappa_u\sqrt[m]{\lambda_u}-\lambda_u<\lambda_l-\lambda_u\le0\end{cases}\tag{17}$$

Therefore, a positive real number $\alpha^*\in(0,\sqrt[m]{\lambda_l})$ exists such that $f(\alpha^*)=0$ and $\beta^*=\kappa_u^{-}[\lambda_l-(\alpha^*)^m]>0$, that is, the system of algebraic Equation (3) has a positive real pair solution in $\hat{\alpha},\hat{\beta}\in\mathrm{R}^+$.

(b) When this condition holds, it follows that

$$\kappa_l\sqrt[m]{\lambda_l}>\lambda_u,\ \ \kappa_u^m\lambda_u\ge\kappa_l^m\lambda_l>\lambda_u^m\ge\lambda_l^m$$

Therefore, in the closed interval $[0,\sqrt[m]{\lambda_l}]$, the function (16) is also used to obtain

$$\begin{cases} f(0) = \kappa_u^{-m}\lambda_l^m - \lambda_u = \kappa_u^{-m}(\lambda_l^m - \kappa_u^m\lambda_u) < 0 \\ f(\sqrt[m]{\lambda_l}) = \kappa_l\sqrt[m]{\lambda_l} - \lambda_u > 0 \end{cases} \tag{18}$$

So, the system of algebraic Equation (3) has a positive real pair solution $\hat{\alpha}, \hat{\beta} \in \mathrm{R}^+$ in $[0, \sqrt[m]{\lambda_l}]$.

In Equation (1), when $\boldsymbol{C} = \boldsymbol{I}$ (identity matrix), $\boldsymbol{B} = \boldsymbol{U}$ (quaternion unitary matrix), we obtain the following corollary.

Corollary 2. Let $\boldsymbol{U} \in \mathbf{Q}^{n\times n}$ be a quaternion unitary matrix, then $\forall m > 0$, the matrix equation $\boldsymbol{X}^m + \boldsymbol{U}^*\boldsymbol{X}\boldsymbol{U} = \boldsymbol{I}$ always exists Hermite positive definite solutions.

Proof When $\boldsymbol{C} = \boldsymbol{I}$, $\boldsymbol{B} = \boldsymbol{U}$, $\kappa_u = \kappa_l = \lambda_u = \lambda_l = 1$ in (2), then, the system of Equation (3) becomes

$$\begin{cases} \alpha^m + \beta = 1 \\ \beta^m + \alpha = 1 \end{cases} \tag{19}$$

It is easy to know that the system of Equation (19) has a positive real number pair solution $\hat{\alpha}, \hat{\beta} \in \mathrm{R}^+$ in $(0,1)$. Based on Theorem 1, the conclusion holds. Using the symmetry of the system of Equation (19), it is known that $\hat{\alpha} = \hat{\beta}$ is the unique solution of the equation $\alpha^m + \alpha - 1 = 0$ in $(0,1)$, it follows that $\boldsymbol{X} = \hat{\alpha}\boldsymbol{I}$ is a positive definite solution of the given equation.

Corollary 3. Let $\boldsymbol{B} \in \mathbf{Q}^{n\times n}, \kappa_u = \lambda_{\max}(\boldsymbol{B}^*\boldsymbol{B}), \kappa_l = \lambda_{\min}(\boldsymbol{B}^*\boldsymbol{B}), m > 0$, If κ_u, κ_l satisfies

$$(\kappa_u^{-m} - 1)(\kappa_l - 1) < 0. \tag{20}$$

Then, the matrix equation $\boldsymbol{X}^m + \boldsymbol{B}^*\boldsymbol{X}\boldsymbol{B} = \boldsymbol{I}$ exists a Hermite positive definite solution.

Proof. When $\boldsymbol{C} = \boldsymbol{I}$, $\lambda_u = \lambda_l = 1$ in (2), hence the system of Equation (3) becomes

$$\begin{cases} \alpha^m + \kappa_u\beta = 1 \\ \beta^m + \kappa_l\alpha = 1 \end{cases} \Leftrightarrow \begin{cases} \beta = \kappa_u^{-1}(1-\alpha^m) \\ \kappa_u^{-m}(1-\alpha^m)^m + \kappa_l\alpha - 1 = 0 \end{cases} \tag{21}$$

It is evident that the function $g(\alpha) = \kappa_u^{-m}(1-\alpha^m)^m + \kappa_l\alpha - 1$ is continuous on the interval $[0,1]$, and when condition (20) holds, we gain

$$g(0)g(1) = (\kappa_u^{-m} - 1)(\kappa_l - 1) < 0.$$

So $g(\alpha)$ has a real root $\hat{\alpha} > 0$ in $(0,1)$, and thus $\hat{\beta} = \kappa_u^{-1}(1-\hat{\alpha}^m) > 0$, It follows from Theorem 1 that, the given equation has a Hermite positive definite solution.

Corollary 4. Let $\boldsymbol{B} \in \mathbf{Q}^{n\times n}, \boldsymbol{C} = \boldsymbol{I}, m > 0$, then the necessary and sufficient condition for the matrix Equation (1) to have a positive definite solution $\boldsymbol{W}$ is that the matrix $\boldsymbol{B}$ has the following decomposition

$$\boldsymbol{B} = \boldsymbol{W}^{-1/2}\boldsymbol{Z} \tag{22}$$

where $\begin{pmatrix} \boldsymbol{W}^{m/2} \\ \boldsymbol{Z} \end{pmatrix}$ is a column unitary orthogonal matrix.

Proof. (Necessity) If $\boldsymbol{W}$ is a positive definite solution of the equation $\boldsymbol{X}^m+\boldsymbol{B}^*\boldsymbol{X}\boldsymbol{B}=\boldsymbol{I}$, such that

$$\begin{aligned}\boldsymbol{W}^m+\boldsymbol{B}^*\boldsymbol{W}\boldsymbol{B}=\boldsymbol{I}\ &\Rightarrow (\boldsymbol{W}^*)^{m/2}\boldsymbol{W}^{m/2}+\boldsymbol{B}^*(\boldsymbol{W}^*)^{1/2}\boldsymbol{W}^{1/2}\boldsymbol{B}=\boldsymbol{I}\\ &\Rightarrow \begin{pmatrix}\boldsymbol{W}^{m/2}\\ \boldsymbol{W}^{1/2}\boldsymbol{B}\end{pmatrix}^*\begin{pmatrix}\boldsymbol{W}^{m/2}\\ \boldsymbol{W}^{1/2}\boldsymbol{B}\end{pmatrix}=\boldsymbol{I}.\end{aligned}\tag{23}$$

We define $\boldsymbol{Z}=\boldsymbol{W}^{1/2}\boldsymbol{B}$, then $\boldsymbol{B}=\boldsymbol{W}^{-1/2}\boldsymbol{Z}$, from (23), we know that $\begin{pmatrix}\boldsymbol{W}^{m/2}\\ \boldsymbol{Z}\end{pmatrix}$ is a column unitary orthogonal matrix.

(Sufficiency) If the matrix $\boldsymbol{B}$ has the decomposition (22), where $\boldsymbol{W}$ is a positive definite matrix and

$$\begin{pmatrix}\boldsymbol{W}^{m/2}\\ \boldsymbol{Z}\end{pmatrix}^*\begin{pmatrix}\boldsymbol{W}^{m/2}\\ \boldsymbol{Z}\end{pmatrix}=\boldsymbol{I}.$$

Substituting $\boldsymbol{X}=\boldsymbol{W}$ into the left side of Equation (1), we get

$$\begin{aligned}\boldsymbol{X}^m+\boldsymbol{B}^*\boldsymbol{X}\boldsymbol{B}&=(\boldsymbol{W}^*)^{m/2}\boldsymbol{W}^{m/2}+\boldsymbol{B}^*(\boldsymbol{W}^*)^{1/2}\boldsymbol{W}^{1/2}\boldsymbol{B}\\ &=(\boldsymbol{W}^*)^{m/2}\boldsymbol{W}^{m/2}+\boldsymbol{Z}^*(\boldsymbol{W}^*)^{-1/2}(\boldsymbol{W}^*)^{1/2}\boldsymbol{W}^{1/2}\boldsymbol{W}^{-1/2}\boldsymbol{Z}.\\ &=(\boldsymbol{W}^*)^{m/2}\boldsymbol{W}^{m/2}+\boldsymbol{Z}^*\boldsymbol{Z}=\boldsymbol{I}\end{aligned}\tag{24}$$

It can be seen that $\boldsymbol{X}=\boldsymbol{W}$ is a positive definite solution to the equation $\boldsymbol{X}^m+\boldsymbol{B}^*\boldsymbol{X}\boldsymbol{B}=\boldsymbol{I}$.

The upper and lower bound estimates for the positive definite solution of Equation (1) are given below.

Theorem 2. Let Equation (1) have a positive definite solution $\boldsymbol{X}$, and $\boldsymbol{C}\geq\boldsymbol{B}^*\boldsymbol{C}^{1/m}\boldsymbol{B}$, $(m>0)$ the

$$(\boldsymbol{C}-\boldsymbol{B}^*\boldsymbol{C}^{1/m}\boldsymbol{B})^{1/m}\leq\boldsymbol{X}\leq\boldsymbol{C}^{1/m}.\tag{25}$$

Proof. Let $\boldsymbol{X}$ be a positive definite solution of Equation (1), such that

$$\boldsymbol{X}^m=\boldsymbol{C}-\boldsymbol{B}^*\boldsymbol{X}\boldsymbol{B}\leq\boldsymbol{C}\ \Rightarrow\boldsymbol{X}\leq\boldsymbol{C}^{1/m}.\tag{26}$$

Furthermore, we have $\boldsymbol{X}\leq\boldsymbol{C}^{1/m}\Rightarrow\boldsymbol{B}^*\boldsymbol{X}\boldsymbol{B}\leq\boldsymbol{B}^*\boldsymbol{C}^{1/m}\boldsymbol{B}$, therefore

$$\boldsymbol{X}^m=\boldsymbol{C}-\boldsymbol{B}^*\boldsymbol{X}\boldsymbol{B}\geq\boldsymbol{C}-\boldsymbol{B}^*\boldsymbol{C}^{1/m}\boldsymbol{B}\ \Rightarrow\boldsymbol{X}\geq(\boldsymbol{C}-\boldsymbol{B}^*\boldsymbol{C}^{1/m}\boldsymbol{B})^{1/m}\tag{27}$$

It can be seen from (26) and (27) that (25) is established.

Corollary 5. If the real numbers in (2) satisfy $\lambda_l>\kappa_u\lambda_u^{1/m}$, then Equation (1) must have a positive definite solution $\boldsymbol{X}$, and

$$(\lambda_l-\kappa_u\lambda_u^{1/m})^{1/m}\boldsymbol{I}\leq\boldsymbol{X}\leq\lambda_u^{1/m}\boldsymbol{I}.\tag{28}$$

Proof. By $\lambda_l>\kappa_u\lambda_u^{1/m}$, based on Theorem 1 and Corollary 1(a), Equation (1) exists a positive definite solution $\boldsymbol{X}$. Then by the proof method of Theorem 2, we obtained

$$\boldsymbol{X}\leq\boldsymbol{C}^{1/m}\leq\lambda_u^{1/m}\boldsymbol{I}\tag{29}$$

and there

$$\begin{aligned} X &\geq (C - B^* C^{1/m} B)^{1/m} \geq (C - \lambda_u^{1/m} B^* B)^{1/m} \\ &\geq (C - \kappa_u \lambda_u^{1/m} I)^{1/m} \geq (\lambda_l I - \kappa_u \lambda_u^{1/m} I)^{1/m} = (\lambda_l - \kappa_u \lambda_u^{1/m})^{1/m} I \end{aligned} \tag{30}$$

It can be seen from (29) and (30) that (28) is established.

Theorem 3. Let Equation (1) exist positive definite solutions X, $B \in \mathbf{Q}^{n \times n}$ nonsingular and $C \leq B^* C^{1/m} B (m > 0)$, then

$$(B^{-1})^* [C - ((B^{-1})^* C B^{-1})^m] B^{-1} \leq X \leq (B^{-1})^* C B^{-1}. \tag{31}$$

Proof. Let X be a positive definite solution of Equation (1). Then by $B \in \mathbf{Q}^{n \times n}$ non-singularity, we gain

$$X = (B^{-1})^* C B^{-1} - (B^{-1})^* X^m B^{-1} \leq (B^{-1})^* C B^{-1} \tag{32}$$

Moreover, from the condition $C \leq B^* C^{1/m} B (m > 0)$, so $C - ((B^{-1})^* C B^{-1})^m \geq 0$, it can be obtained from (32)

$$X^m \leq ((B^{-1})^* C B^{-1})^m$$

thus have

$$\begin{aligned} X &= (B^{-1})^* C B^{-1} - (B^{-1})^* X^m B^{-1} \\ &\geq (B^{-1})^* C B^{-1} - (B^{-1})^* ((B^{-1})^* C B^{-1})^m B^{-1} = (B^{-1})^* [C - ((B^{-1})^* C B^{-1})^m] B^{-1} \end{aligned} \tag{33}$$

From (32) and (33), it can be seen that (31) is established.

Corollary 6. Under the condition of Theorem 3, if the real number in (2) satisfy $\lambda_u < \kappa_l \lambda_l^{1/m}$, then Equation (1) must have a positive definite solution X, and

$$(\lambda_l - (\lambda_u \kappa_l^{-1})^m) \kappa_u^{-1} I \leq X \leq \lambda_u \kappa_l^{-1} I. \tag{34}$$

Proof. By $\lambda_u < \kappa_l \lambda_l^{1/m}$, According to Theorem 1 and Corollary 1, there exists a positive definite solution X to Equation (1). Then by the proof method of Theorem 3, we get

$$X \leq (B^{-1})^* C B^{-1} \leq \lambda_u (B^{-1})^* B^{-1} = \lambda_u (B B^*)^{-1} \leq \lambda_u \kappa_l^{-1} I \tag{35}$$

therefore, there is

$$((B^{-1})^* C B^{-1})^m \leq (\lambda_u \kappa_l^{-1})^m I$$

then

$$\begin{aligned} X &\geq (B^{-1})^* [C - ((B^{-1})^* C B^{-1})^m] B^{-1} \\ &\geq (B^{-1})^* [\lambda_l I - (\lambda_u \kappa_l^{-1})^m I] B^{-1} \geq (\lambda_l - (\lambda_u \kappa_l^{-1})^m)(B B^*)^{-1} \geq (\lambda_l - (\lambda_u \kappa_l^{-1})^m) \kappa_u^{-1} I \end{aligned} \tag{36}$$

According to (35) and (36), (34) holds.

3. Iterative Method of Positive Definite Solution

It follows from the results presented in Section 2 that, when the coefficient matrices B, C of Equation (1) satisfy the given conditions, we can construct the positive definite solution iteration scheme of Equation (1).

(I) Under the condition of Theorem 1, when $\alpha \leq \beta$, the iterative scheme is established as follows:

$$\begin{cases} X_{k+1} = (C - B^* X_k B)^{1/m} \\ X_0 = \alpha I_n \end{cases} \Leftrightarrow \begin{cases} X_{k+1}^\sigma = [C^\sigma - (B^*)^\sigma (X_k)^\sigma B^\sigma]^{1/m} \\ X_0^\sigma = \alpha I_{2n} \end{cases} \tag{37}$$

(II) Under the condition of Theorem 1, when $\alpha > \beta$, the iterative scheme is established as follows:

$$\begin{cases} X_{k+1} = (B^{-1})^* (C - X_k^m) B^{-1} \\ X_0 = \alpha I_n \end{cases} \Leftrightarrow \begin{cases} X_{k+1}^\sigma = (B^{-*})^\sigma (C^\sigma - (X_k^m)^\sigma)(B^{-1})^\sigma \\ X_0^\sigma = \alpha I_{2n} \end{cases} \tag{38}$$

(III) Under the condition of Theorem 2 and Corollary 5, when $\lambda_l > \kappa_u \lambda_u^{1/m}$, the iterative scheme is established as follows:

$$\begin{cases} X_{k+1} = (C - B^* X_k B)^{1/m} \\ X_0 = C^{1/m} \end{cases} \Leftrightarrow \begin{cases} X_{k+1}^\sigma = [C^\sigma - (B^*)^\sigma (X_k)^\sigma B^\sigma]^{1/m} \\ X_0^\sigma = (C^\sigma)^{1/m} \end{cases} \tag{39}$$

(IV) Under the condition of Theorem 3 and Corollary 6, when $\lambda_u < \kappa_l \lambda_l^{1/m}$, the iterative scheme is established as follows:

$$\begin{cases} X_{k+1} = (B^{-1})^* (C - X_k^m) B^{-1} \\ X_0 = (B^{-1})^* C B^{-1} \end{cases} \Leftrightarrow \begin{cases} X_{k+1}^\sigma = (B^{-*})^\sigma (C^\sigma - (X_k^m)^\sigma)(B^{-1})^\sigma \\ X_0^\sigma = (B^{-*})^\sigma C^\sigma (B^{-1})^\sigma \end{cases} \tag{40}$$

where $(\bullet)^\sigma$ denotes the complex operator of the quaternion matrix $(\bullet)$. In the actual calculation, due to the non-commutative reason of quaternion multiplication, we use the right iterative format above (37), (38), (39), (40) to calculate in the Matlab software. Finally, the kth approximate solution X_k is reduced back to $X_k = X_{k1} + X_{k2}\mathrm{j}$, which is the approximate solution of Equation (1). According to the relation between a quaternion matrix and the Frobenius norm of its complex representation matrix, the residual term norm of the first approximate solution of Equation (1) is

$$R_k = \left\| X_k^m + B^* X_k B - C \right\| = \frac{1}{\sqrt{2}} \left\| (X_k^m)^\sigma + (B^*)^\sigma X_k^\sigma B^\sigma - C^\sigma \right\|.$$

Note: The selection of the initial matrix of the iteration.

(I) From Theorem 2 and Corollary 5, when the condition $C \geq B^* C^{1/m} B (m > 0)$ or the real numbers in (2) satisfy $\lambda_l > \kappa_u \lambda_u^{1/m}$, the initial matrix can be selected as $X_0 = C^{1/m}$, or $X_0 = (\lambda_l - \kappa_u \lambda_u^{1/m})^{1/m} I$. At this time, it is not necessary to solve α, β in Equation (3).

(II) From Theorem 3 and Corollary 6, when the condition $B \in \mathbf{Q}^{n \times n}$ is nonsingular, and the real numbers in $C \leq B^* C^{1/m} B (m > 0)$ or (2) satisfy $\lambda_u < \kappa_l \lambda_l^{1/m}$, the initial matrix can be selected as $X_0 = (B^{-1})^* C B^{-1}$, or $X_0 = (\lambda_l - (\lambda_u \kappa_l^{-1})^m) \kappa_u^{-1} I$. At this time, it is also not necessary to solve α, β in Equation (3).

4. Numerical Examples

This section mainly focuses on the characteristics of the coefficient matrix of Equation (1), combined with the iterative formula constructed in Section 3. It uses numerical examples to illustrate the effectiveness and feasibility of the results in this paper. Firstly, based on the results of Theorem 1, two examples are used to illustrate that the iterative formula can be effectively used to calculate the positive definite solution of the equation when the equation satisfies the given conditions.

Example 4.1. Let $m = 2.5$, considering the quaternion matrix Equation (1), two n-order quaternion matrices are given as follows:

$$\boldsymbol{B} = \begin{pmatrix} 0.1 & 0.03\mathrm{i}+0.01\mathrm{k} & & \\ -0.03\mathrm{i}-0.01\mathrm{k} & 0.1 & \ddots & \\ & \ddots & \ddots & 0.03\mathrm{i}+0.01\mathrm{k} \\ & & -0.03\mathrm{i}-0.01\mathrm{k} & 0.1 \end{pmatrix},$$

$$\boldsymbol{C} = \begin{pmatrix} 2 & 0.3\mathrm{i}+0.02\mathrm{k} & & \\ -0.3\mathrm{i}-0.02\mathrm{k} & 2 & \ddots & \\ & \ddots & \ddots & 0.3\mathrm{i}+0.02\mathrm{k} \\ & & -0.3\mathrm{i}-0.02\mathrm{k} & 2 \end{pmatrix}$$

Establish an appropriate iterative scheme to find the positive definite solution of Equation (1)

Solution: From the complex decomposition formula $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2\mathrm{j}, \boldsymbol{C} = \boldsymbol{C}_1 + \boldsymbol{C}_2\mathrm{j}$ of the quaternion matrix, we can get

$$\boldsymbol{B}_1 = \begin{pmatrix} 0.1 & 0.03\mathrm{i} & & \\ -0.03\mathrm{i} & 0.1 & \ddots & \\ & \ddots & \ddots & 0.03\mathrm{i} \\ & & -0.03\mathrm{i} & 0.1 \end{pmatrix}, \quad \boldsymbol{B}_2 = \begin{pmatrix} 0 & -0.01\mathrm{i} & & \\ 0.01\mathrm{i} & 0 & \ddots & \\ & \ddots & \ddots & -0.01\mathrm{i} \\ & & 0.01\mathrm{i} & 0 \end{pmatrix},$$

$$\boldsymbol{C}_1 = \begin{pmatrix} 2 & 0.3\mathrm{i} & & \\ 0.3\mathrm{i} & 2 & \ddots & \\ & \ddots & \ddots & 0.3\mathrm{i} \\ & & -0.3\mathrm{i} & 2 \end{pmatrix}, \quad \boldsymbol{C}_2 = \begin{pmatrix} 0 & -0.02\mathrm{i} & & \\ 0.02\mathrm{i} & 0 & \ddots & \\ & \ddots & \ddots & -0.02\mathrm{i} \\ & & 0.02\mathrm{i} & 0 \end{pmatrix},$$

Direct calculation shows that

$$\lambda_{\min}(C) \doteq 1.4792, \lambda_{\max}(C) \doteq 2.5208,$$
$$\lambda_{\min}(B^*B) \doteq 0.002, \lambda_{\max}(B^*B) \doteq 0.024, \alpha \doteq 0.56796104, \beta \doteq 1.12941416,$$

Then, using Theorem 1, Equation (1) has a positive definite solution, which is solved by iteration (37). when $n = 100, 500, 1000$, The results of the iterations are shown in **Table 1**.

Example 4.2. Let $m = 0.5$, Considering the quaternion matrix Equation (1), two n-order quaternion matrices are given as follows:

Table 1. Iterative calculation results under different matrix orders

Order of matrix n	Iterative calculation results under different matrix orders		
	iter	$\tilde{R}_k$	$t(s)$
100	12	1.0752e−12	0.2660
500	17	2.9932e−12	6.5690
1000	22	5.0875e−12	40.0360

$$B=\begin{pmatrix} 0.1 & 0.07\mathrm{i}+0.01\mathrm{k} & & \\ -0.07\mathrm{i}-0.01\mathrm{k} & 0.1 & \ddots & \\ & \ddots & \ddots & 0.07\mathrm{i}+0.01\mathrm{k} \\ & & -0.07\mathrm{i}-0.01\mathrm{k} & 0.1 \end{pmatrix},$$

$$C=\begin{pmatrix} 0.6 & 0.06\mathrm{i}-0.05\mathrm{k} & & \\ -0.06\mathrm{i}+0.05k & 0.6 & \ddots & \\ & \ddots & \ddots & 0.06\mathrm{i}-0.05\mathrm{k} \\ & & -0.06\mathrm{i}+0.05\mathrm{k} & 0.6 \end{pmatrix}$$

Solution: From the complex decomposition formula $\boldsymbol{B}=\boldsymbol{B}_1+\boldsymbol{B}_2\mathrm{j}, \boldsymbol{C}=\boldsymbol{C}_1+\boldsymbol{C}_2\mathrm{j}$ of the quaternion matrix, we can get

$$\boldsymbol{B}_1=\begin{pmatrix} 0.1 & 0.07\mathrm{i} & & \\ -0.07\mathrm{i} & 0.1 & \ddots & \\ & \ddots & \ddots & 0.07\mathrm{i} \\ & & -0.07\mathrm{i} & 0.1 \end{pmatrix}, \quad \boldsymbol{B}_2=\begin{pmatrix} 0 & -0.01\mathrm{i} & & \\ 0.01\mathrm{i} & 0 & \ddots & \\ & \ddots & \ddots & -0.01\mathrm{i} \\ & & 0.01\mathrm{i} & 0 \end{pmatrix},$$

$$\boldsymbol{C}_1=\begin{pmatrix} 0.6 & 0.06\mathrm{i} & & \\ -0.06\mathrm{i} & 0.6 & \ddots & \\ & \ddots & \ddots & 0.06\mathrm{i} \\ & & -0.06\mathrm{i} & 0.6 \end{pmatrix}, \quad \boldsymbol{C}_2=\begin{pmatrix} 0 & 0.05\mathrm{i} & & \\ -0.05\mathrm{i} & 0 & \ddots & \\ & \ddots & \ddots & 0.05\mathrm{i} \\ & & -0.05\mathrm{i} & 0 \end{pmatrix}$$

Direct calculation shows that

$$\lambda_l=\lambda_{\min}(\boldsymbol{C})\doteq 0.4647, \lambda_u=\lambda_{\max}(\boldsymbol{C})\doteq 0.7353,$$
$$\kappa_l=\lambda_{\min}(\boldsymbol{B}^*\boldsymbol{B})\doteq 0.0005, \kappa_u=\lambda_{\max}(\boldsymbol{B}^*\boldsymbol{B})\doteq 0.0495$$

As inequality $\lambda_l>\kappa_u\lambda_u^{1/m}$ holds, From Theorem 2, an Equation (1) has a positive definite solution, which is solved by iteration (39). when $n=100,500,1000$, The results of the iterations are shown in **Table 2**.

From the comparison of the above two examples, it can be seen that when the inequality $\lambda_l>\kappa_u\lambda_u^{1/m}$ holds, the corresponding iteration has a smaller CPU running time.

5. Conclusion

The criteria for the existence of Hermite positive definite solutions of a class of nonlinear matrix Equation (1) on the quaternion field and the iterative solution method are given. The maximum and minimum eigenvalues of matrices $\boldsymbol{B}^*\boldsymbol{B}$ and $\boldsymbol{C}$ are mainly used to create the system of algebraic Equation (3) and

Table 2. Iterative calculation results under different matrix orders

Order of matrix n	Iterative calculation results under different matrix orders		
	iter	$\tilde{R}_k$	$t(s)$
100	8	1.9397e−10	0.0100
500	8	4.4266e−10	1.0740
1000	9	4.8133e−11	7.2520

through discussion, the existence of its positive real number solutions, Obtaining some necessary and sufficient conditions for the existence of Hermite positive definite solutions of matrix equations, thus, the existence interval of the solution and the upper and lower bounds estimation formula of the solution are obtained. At the same time, according to the size relationship of the positive real solution α, β of the equation group (3), the iterative (37), (38), (39), (40) of convergence are constructed respectively. Two numerical examples verify the effectiveness and feasibility of the given iteration. It is concluded that when the eigenvalues satisfy the inequality $\lambda_l > \kappa_u \lambda_u^{1/m}$, the corresponding iterative formula has a higher convergence speed. The results can be effectively used to judge and calculate the Hermitian positive definite solution of the quaternion matrix Equation (1), which extends the problem of solving non-integer order matrix equations in complex fields.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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