Journal of Advances in Mathematics and Computer Science

Volume 39, Issue 2, Page 11-19, 2024; Article no.JAMCS.112216
ISSN: 2456-9968
(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

On unique Fixed-Point Theorems via Interpolative and Rational Contractions in Super Metric Spaces

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2024/v39i21865

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/112216

Original Research Article

Received: 01/12/2023
Accepted: 05/02/2024
Published: 13/02/2024

Abstract

In the present research paper, Kannan, Reich, and Dass-Gupta-type contractions are defined and discussed in the framework of super metric space. Further, some fixed-point results are proved using the notion of interpolation.

Keywords: Fixed point; iterative methods; interpolative contraction; rational contraction; super metric space.

2010 Mathematics Subject Classification: 47H10, 54H25.

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1 Introduction

One of the appealing areas of nonlinear functional analysis is metric fixed-point theory. In light of Banach's pioneering fixed-point theorem, several findings and publications on the topic have been made during the past 100 years. Essentially, there are two widely accepted theories about how to advance the metric fixed point: the first is changing (weakening) the constraints on the contraction mapping, and the second is altering the abstract structure. Metric spaces have already seen several generalizations and extensions. These include the quasi-metric space, the b-metric space, the symmetric space, the fuzzy metric space, the dislocated metric space, the partial metric space, the 2-metric space, the modular metric space, the cone metric space, the ultra-metric space, and a variety of other combinations of these.

It is important to note that the fixed-point theory is very practical and helpful in finding solutions to numerous issues in a variety of industries. As a result, this topic has been the subject of extensive research, the findings of which have been disseminated in the form of articles and books. These discoveries highlight the fact that the fixed-point theory is congested and constrained. As an illustration, most of the results for cone metric spaces are equivalent to the comparable results when standard metric space is used. Regarding the G-metric space, the same result may be drawn. The Banach contraction principle was then widely generalized in the literature see [1-18]. Both pure and applied mathematics make extensive use of it. Kannan [14] defined a new variation of this theory in 1968 and eliminated the continuity condition from it. Kannan fixed-point theorem is the first significant variant of the outstanding result of Banach on the metric fixed-point theory. Kannan's theorem has been generalized in different ways. In the present note, we zoom in on one of the recent generalizations that was proposed by Karapınar [4] as interpolative Kannan-type contraction. It was indicated in Karapinar [4] that each interpolative Kannan-type contraction in a complete metric space admits a fixed point. Erdal Karapinar and Andreea Fulga [5] introduced super-metric space. We were able to derive some fixed-point theorems in this structure, and we believe that this method could assist in alleviating the congestion and squeezing problems noted before.

We prove some fixed-point theorems for interpolative contraction and interpolative rational contraction in super metric space. Our findings extend the contractions of the metric space to a super metric space by Kannan's contraction, Riech's contraction and Dass-Gupta's rational contraction.

2 Preliminaries

We begin this section with the definition of the super metric.

Definition 2.1 (see [5]) Let $\mathfrak{D}$ is a non-empty set. We say that a function $\eta: \mathfrak{D} \times \mathfrak{D} \to [0, +\infty)$ is a super metric if it satisfies the following axioms:

(s1). $\forall\ \sigma, \varsigma \in \mathfrak{D}$, if $\eta(\sigma, \varsigma) = 0$, *then* $\sigma = \varsigma$.
(s2). $\forall\ \sigma, \varsigma \in \mathfrak{D}, \eta(\sigma, \varsigma) = \eta(\varsigma, \sigma)$.
(s3). *There exists* $s \geq 1$ *such that for every* $\varsigma \in \mathfrak{D}$, *there exist distinct sequences* $\{\sigma_n\}, \{\varsigma_n\} \subset \mathfrak{D}$, *with* $\eta(\sigma_n, \varsigma_n) \to 0$ *when* $n \to \infty$, *such that*

$$\lim_{n\to\infty} \sup \eta(\varsigma_n, \varsigma) \leq s \lim_{n\to\infty} \sup \eta(\sigma_n, \varsigma)$$

The tripled $(\mathfrak{D}, \eta, s)$ is called a super metric space.

Definition 2.2 (see [5]) On a super metric space $(\mathfrak{D}, \eta, s)$, a sequence $\{\sigma_n\}$:

(i). converges to σ in $\mathfrak{D}$ if and only *if* $\lim_{n\to\infty} \eta(\sigma_n, \sigma) = 0$.
(ii). is a Cauchy sequence in $\mathfrak{D}$ if and only if $\lim_{n\to\infty} \sup\{\eta(\sigma_n, \sigma_\sigma) : \sigma > n\} = 0$.

Proposition 2.3 (see [5]) On a super metric space, the limit of a convergent sequence is unique.

Definition 2.4 (see [5]) We say that a super metric space $(\mathfrak{D}, \eta, s)$ is complete if and only if every Cauchy sequence is convergent in $\mathfrak{D}$.

Example 2.5 (see [5]) Let the set $\mathfrak{D} = \mathbb{R}, s = 2$, *and* $\eta: \mathfrak{D} \times \mathfrak{D} \to [0, +\infty)$ be an application defined as follows*:*

$\eta(\sigma, \varsigma) = (\sigma - \varsigma)^2$, for $\sigma, \varsigma \in \mathbb{R} \backslash \{1\}$
$\eta(1, \varsigma) = \eta(\varsigma, 1) = (1 - \varsigma^3)^2$, for $\varsigma \in \mathbb{R}$.

Then, the tripled $(\mathfrak{D}, \eta, s)$ forms a super metric space.

Example 2.6 (see [5]) *Let the set* $\mathfrak{D} = [0, +\infty]$ *and* $\eta: \mathfrak{D} \times \mathfrak{D} \to [0, +\infty)$ *be a function, defined as follows:*

$\eta(\sigma, \varsigma) = \frac{|\sigma\varsigma - 1|}{\sigma + \varsigma + 1}$, *for* $\sigma, \varsigma \in [0,1) \cup (1, +\infty], \sigma \neq \varsigma$,

$\eta(\sigma, \varsigma) = 0$, *for* $\sigma, \varsigma \in [0, +\infty), \sigma = \varsigma$,

$\eta(\sigma, 1) = \eta(1, \sigma) = |\sigma - 1|$, *for* $\sigma \in [0, +\infty]$.

We can easily see that η forms a super metric on $\mathfrak{D}$.

Proposition 2.7 (see [5]) *Let* $\Gamma: \mathfrak{D} \to \mathfrak{D}$ be an asymptotically regular mapping on a complete super metric space $(\mathfrak{D}, \eta, s)$. Then, the Picard iteration $\{\Gamma^n \sigma\}$ for the initial point $\sigma \in \mathbb{R}$ is a convergent sequence on $\mathfrak{D}$.

Theorem 2.8 (see [5]) Let $(\mathfrak{D}, \eta, s)$ be a complete super-metric space and let $\Gamma: \mathfrak{D} \to \mathfrak{D}$ be a mapping. Suppose that $0 < \alpha < 1$ such that

$$\eta(\Gamma\sigma, \Gamma\varsigma) \leq \eta(\sigma, \varsigma)$$

for all $(\sigma, \varsigma) \in \mathfrak{D}$. Then Γ has a unique fixed point in $\mathfrak{D}$.

Theorem 2.9 (see [5]) Let (D, η, s) be a complete super metric space and $\Gamma: D \to D$ be a mapping, such that there exist $\alpha \in [0, 1)$ and that

$$\eta(\Gamma\sigma, \Gamma\varsigma) \leq k \max\left\{\eta(\sigma, \varsigma), \frac{\eta(\sigma, \Gamma\sigma)\eta(\varsigma, \Gamma\varsigma)}{\eta(\sigma, \varsigma) + 1}\right\}$$

Then, Γ has a unique fixed point.

3 Main Results

We start with the following definition.

Definition 3.1 Let $(\mathfrak{D}, \eta, s)$ be a *super* metric space and $\Gamma: \mathfrak{D} \to \mathfrak{D}$ a self-map. Then Γ is called a (λ, α)-interpolative Kannan contraction, if there exist $\lambda \in [0,1), \alpha \in (0,1)$ such that

$$\eta(\Gamma\sigma, \Gamma\varsigma) \leq \lambda\big(\eta(\sigma, \Gamma\sigma)\big)^{\alpha}\big(\eta(\varsigma, \Gamma\varsigma)\big)^{1-\alpha} \tag{3.1}$$

for all $\sigma, \varsigma \in \mathfrak{D}$, with $\sigma \neq \varsigma$.

Definition 3.2 Let $(\mathfrak{D}, \eta, s)$ be a *super* metric space and $\Gamma: \mathfrak{D} \to \mathfrak{D}$ a self-map. Then Γ is called a (λ, α, β)-interpolative Kannan contraction, if there exist $\lambda \in [0,1), \alpha, \beta \in (0,1), \alpha + \beta < 1$ such that

$$\eta(\Gamma\sigma, \Gamma\varsigma) \leq \lambda\big(\eta(\sigma, \Gamma\sigma)\big)^{\alpha}\big(\eta(\varsigma, \Gamma\varsigma)\big)^{\beta} \tag{3.2}$$

for all $\sigma, \varsigma \in \mathfrak{D}$, with $\sigma \neq \varsigma$.

Definition 3.3 Let $(\mathfrak{D}, \eta, s)$ be a *super* metric space and $\Gamma: \mathfrak{D} \to \mathfrak{D}$ a self-map. Then Γ is called a $(\lambda, \alpha, \beta, \gamma)$-interpolative Reich contraction, if there exist $\lambda \in [0,1), \alpha, \beta, \gamma \in (0,1)$, $\alpha + \beta + \gamma < 1$ such that

$$\eta(\Gamma\sigma, \Gamma\varsigma) \leq \lambda\big(\eta(\sigma, \varsigma)\big)^{\alpha}\big(\eta(\sigma, \Gamma\sigma)\big)^{\beta}\big(\eta(\varsigma, \Gamma\varsigma)\big)^{\gamma} \tag{3.3}$$

for all $\sigma, \varsigma \in \mathfrak{D}$, with $\sigma \neq \varsigma$.

Definition 3.4 Let $(\mathfrak{D}, \eta, s)$ be a *super* metric space and $\Gamma: \mathfrak{D} \to \mathfrak{D}$ a self-map. Then Γ is called a (λ, α, β)-interpolative Dass-Gupta rational contraction, if there exist $\lambda \in [0,1), \alpha, \beta \in (0,1)$, $\alpha + \beta < 1$ such that

$$\eta(\Gamma\sigma, \Gamma\varsigma) \leq \lambda\big(\eta(\sigma, \varsigma)\big)^{\alpha}\left(\frac{[1+\eta(\sigma,\Gamma\sigma)]\eta(\varsigma,\Gamma\varsigma)}{1+\eta(\sigma,\varsigma)}\right)^{\beta} \tag{3.4}$$

for all $\sigma, \varsigma \in \mathfrak{D}$, with $\sigma \neq \varsigma$.

Our first result as follows.

Theorem 3.5 Let $(\mathfrak{D}, \eta, s)$ be a complete *super* metric space and $\Gamma: \mathfrak{D} \to \mathfrak{D}$ be a (λ, α)-interpolative Kannan contraction. Then, Γ has a unique fixed point.

Proof Let $\sigma_0 \in \mathfrak{D}$ and let $\Gamma\sigma_0 = \sigma_1$. If $\sigma_0 = \sigma_1$ then σ_1 is the fixed point and the proof is completed. Henceforward, assume that $\sigma_0 \neq \sigma_1$. Thus, $\eta(\sigma_0, \sigma_1) > 0$. Thus, without loss of generality, for each nonnegative integer n, we can define

$$\sigma_{n+1} = \Gamma\sigma_n \tag{3.5}$$

such that $\sigma_{n+1} \neq \Gamma\sigma_n$. So $\eta(\sigma_n, \sigma_{n+1}) > 0$, for all $n \in \mathbb{N}$. From inequality (3.1), we have

$$\begin{aligned}\eta(\sigma_n, \sigma_{n+1}) &= \eta(\Gamma\sigma_{n-1}, \Gamma\sigma_n)\\ &\leq \lambda\big(\eta(\sigma_{n-1}, \Gamma\sigma_{n-1})\big)^{\alpha}\big(\eta(\sigma_n, \Gamma\sigma_n)\big)^{1-\alpha}\\ &= \lambda\big(\eta(\sigma_{n-1}, \sigma_n)\big)^{\alpha}\big(\eta(\sigma_n, \sigma_{n+1})\big)^{1-\alpha}.\end{aligned}$$

Thus,

$$\big(\eta(\sigma_n, \sigma_{n+1})\big)^{\alpha} \leq \lambda\big(\eta(\sigma_{n-1}, \sigma_n)\big)^{\alpha}$$

Therefore, the above inequality gives,

$$\eta(\sigma_n, \sigma_{n+1}) \leq \lambda^{\frac{1}{\alpha}}\eta(\sigma_{n-1}, \sigma_n) \leq \lambda\eta(\sigma_{n-1}, \sigma_n) \tag{3.6}$$

So, inequality (3.6) implies that

$$\eta(\sigma_n, \sigma_{n+1}) \leq \lambda\eta(\sigma_{n-1}, \sigma_n) \leq \lambda^2\eta(\sigma_{n-1}, \sigma_n) \leq \cdots \leq \lambda^n\eta(\sigma_0, \sigma_1) \tag{3.7}$$

Taking the limit n tends to infinity in inequality (3.7), we get

$$\lim_{n\to\infty} \eta(\sigma_n, \sigma_{n+1}) = 0. \tag{3.8}$$

In what follows, we want to show that the sequence $\{\sigma_n\}$ is a Cauchy sequence. Now suppose that, $m, n \in N$ with $m > n$. If $\sigma_n = \sigma_m$, then $\Gamma^m\sigma_0 = \Gamma^n\sigma_0$. This implies that $\Gamma^{m-n}(\Gamma^n\sigma_0) = \Gamma^n\sigma_0$. Thus, we have $\Gamma^n\sigma_0$ is the fixed point of Γ^{m-n}. Also,

$$\Gamma\big(\Gamma^{m-n}(\Gamma^n\sigma_0)\big) = \Gamma^{m-n}\big(\Gamma(\Gamma^n\sigma_0)\big) = \Gamma(\Gamma^n\sigma_0) \tag{3.9}$$

It means that, $\Gamma(\Gamma^n\sigma_0)$ is the fixed point of Γ^{m-n}. Thus, $\Gamma(\Gamma^n\sigma_0) = \Gamma^n\sigma_0$. So, $\Gamma^n\sigma_0$ is the fixed point of Γ. Now, suppose that $\sigma_n \neq \sigma_m$. Then from inequality (3.8) and using (s3), we get

$$\limsup_{n\to\infty} \eta(\sigma_n, \sigma_{n+2}) \leq s \limsup_{n\to\infty} \eta(\sigma_{n+1}, \sigma_{n+2}) \leq s \limsup_{n\to\infty}\{\lambda^{n+1}\eta(\sigma_0, \sigma_1)\} = 0. \tag{3.10}$$

Hence, $\limsup_{n\to\infty} \eta(\sigma_n, \sigma_{n+2}) = 0$. Similarly, we have

$$\limsup_{n\to\infty} \eta(\sigma_n, \sigma_{n+3}) \leq s \limsup_{n\to\infty} \eta(\sigma_{n+2}, \sigma_{n+3}) \leq s \limsup_{n\to\infty}\{\lambda^{n+2}\eta(\sigma_0, \sigma_1)\} = 0. \tag{3.11}$$

Inductively, one can conclude that $\limsup_{n\to\infty}\{\eta(\sigma_n, \sigma_m): m > n\} = 0$. Thus $\{\sigma_n\}$ is a Cauchy sequence in a complete super metric space $(\mathfrak{D}, \eta, s)$, the sequence $\{\sigma_n\}$ converges to $\sigma^* \in \mathfrak{D}$. We claim that σ^* is the fixed point of Γ. On the contrary, assume $\eta(\sigma^*, \Gamma\sigma^*) > 0$. Note that

$$\begin{aligned}\eta(\sigma_{n+1}, \Gamma\sigma^*) &= \eta(\Gamma\sigma_n, \Gamma\sigma^*)\\ &\leq \lambda\big(\eta(\sigma^*, \Gamma\sigma^*)\big)^{\alpha}\big(\eta(\sigma_n, \Gamma\sigma_n)\big)^{1-\alpha}\\ &= \lambda\big(\eta(\sigma^*, \Gamma\sigma^*)\big)^{\alpha}\big(\eta(\sigma_n, \sigma_{n+1})\big)^{1-\alpha} \to 0 \quad \text{as } n \to 0.\end{aligned} \tag{3.12}$$

Thus, $\eta(\sigma_{n+1}, \Gamma\sigma^*) = 0$. If there is $N > 0$ such that for all $n > N$ such that $\sigma_{N+1} = \sigma^*$, then we can conclude that $\eta(\sigma^*, \Gamma\sigma^*) = 0$ and so σ^* is the fixed point for $\Gamma\sigma^*$. Otherwise, suppose that for all $n \in \mathbb{N}, \sigma_n \neq \sigma^*$. Thus, we have,

$$\eta(\sigma^*, \Gamma\sigma^*) \leq \limsup_{n\to\infty} \eta(\sigma_{n+1}, \Gamma\sigma^*) \tag{3.13}$$

and one can conclude that $\eta(\sigma^*, \Gamma\sigma^*) = 0$, which is a contradiction. Hence $\sigma^* = \Gamma\sigma^*$ is the fixed point of Γ in $\mathfrak{D}$. We shall now prove the uniqueness of the fixed point. If $\varsigma^* \in \mathfrak{D}$ is another fixed point of Γ, that is, $\Gamma\varsigma^* = \varsigma^*$, then we get

$$\eta(\sigma^*, \varsigma^*) = \eta(\Gamma\sigma^*, \Gamma\varsigma^*) \leq \lambda\big(\eta(\sigma^*, \Gamma\sigma^*)\big)^{\alpha}\big(\eta(\varsigma^*, \Gamma\varsigma^*)\big)^{1-\alpha} \leq 0$$

which is a contradiction, and hence, $\sigma^* = \varsigma^*$.

Theorem 3.6 Let $(\mathfrak{D}, \eta, s)$ be a complete *super* metric space and $\Gamma: \mathfrak{D} \to \mathfrak{D}$ be a (λ, α, β)-interpolative Kannan contraction. Then Γ has a unique fixed point.

Proof Following the steps of proof of Theorem 3.5, we construct the sequence $\{\sigma_n\}$ by iterating

$$\sigma_{n+1} = \Gamma\,\sigma_n, \forall\, n \in N,$$

where $\sigma_0 \in \mathfrak{D}$ is arbitrary starting point. Then, by (3.2), we have

$$\begin{aligned}\eta(\sigma_n, \sigma_{n+1}) &= \eta(\Gamma\sigma_{n-1}, \Gamma\sigma_n)\\ &\leq \lambda\big(\eta(\sigma_{n-1}, \Gamma\sigma_{n-1})\big)^{\alpha}\big(\eta(\sigma_n, \Gamma\sigma_n)\big)^{\beta}\\ &= \lambda\big(\eta(\sigma_{n-1}, \sigma_n)\big)^{\alpha}\big(\eta(\sigma_n, \sigma_{n+1})\big)^{\beta}\end{aligned}$$

Since $\alpha < 1 - \beta$, the last inequality gives

$$\eta(\sigma_n, \sigma_{n+1})^{1-\beta} \leq \lambda\eta(\sigma_{n-1}, \sigma_n)^{\alpha} \leq \lambda\eta(\sigma_{n-1}, \sigma_n)^{1-\beta} \tag{3.14}$$

Hence

$$\eta(\sigma_n, \sigma_{n+1}) \leq \lambda^{\frac{1}{1-\beta}}\eta(\sigma_{n-1}, \sigma_n) \leq \lambda\eta(\sigma_{n-1}, \sigma_n)$$

and then

$$\eta(\sigma_n, \sigma_{n+1}) \leq \lambda^n\eta(\sigma_0, \sigma_1) \tag{3.15}$$

and in taking the limit from the above inequality, we get

$$\lim_{n\to\infty} \eta(\sigma_n, \sigma_{n+1}) = 0. \tag{3.16}$$

As already elaborated in the proof of Theorem 3.5, the classical procedure leads to the existence of a fixed-point $\sigma^* \in \mathfrak{D}$. Now, we prove the uniqueness of σ^*. If $\varsigma^* \in \mathfrak{D}$ is another fixed point of Γ, that is, $\Gamma\varsigma^* = \varsigma^*$, then from (3.2), we get

$$\eta(\sigma^*, \varsigma^*) = \eta(\Gamma\sigma^*, \Gamma\varsigma^*) \leq \lambda\big(\eta(\sigma^*, \Gamma\sigma^*)\big)^{\alpha}\big(\eta(\varsigma^*, \Gamma\varsigma^*)\big)^{\beta} \leq 0 \tag{3.17}$$

This yields that $\sigma^* = \varsigma^*$. This completes the proof.

Theorem 3.7 Let $(\mathfrak{D}, \eta, s)$ be a complete *super* metric space and $\Gamma: \mathfrak{D} \to \mathfrak{D}$ be a $(\lambda, \alpha, \beta, \gamma)$-interpolative Reich contraction. Then Γ has a unique fixed point.

Proof Following the steps of proof of Theorem 3.5, we construct the sequence $\{\sigma_n\}$ by iterating

$$\sigma_{n+1} = \Gamma\, \sigma_n, \forall\, n \in \mathbb{N},$$

where $\sigma_0 \in \mathfrak{D}$ is arbitrary starting point. Then, by (3.3), we have

$$\begin{aligned}
\eta(\sigma_n, \sigma_{n+1}) &= \eta(\Gamma\sigma_{n-1}, \Gamma\sigma_n) \\
&\leq \lambda\big(\eta(\sigma_{n-1}, \sigma_n)\big)^{\alpha}\big(\eta(\sigma_{n-1}, \Gamma\sigma_{n-1})\big)^{\beta}\big(\eta(\sigma_n, \Gamma\sigma_n)\big)^{\gamma} \\
&= \lambda\big(\eta(\sigma_{n-1}, \sigma_n)\big)^{\alpha+\beta}\big(\eta(\sigma_n, \sigma_{n+1})\big)^{\gamma}
\end{aligned}$$

Since $\alpha + \beta < 1 - \gamma$, the last inequality gives

$$\big(\eta(\sigma_n, \sigma_{n+1})\big)^{1-\gamma} \leq \lambda\big(\eta(\sigma_{n-1}, \sigma_n)\big)^{\alpha+\beta} \leq \lambda\big(\eta(\sigma_{n-1}, \sigma_n)\big)^{1-\gamma} \tag{3.18}$$

Hence

$$\eta(\sigma_n, \sigma_{n+1}) \leq \lambda^{\frac{1}{1-\gamma}}\eta(\sigma_{n-1}, \sigma_n) \leq \lambda\eta(\sigma_{n-1}, \sigma_n)$$

and then

$$\eta(\sigma_n, \sigma_{n+1}) \leq \lambda^n\eta(\sigma_0, \sigma_1) \tag{3.19}$$

and in taking the limit from the above inequality, we get

$$\lim_{n\to\infty} \eta(\sigma_n, \sigma_{n+1}) = 0. \tag{3.20}$$

As already elaborated in the proof of Theorem 3.5, the classical procedure leads to the existence of a fixed-point $\sigma^* \in \mathfrak{D}$. Now, we prove the uniqueness of σ^*. If $\varsigma^* \in \mathfrak{D}$ is another fixed point of Γ, that is, $\Gamma\varsigma^* = \varsigma^*$, then from (3.3), we get

$$\begin{aligned} \eta(\sigma^*, \varsigma^*) &= \eta(\Gamma\sigma^*, \Gamma\varsigma^*) \\ &\leq \lambda\left(\eta(\sigma^*, \varsigma^*)\right)^{\alpha} \left(\eta(\sigma^*, \Gamma\sigma^*)\right)^{\beta} \left(\eta(\varsigma^*, \Gamma\varsigma^*)\right)^{\gamma} \leq 0 \end{aligned} \tag{3.21}$$

This yields that $\sigma^* = \varsigma^*$. This completes the proof.

Theorem 3.8 Let $(\mathfrak{D}, \eta, s)$ be a complete *super* metric space and $\Gamma: \mathfrak{D} \to \mathfrak{D}$ be a (λ, α, β)-interpolative Dass-Gupta rational contraction. Then Γ has a unique fixed point.

Proof Following the steps of proof of Theorem 3.5, we construct the sequence $\{\sigma_n\}$ by iterating

$$\sigma_{n+1} = \Gamma\, \sigma_n, \forall\, n \in \mathbb{N},$$

where $\sigma_0 \in \mathfrak{D}$ is arbitrary starting point. Then, by (3.4), we have

$$\begin{aligned} \eta(\sigma_n, \sigma_{n+1}) &= \eta(\Gamma\sigma_{n-1}, \Gamma\sigma_n) \\ &\leq \lambda\left(\eta(\sigma_{n-1}, \sigma_n)\right)^{\alpha} \left(\frac{[1+\eta(\sigma_{n-1},\Gamma\sigma_{n-1})]\eta(\sigma_n,\Gamma\sigma_n)}{1+\eta(\sigma_{n-1},\sigma_n)}\right)^{\beta} \\ &\leq \lambda\left(\eta(\sigma_{n-1}, \sigma_n)\right)^{\alpha} \left(\frac{[1+\eta(\sigma_{n-1},\sigma_n)]\eta(\sigma_n,\sigma_{n+1})}{1+\eta(\sigma_{n-1},\sigma_n)}\right)^{\beta} \\ &= \lambda\left(\eta(\sigma_{n-1}, \sigma_n)\right)^{\alpha} \left(\eta(\sigma_n, \sigma_{n+1})\right)^{\beta} \end{aligned}$$

Since $\alpha + \beta < 1$, the last inequality gives

$$\left(\eta(\sigma_n, \sigma_{n+1})\right)^{1-\beta} \leq \lambda\left(\eta(\sigma_{n-1}, \sigma_n)\right)^{\alpha} \leq \lambda\left(\eta(\sigma_{n-1}, \sigma_n)\right)^{1-\beta}$$

i.e. $$\eta(\sigma_n, \sigma_{n+1}) \leq \lambda^{\frac{1}{1-\beta}} \eta(\sigma_{n-1}, \sigma_n) \leq \lambda\eta(\sigma_{n-1}, \sigma_n)$$

and then

$$\eta(\sigma_n, \sigma_{n+1}) \leq \lambda^n \eta(\sigma_0, \sigma_1) \tag{3.22}$$

and in taking the limit from the above inequality, we get

$$\lim_{n\to\infty} \eta(\sigma_n, \sigma_{n+1}) = 0. \tag{3.23}$$

As already elaborated in the proof of Theorem 3.5, the classical procedure leads to the existence of a fixed-point $\sigma^* \in \mathfrak{D}$. Now, we prove the uniqueness of σ^*. If $\varsigma^* \in \mathfrak{D}$ is another fixed point of Γ, that is, $\Gamma\varsigma^* = \varsigma^*$, then from (3.4), we get

$$\begin{aligned}\eta(\sigma^*,\varsigma^*) &= \eta(\Gamma\sigma^*,\Gamma\varsigma^*)\\ &\leq \lambda\big(\eta(\sigma^*,\varsigma^*)\big)^{\alpha}\left(\frac{[1+\eta(\sigma^*,\Gamma\sigma^*)]\eta(\varsigma^*,\Gamma\varsigma^*)}{1+\eta(\sigma^*,\varsigma^*)}\right)^{\beta}\\ &= 0\end{aligned}$$

This yields that $\sigma^* = \varsigma^*$. This completes the proof.

4 Conclusion

In this paper, using the new framework of super metric spaces, we introduced the concept of (λ, α)-interpolative Kannan contraction, $(\lambda\ \alpha, \beta)$-interpolative Kannan contraction and $(\lambda, \alpha, \beta, \gamma)$-interpolative Riech contraction and $(\lambda\ \alpha, \beta)$-interpolative Dass-Gupta rational contraction and proved the existence of fixed points for self-mapping.

Competing Interests

Authors have declared that no competing interests exist.

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