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Characterization of Almost η-Ricci Solitons With Respect to Schouten-van Kampen Connection on Sasakian Manifolds

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, we investigate Sasakian manifolds that admit almost η-Ricci solitons with respect to the Schouten-van Kampen connection using certain curvature tensors. Concepts of Ricci pseudosymmetry for Sasakian manifolds admitting η-Ricci solitons are introduced based on the selection of specific curvature tensors such as Riemann, concircular, projective, pseudo-projective, $\mathcal{M}$-projective, and W_2 tensors. Subsequently, necessary conditions are established for a Sasakian manifold admitting η-Ricci soliton with respect to the Schouten-van Kampen connection to be Ricci semisymmetric, based on the choice of curvature tensors. Characterizations are then derived, and classifications are made under certain conditions.

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1 Introduction

The concept of Ricci flow was introduced by Hamilton in 1982. By employing this concept, Hamilton discovered the canonical metric on a smooth manifold. Subsequently, Ricci flow has become a powerful tool for studying Riemannian manifolds, especially those with positive curvature. Perelman utilized Ricci flow and its surgery to prove the Poincaré conjecture [1],[2].

A Ricci soliton emerges as the endpoint of solitons during the Ricci flow process. A solution to the Ricci flow is labeled a Ricci soliton if it progresses exclusively under a one-parameter group of diffeomorphisms and scaling.

Over the last two decades, the geometry of Ricci solitons has attracted significant attention from numerous mathematicians. Its importance escalated notably after Perelman employed Ricci solitons to resolve the long-standing Poincaré conjecture, posed in 1904. In [3], Sharma examined Ricci solitons in contact geometry. Subsequently, Ricci solitons in contact metric manifolds have been investigated by various authors in [4]-[16].

The Schouten-van Kampen connection was primarily introduced for the detailed examination of non-holomorphic manifolds in [17, 18]. Initially addressed by Bejancu in foliated manifolds [19], it was later discussed in almost contact and almost paracontact manifolds in [20].

Sasaki defined Sasakian manifolds as the one-dimensional form of Kahler manifolds in [21]. After Cartan studied symmetric Riemannian manifolds in [22], Takahashi explored different notions of symmetry in Sasakian manifolds in [23]. Subsequently, G. Ghosh began to investigate Sasakian manifolds based on the Schouten-van Kampen connection in [24].

In this paper, we investigate Sasakian manifolds that admit almost η-Ricci solitons with respect to the Schouten-van Kampen connection using certain curvature tensors. Concepts of Ricci pseudosymmetry for Sasakian manifolds admitting η-Ricci solitons are introduced based on the selection of specific curvature tensors such as Riemann, concircular, projective, pseudo-projective, $\mathcal{M}$-projective, and W_2 tensors. Subsequently, necessary conditions are established for a Sasakian manifold admitting η-Ricci soliton with respect to the Schouten-van Kampen connection to be Ricci semisymmetric, based on the choice of curvature tensors. Characterizations are then derived, and classifications are made under certain conditions.

From this section onwards, the Schouten-van Kampen connection will be denoted as the $\mathcal{S}-\mathcal{VK}$ connection.

2 Preliminaries

Let Φ be a $(2n+1)$-dimensional Sasakian manifold. In this case, it is evident that the quadruple (ϕ, ξ, η, g) defined on Φ, satisfying the following relations, holds on the Sasakian manifold.

$$\phi^2\mathcal{Q}_1 = -\mathcal{Q}_1 + \eta\left(\mathcal{Q}_1\right)\xi, \eta(\xi) = 1, \eta\left(\phi\mathcal{Q}_1\right) = 0 \text{ and } \phi\xi = 0, \tag{1}$$

$$g\left(\mathcal{Q}_1, \mathcal{Q}_2\right) = g\left(\phi\mathcal{Q}_1, \phi\mathcal{Q}_2\right) + \eta(\mathcal{Q}_1)\eta(\mathcal{Q}_2), \tag{2}$$

$$g\left(\phi\mathcal{Q}_1, \mathcal{Q}_2\right) = -g\left(\mathcal{Q}_1, \phi\mathcal{Q}_2\right), g\left(\mathcal{Q}_1, \xi\right) = \eta(\mathcal{Q}_1), \tag{3}$$

for all vectors field $\mathcal{Q}_1, \mathcal{Q}_2$ in [25, 26].

$$\left(\nabla_{\mathcal{Q}_1}\phi\right)\mathcal{Q}_2 = g\left(\mathcal{Q}_1, \mathcal{Q}_2\right)\xi - \eta\left(\mathcal{Q}_2\right)\mathcal{Q}_1, \tag{4}$$

for all vectors field $\mathcal{Q}_1, \mathcal{Q}_2$, where ∇ is the Levi-Civita connection of the Riemannian metric. Thus, for a $(2n+1)$-dimensional Sasakian manifold, the Riemann curvature tensor, Ricci curvature tensor, and Ricci operator satisfy some fundamental relations as follows.

$$\nabla_{\mathcal{Q}_1}\xi = -\phi\mathcal{Q}_1, \tag{5}$$

$$(\nabla_{\mathcal{Q}_1}\eta)\,\mathcal{Q}_2 = g(\mathcal{Q}_1, \phi\mathcal{Q}_2), \tag{6}$$

$$R(\xi, \mathcal{Q}_1)\,\mathcal{Q}_2 = g(\mathcal{Q}_1, \mathcal{Q}_2)\,\xi - \eta(\mathcal{Q}_2)\,\mathcal{Q}_1, \tag{7}$$

$$R(\mathcal{Q}_1, \xi)\,\mathcal{Q}_2 = -g(\mathcal{Q}_1, \mathcal{Q}_2)\,\xi + \eta(\mathcal{Q}_2)\,\mathcal{Q}_1, \tag{8}$$

$$R(\mathcal{Q}_1, \mathcal{Q}_2)\,\xi = \eta(\mathcal{Q}_2)\,\mathcal{Q}_1 - \eta(\mathcal{Q}_1)\,\mathcal{Q}_2, \tag{9}$$

$$S(\mathcal{Q}_1, \xi) = 2n\eta(\mathcal{Q}_1), \tag{10}$$

$$Q\xi = 2n\xi, \tag{11}$$

$$S(\phi\mathcal{Q}_1, \phi\mathcal{Q}_2) = S(\mathcal{Q}_1, \mathcal{Q}_2) - 2n\eta(\mathcal{Q}_1)\,\eta(\mathcal{Q}_2), \tag{12}$$

for all vectors field $\mathcal{Q}_1, \mathcal{Q}_2$ in [27, 28, 29].

3 Curvature Tensor and Ricci Tensor with Respect to the $\mathcal{S}-\mathcal{VK}$ Connection

The $\mathcal{S}-\mathcal{VK}$ connection $\mathring{\nabla}$ is given by [20],

$$\mathring{\nabla}_{\mathcal{Q}_1}\mathcal{Q}_2 = \nabla_{\mathcal{Q}_1}\mathcal{Q}_2 - \eta(\mathcal{Q}_2)\,\nabla_{\mathcal{Q}_1}\xi - (\nabla_{\mathcal{Q}_1}\eta)(\mathcal{Q}_2)\,\xi, \tag{13}$$

for any $\mathcal{Q}_1, \mathcal{Q}_2$ tangent to Φ. With the help of (5) and (6), the above equation takes the form

$$\mathring{\nabla}_{\mathcal{Q}_1}\mathcal{Q}_2 = \nabla_{\mathcal{Q}_1}\mathcal{Q}_2 + g(\mathcal{Q}_1, \phi\mathcal{Q}_2)\,\xi + \eta(\mathcal{Q}_2)\,\phi\mathcal{Q}_1. \tag{14}$$

Putting $\mathcal{Q}_2 = \xi$ in (14) and using (1), we have

$$\mathring{\nabla}_{\mathcal{Q}_1}\xi = \nabla_{\mathcal{Q}_1}\xi + \eta(\mathcal{Q}_1)\,\xi - \mathcal{Q}_1. \tag{15}$$

Using (5) in (15), we get

$$\mathring{\nabla}_{\mathcal{Q}_1}\xi = 0. \tag{16}$$

Let R and $\mathring{R}$ denote the curvature tensor ∇ and $\mathring{\nabla}$ respectively. Then

$$\mathring{R}(\mathcal{Q}_1, \mathcal{Q}_2)\,\mathcal{Q}_3 = \mathring{\nabla}_{\mathcal{Q}_1}\mathring{\nabla}_{\mathcal{Q}_2}\mathcal{Q}_3 - \mathring{\nabla}_{\mathcal{Q}_2}\mathring{\nabla}_{\mathcal{Q}_1}\mathcal{Q}_3 - \mathring{\nabla}_{[\mathcal{Q}_1,\mathcal{Q}_2]}\mathcal{Q}_3. \tag{17}$$

Using (14) in (17), we yields

$$\begin{aligned}\mathring{R}(\mathcal{Q}_1, \mathcal{Q}_2)\,\mathcal{Q}_3 &= R(\mathcal{Q}_1, \mathcal{Q}_2)\,\mathcal{Q}_3 - g(\mathcal{Q}_2, \phi\mathcal{Q}_3)\,\mathcal{Q}_1\\ &+g(\mathcal{Q}_1, \phi\mathcal{Q}_3)\,\phi\mathcal{Q}_2 - \eta(\mathcal{Q}_2)\,\eta(\mathcal{Q}_3)\,\mathcal{Q}_1\\ &+\eta(\mathcal{Q}_1)\,\eta(\mathcal{Q}_3)\,\mathcal{Q}_2.\end{aligned} \tag{18}$$

If we choose $\mathcal{Q}_1 = \xi, \mathcal{Q}_2 = \xi$ and $\mathcal{Q}_3 = \xi$ respectively in (18), we get

$$\mathring{R}(\xi, \mathcal{Q}_2)\,\mathcal{Q}_3 = g(\mathcal{Q}_2, \mathcal{Q}_3)\,\xi - g(\mathcal{Q}_2, \phi\mathcal{Q}_3)\,\xi - \eta(\mathcal{Q}_2)\,\eta(\mathcal{Q}_3)\,\xi, \tag{19}$$

$$\mathring{R}(\mathcal{Q}_1, \xi)\,\mathcal{Q}_3 = g(\mathcal{Q}_1, \mathcal{Q}_3)\,\xi + \eta(\mathcal{Q}_1)\,\eta(\mathcal{Q}_3)\,\xi, \tag{20}$$

$$\mathring{R}(\mathcal{Q}_1, \mathcal{Q}_2)\,\xi = 0 \tag{21}$$

Taking inner product with W of (18) we get

$$\begin{aligned}g\left(\mathring{R}(\mathcal{Q}_1, \mathcal{Q}_2)\,\mathcal{Q}_3, W\right) &= g(R(\mathcal{Q}_1, \mathcal{Q}_2)\,\mathcal{Q}_3, W) - g(\mathcal{Q}_2, \phi\mathcal{Q}_3)\,g(\phi\mathcal{Q}_1, W)\\ g(\mathcal{Q}_1, \phi\mathcal{Q}_3)\,g(\phi\mathcal{Q}_2, W) &- g(\mathcal{Q}_1, W)\,\eta(\mathcal{Q}_2)\,\eta(\mathcal{Q}_3) + g(\mathcal{Q}_2, W)\,\eta(\mathcal{Q}_1)\,\eta(\mathcal{Q}_3).\end{aligned} \tag{22}$$

If we select the local orthonormal basis of the tangent space of the manifold Φ as $\{e_1, e_2, ..., e_{2n+1}\}$, we get

$$\mathring{S}(\mathcal{Q}_2, \mathcal{Q}_3) = S(\mathcal{Q}_2, \mathcal{Q}_3) + g(\mathcal{Q}_2, \mathcal{Q}_3) - (2n+1)\eta(\mathcal{Q}_2)\eta(\mathcal{Q}_3), \tag{23}$$

where $\mathring{S}$ and S are the Ricci tensor of Φ with respect to $\mathcal{S}-\mathcal{VK}$ connection and Levi-Civita connection, respectively.

Let $\mathring{r}$ and r denote the scalar curvature of Φ with respect to $\mathcal{S}-\mathcal{VK}$ connection and Levi-Civita connection respectively. If we choose $\mathcal{Q}_2 = \mathcal{Q}_3 = e_i$ in (23) and taking summation over i, $1 \leq i \leq 2n+1$, we have

$$\mathring{r} = r. \tag{24}$$

Obviously,

$$\mathring{R}(\mathcal{Q}_1, \mathcal{Q}_2)\mathcal{Q}_3 = -\mathring{R}(\mathcal{Q}_2, \mathcal{Q}_1)\mathcal{Q}_3,$$

$$\mathring{R}(\mathcal{Q}_1, \mathcal{Q}_2)\mathcal{Q}_3 + \mathring{R}(\mathcal{Q}_2, \mathcal{Q}_3)\mathcal{Q}_1 + \mathring{R}(\mathcal{Q}_3, \mathcal{Q}_1)\mathcal{Q}_2 = 0,$$

and the Ricci tensor $\mathring{S}$ is symmetric [24].

On the other hand, if the Sasakian manifold is a Ricci flat according to the $\mathcal{S}-\mathcal{VK}$ connection, we can state the following theorem, as can be seen from [24].

Theorem 1. *If Φ^{2n+1} is Ricci flat under the $\mathcal{S}-\mathcal{VK}$ connection, then it is an η-Einstein manifold, and vice versa.*

4 Almost $\eta-$Ricci Solitons Admiting Sasakian Manifolds According to $\mathcal{S}-\mathcal{VK}$ Connection

η-Ricci solitons on a Riemannian manifold were introduced by J.T. Cho and M. Kimura in [30] as a quadruple (g, ξ, λ, μ) satisfying,

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{25}$$

which is a generalization of the triple (g, ξ, λ) on Φ satisfying

$$L_\xi g + 2S + 2\lambda g = 0, \tag{26}$$

where λ and μ are real constants and η is the dual of ξ and S denotes the Ricci tensor of Φ. Furthermore if λ and μ are smooth functions on Φ, then it called almost $\eta-$Ricci soliton on Φ [30].

We can classify an almost η-Ricci soliton on the manifold Φ according to the sign of λ as follows:

· If λ is negative, it is called shrinking.
· If λ is zero, it is called steady.
· If λ is positive, it is called expanding.

Let Φ be a Riemannian manifold, $\mathcal{B}$ is $(0,k)-$type tensor field and $\mathcal{A}$ is $(0,2)-$type tensor field. In this case, Tachibana tensor field $Q(\mathcal{A}, \mathcal{B})$ is defined as

$$Q(\mathcal{A}, \mathcal{B})(H_1, ..., H_k; \mathcal{Q}_1, \mathcal{Q}_2) = -\mathcal{B}((\mathcal{Q}_1 \wedge_{\mathcal{A}} \mathcal{Q}_2)H_1, ..., H_k) -$$

$$... - \mathcal{B}(H_1, ..., H_{k-1}, (\mathcal{Q}_1 \wedge_{\mathcal{A}} \mathcal{Q}_2)H_k),$$

where,

$$(\mathcal{Q}_1 \wedge_{\mathcal{A}} \mathcal{Q}_2)\mathcal{Q}_3 = \mathcal{A}(\mathcal{Q}_2, \mathcal{Q}_3)\mathcal{Q}_1 - \mathcal{A}(\mathcal{Q}_1, \mathcal{Q}_3)\mathcal{Q}_2,$$

$k \geq 1, H_1, H_2, ..., H_k, \mathcal{Q}_1, \mathcal{Q}_2 \in \Gamma(TM)$.

Now let (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection. Then we have

$$\left(\mathring{L}_\xi g\right)(\mathcal{Q}_1,\mathcal{Q}_2)=\mathring{L}_\xi g(\mathcal{Q}_1,\mathcal{Q}_2)-g\left(\mathring{L}_\xi\mathcal{Q}_1,\mathcal{Q}_2\right)-g\left(\mathcal{Q}_1,\mathring{L}_\xi\mathcal{Q}_2\right)$$

$$=\xi g(\mathcal{Q}_1,\mathcal{Q}_2)-g([\xi,\mathcal{Q}_1],\mathcal{Q}_2)-g(\mathcal{Q}_1,[\xi,\mathcal{Q}_2])$$

$$=g\left(\mathring{\nabla}_\xi\mathcal{Q}_1,\mathcal{Q}_2\right)+g\left(\mathcal{Q}_1,\mathring{\nabla}_\xi\mathcal{Q}_2\right)-g\left(\mathring{\nabla}_\xi\mathcal{Q}_1,\mathcal{Q}_2\right)$$

$$+g\left(\mathring{\nabla}_{\mathcal{Q}_1}\xi,\mathcal{Q}_2\right)-g\left(\mathring{\nabla}_\xi\mathcal{Q}_2,\mathcal{Q}_1\right)+g\left(\mathcal{Q}_1,\mathring{\nabla}_{\mathcal{Q}_2}\xi\right),$$

for all $\mathcal{Q}_1,\mathcal{Q}_2\in\Gamma(T\Phi)$. By using (16), we have

$$\left(\mathring{L}_\xi g\right)(\mathcal{Q}_1,\mathcal{Q}_2)=0. \tag{27}$$

Thus, in a Sasakian manifold, from (26) and (27), we have

$$\mathring{S}(\mathcal{Q}_1,\mathcal{Q}_2)+\mathring{\lambda}g(\mathcal{Q}_1,\mathcal{Q}_2)+\mathring{\mu}\eta(\mathcal{Q}_1)\eta(\mathcal{Q}_2)=0. \tag{28}$$

Thus, we can easily express the following result.

Corollary 1. *Let Φ^{2n+1} be an $(2n+1)-$dimensional Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection. The $(2n+1)-$dimensional Sasakian manifold admitting almost $\eta-$Ricci soliton $\left(\Phi^{2n+1},g,\xi,\lambda,\mu\right)$ is an $\eta-$Einstein manifold.*

For $\mathcal{Q}_2=\xi$ in (28), this implies that

$$\mathring{S}(\xi,\mathcal{Q}_1)=-\left(\mathring{\lambda}+\mathring{\mu}\right)\eta(\mathcal{Q}_1). \tag{29}$$

Taking into account of (23) and (29), we conclude that

$$\mathring{\lambda}+\mathring{\mu}=0. \tag{30}$$

Definition 1. *Let Φ^{2n+1} be an $(2n+1)-$dimensional Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection. If $\mathring{R}\cdot\mathring{S}$ and $\mathring{Q}\left(g,\mathring{S}\right)$ are linearly dependent, then the Φ^{2n+1} is said to be* ***Ricci pseudosymmetric manifold*** *according to $\mathcal{S}-\mathcal{VK}$ connection.*

In particular, if $\mathring{R}\cdot\mathring{S}=0$, the manifold Φ^{2n+1} is said to be **Ricci semisymmetric** according to $\mathcal{S}-\mathcal{VK}$ connection.

Theorem 2. *Let Φ^{2n+1}be Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Φ^{2n+1}. If Φ^{2n+1} is a Ricci pseudosymmetric, then Φ^{2n+1} is either a Ricci semisymmetric manifold or steady.*

Proof. Let's assume that Sasakian manifold Φ^{2n+1} be Ricci pseudosymmetric manifold according to $\mathcal{S}-\mathcal{VK}$ connection and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Φ^{2n+1}. That's mean

$$\left(\mathring{R}(\mathcal{Q}_1,\mathcal{Q}_2)\cdot\mathring{S}\right)(\mathcal{Q}_3,\mathcal{Q}_4)=\mathring{L}_1\mathring{Q}\left(g,\mathring{S}\right)(\mathcal{Q}_3,\mathcal{Q}_4;\mathcal{Q}_1,\mathcal{Q}_2),$$

for all $\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{Q}_3,\mathcal{Q}_4\in\Gamma\left(T\Phi^{2n+1}\right)$. From the last equation, we can easily write

$$\begin{aligned}&\mathring{S}\left(\mathring{R}(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_3,\mathcal{Q}_4\right)+\mathring{S}\left(\mathcal{Q}_3,\mathring{R}(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_4\right)\\&=\mathring{L}_1\left\{\mathring{S}((\mathcal{Q}_1\wedge_g\mathcal{Q}_2)\mathcal{Q}_3,\mathcal{Q}_4)+\mathring{S}(\mathcal{Q}_3,(\mathcal{Q}_1\wedge_g\mathcal{Q}_2)\mathcal{Q}_4)\right\}.\end{aligned} \tag{31}$$

If we choose $\mathcal{Q}_4 = \xi$ in (31) , we get

$$\mathring{S}\left(\mathring{R}\left(\mathcal{Q}_1, \mathcal{Q}_2\right)\mathcal{Q}_3, \xi\right) + \mathring{S}\left(\mathcal{Q}_3, \mathring{R}\left(\mathcal{Q}_1, \mathcal{Q}_2\right)\xi\right)$$
$$= \mathring{L}_1\left\{\mathring{S}\left(g\left(\mathcal{Q}_2, \mathcal{Q}_3\right)\mathcal{Q}_1 - g\left(\mathcal{Q}_1, \mathcal{Q}_3\right)\mathcal{Q}_2, \xi\right)\right. \tag{32}$$
$$\left. + \mathring{S}\left(\mathcal{Q}_3, \eta\left(\mathcal{Q}_2\right)\mathcal{Q}_1 - \eta\left(\mathcal{Q}_1\right)\mathcal{Q}_2\right)\right\}.$$

If we make use of (21) and (23) in (32) , we have

$$\mathring{L}_1\mathring{S}\left(\mathcal{Q}_3, \eta\left(\mathcal{Q}_2\right)\mathcal{Q}_1 - \eta\left(\mathcal{Q}_1\right)\mathcal{Q}_2\right) = 0 \tag{33}$$

If we use (28) in the (33), we get

$$\mathring{\lambda}_1\mathring{L}_1 g\left(\mathcal{Q}_3, \eta\left(\mathcal{Q}_2\right)\mathcal{Q}_1 - \eta\left(\mathcal{Q}_1\right)\mathcal{Q}_2\right) = 0.$$

It is clear from the last equality that the proof of the theorem is completed. □

For an $(2n+1)-$dimensional Φ Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection, the concircular curvature tensor is defined as

$$\mathring{C}\left(\mathcal{Q}_1, \mathcal{Q}_2\right)\mathcal{Q}_3 = \mathring{R}\left(\mathcal{Q}_1, \mathcal{Q}_2\right)\mathcal{Q}_3 - \frac{\mathring{r}}{2n\left(2n+1\right)}\left[g\left(\mathcal{Q}_2, \mathcal{Q}_3\right)\mathcal{Q}_1 - g\left(\mathcal{Q}_1, \mathcal{Q}_3\right)\mathcal{Q}_2\right]. \tag{34}$$

If we choose $\mathcal{Q}_3 = \xi$ in (34) , we can write

$$\mathring{C}\left(\mathcal{Q}_1, \mathcal{Q}_2\right)\xi = \frac{\mathring{r}}{2n\left(2n+1\right)}\left[\eta\left(\mathcal{Q}_1\right)\mathcal{Q}_2 - \eta\left(\mathcal{Q}_2\right)\mathcal{Q}_1\right]. \tag{35}$$

Thus we have the following theorem.

Theorem 3. *Let Φ^{2n+1} be Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Φ^{2n+1}. If Φ^{2n+1} is a concircular Ricci pseudosymmetric, then Φ^{2n+1} is a steady or*

$$\mathring{L}_2 = \frac{\mathring{r}}{2n\left(2n+1\right)}.$$

Proof. Let's assume that Sasakian manifold Φ^{2n+1} be concircular Ricci pseudosymmetric according to $\mathcal{S}-\mathcal{VK}$ connection and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Φ^{2n+1}. That's mean

$$\left(\mathring{C}\left(\mathcal{Q}_1, \mathcal{Q}_2\right)\cdot\mathring{S}\right)\left(\mathcal{Q}_3, \mathcal{Q}_4\right) = \mathring{L}_2\mathring{Q}\left(g, \mathring{S}\right)\left(\mathcal{Q}_3, \mathcal{Q}_4; \mathcal{Q}_1, \mathcal{Q}_2\right),$$

for all $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4 \in \Gamma\left(T\Phi^{2n+1}\right)$. From the last equation, we can easily write

$$\mathring{S}\left(\mathring{C}\left(\mathcal{Q}_1, \mathcal{Q}_2\right)\mathcal{Q}_3, \mathcal{Q}_4\right) + \mathring{S}\left(\mathcal{Q}_3, \mathring{C}\left(\mathcal{Q}_1, \mathcal{Q}_2\right)\mathcal{Q}_4\right)$$
$$= \mathring{L}_2\left\{\mathring{S}\left(\left(\mathcal{Q}_1 \wedge_g \mathcal{Q}_2\right)\mathcal{Q}_3, \mathcal{Q}_4\right) + \mathring{S}\left(\mathcal{Q}_3, \left(\mathcal{Q}_1 \wedge_g \mathcal{Q}_2\right)\mathcal{Q}_4\right)\right\}. \tag{36}$$

If we choose $\mathcal{Q}_4 = \xi$ in (36) , we get

$$\mathring{S}\left(\mathring{C}\left(\mathcal{Q}_1, \mathcal{Q}_2\right)\mathcal{Q}_3, \xi\right) + \mathring{S}\left(\mathcal{Q}_3, \mathring{C}\left(\mathcal{Q}_1, \mathcal{Q}_2\right)\xi\right)$$
$$= \mathring{L}_2\left\{\mathring{S}\left(g\left(\mathcal{Q}_2, \mathcal{Q}_3\right)\mathcal{Q}_1 - g\left(\mathcal{Q}_1, \mathcal{Q}_3\right)\mathcal{Q}_2, \xi\right)\right. \tag{37}$$
$$\left. + \mathring{S}\left(\mathcal{Q}_3, \eta\left(\mathcal{Q}_2\right)\mathcal{Q}_1 - \eta\left(\mathcal{Q}_1\right)\mathcal{Q}_2\right)\right\}.$$

By using of (23) and (35) in (37) , we have

$$\left[\mathring{L}_2+\frac{\mathring{r}}{2n\left(2n+1\right)}\right]\mathring{S}\left(\mathcal{Q}_3,\eta\left(\mathcal{Q}_1\right)\mathcal{Q}_2-\eta\left(\mathcal{Q}_2\right)\mathcal{Q}_1\right)=0. \tag{38}$$

If we use (28) in the (38), we can write

$$\mathring{\lambda}\left[\mathring{L}_2+\frac{\mathring{r}}{2n\left(2n+1\right)}\right]g\left(\mathcal{Q}_3,\eta\left(\mathcal{Q}_1\right)\mathcal{Q}_2-\eta\left(\mathcal{Q}_2\right)\mathcal{Q}_1\right)=0.$$

It is clear from the last equality that the proof of the theorem is completed. □

We can give the result obtained from this theorem as follows.

Corollary 2. *Let Φ^{2n+1} be Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Φ^{2n+1}. If Φ^{2n+1} is a concircular Ricci semisymmetric, then Φ^{2n+1} is either manifold with scalar curvature $r=0$ or steady.*

For an $(2n+1)-$dimensional Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection, the projective curvature tensor is defined as

$$\mathring{P}\left(\mathcal{Q}_1,\mathcal{Q}_2\right)\mathcal{Q}_3=\mathring{R}\left(\mathcal{Q}_1,\mathcal{Q}_2\right)\mathcal{Q}_3-\frac{1}{2n}\left[\mathring{S}\left(\mathcal{Q}_2,\mathcal{Q}_3\right)\mathcal{Q}_1-\mathring{S}\left(\mathcal{Q}_1,\mathcal{Q}_3\right)\mathcal{Q}_2\right]. \tag{39}$$

If we choose $\mathcal{Q}_3=\xi$ in (39) , we can write

$$\mathring{P}\left(\mathcal{Q}_1,\mathcal{Q}_2\right)\xi=0. \tag{40}$$

Theorem 4. *Let Φ^{2n+1} be Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Φ^{2n+1}. If Φ^{2n+1} is a projective Ricci pseudosymmetric, then Φ^{2n+1} is either projective Ricci semisymmetric or steady.*

Proof. Let's assume that Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection be projective Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Φ^{2n+1}. Then we have

$$\left(\mathring{P}\left(\mathcal{Q}_1,\mathcal{Q}_2\right)\cdot\mathring{S}\right)\left(\mathcal{Q}_3,\mathcal{Q}_4\right)=\mathring{L}_3\mathring{Q}\left(g,\mathring{S}\right)\left(\mathcal{Q}_3,\mathcal{Q}_4;\mathcal{Q}_1,\mathcal{Q}_2\right),$$

for all $\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{Q}_3,\mathcal{Q}_4\in\Gamma\left(T\Phi^{2n+1}\right)$. From the last equation, we can easily write

$$\begin{aligned}&\mathring{S}\left(\mathring{P}\left(\mathcal{Q}_1,\mathcal{Q}_2\right)\mathcal{Q}_3,\mathcal{Q}_4\right)+\mathring{S}\left(\mathcal{Q}_3,\mathring{P}\left(\mathcal{Q}_1,\mathcal{Q}_2\right)\mathcal{Q}_4\right)\\&=\mathring{L}_3\left\{\mathring{S}\left(\left(\mathcal{Q}_1\wedge_g\mathcal{Q}_2\right)\mathcal{Q}_3,\mathcal{Q}_4\right)+\mathring{S}\left(\mathcal{Q}_3,\left(\mathcal{Q}_1\wedge_g\mathcal{Q}_2\right)\mathcal{Q}_4\right)\right\}.\end{aligned} \tag{41}$$

If we choose $\mathcal{Q}_4=\xi$ in (41) , we get

$$\begin{aligned}&\mathring{S}\left(\mathring{P}\left(\mathcal{Q}_1,\mathcal{Q}_2\right)\mathcal{Q}_3,\xi\right)+\mathring{S}\left(\mathcal{Q}_3,\mathring{P}\left(\mathcal{Q}_1,\mathcal{Q}_2\right)\xi\right)\\&=\mathring{L}_3\left\{\mathring{S}\left(g\left(\mathcal{Q}_2,\mathcal{Q}_3\right)\mathcal{Q}_1-g\left(\mathcal{Q}_1,\mathcal{Q}_3\right)\mathcal{Q}_2,\xi\right)\right.\\&\left.+\mathring{S}\left(\mathcal{Q}_3,\eta\left(\mathcal{Q}_2\right)\mathcal{Q}_1-\eta\left(\mathcal{Q}_1\right)\mathcal{Q}_2\right)\right\}.\end{aligned} \tag{42}$$

If we make use of (22) and (40) in (42) , we have

$$\mathring{L}_3\mathring{S}\left(\mathcal{Q}_3,\eta\left(\mathcal{Q}_2\right)\mathcal{Q}_1-\eta\left(\mathcal{Q}_1\right)\mathcal{Q}_2\right)=0. \tag{43}$$

If we use (28) in (43), we get

$$\mathring{\lambda}\mathring{L}_3g\left(\mathcal{Q}_3,\eta\left(\mathcal{Q}_2\right)\mathcal{Q}_1-\eta\left(\mathcal{Q}_1\right)\mathcal{Q}_2\right)=0.$$

It is clear from the last equality that the proof of the theorem is completed. □

For an $(2n+1)$ −dimensional Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection, the pseudo-projective curvature tensor is defined as

$$\begin{aligned}\mathring{P}_*(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_3 &= a_0\mathring{R}(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_3 + a_1\left[\mathring{S}(\mathcal{Q}_2,\mathcal{Q}_3)\mathcal{Q}_1 - \mathring{S}(\mathcal{Q}_1,\mathcal{Q}_3)\mathcal{Q}_2\right]\\ &-\tfrac{\mathring{r}}{2n+1}\left(\tfrac{a_0}{2n}+a_1\right)\left[g(\mathcal{Q}_2,\mathcal{Q}_3)\mathcal{Q}_1 - g(\mathcal{Q}_1,\mathcal{Q}_3)\mathcal{Q}_2\right].\end{aligned} \tag{44}$$

If we choose $\mathcal{Q}_3=\xi$ in (44), we can write

$$\mathring{P}_*(\mathcal{Q}_1,\mathcal{Q}_2)\xi = \frac{\mathring{r}}{2n+1}\left(\frac{a_0}{2n}+a_1\right)\left[\eta(\mathcal{Q}_1)\mathcal{Q}_2 - \eta(\mathcal{Q}_2)\mathcal{Q}_1\right]. \tag{45}$$

Theorem 5. *Let Φ^{2n+1} be Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Φ^{2n+1}. If Φ^{2n+1} is a pseudo-projective Ricci pseudosymmetric, then Φ^{2n+1} is either steady or*

$$\mathring{L}_4 = \frac{-\mathring{r}}{2n+1}\left(\frac{a_0}{2n}+a_1\right).$$

Proof. Let's assume that Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection be pseudo-projective Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Φ^{2n+1}. Then we have

$$\left(\mathring{P}_*(\mathcal{Q}_1,\mathcal{Q}_2)\cdot\mathring{S}\right)(\mathcal{Q}_3,\mathcal{Q}_4) = \mathring{L}_4\mathring{Q}\left(g,\mathring{S}\right)(\mathcal{Q}_3,\mathcal{Q}_4;\mathcal{Q}_1,\mathcal{Q}_2),$$

for all $\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{Q}_3,\mathcal{Q}_4\in\Gamma\left(T\Phi^{2n+1}\right)$. From the last equation, we can easily write

$$\begin{aligned}&\mathring{S}\left(\mathring{P}_*(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_3,\mathcal{Q}_4\right) + \mathring{S}\left(\mathcal{Q}_3,\mathring{P}_*(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_4\right)\\ &= \mathring{L}_4\left\{\mathring{S}\left((\mathcal{Q}_1\wedge_g\mathcal{Q}_2)\mathcal{Q}_3,\mathcal{Q}_4\right) + \mathring{S}\left(\mathcal{Q}_3,(\mathcal{Q}_1\wedge_g\mathcal{Q}_2)\mathcal{Q}_4\right)\right\}.\end{aligned} \tag{46}$$

If we choose $\mathcal{Q}_4=\xi$ in (46), we get

$$\begin{aligned}&\mathring{S}\left(\mathring{P}_*(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_3,\xi\right) + \mathring{S}\left(\mathcal{Q}_3,\mathring{P}_*(\mathcal{Q}_1,\mathcal{Q}_2)\xi\right)\\ &= \mathring{L}_4\left\{\mathring{S}\left(g(\mathcal{Q}_2,\mathcal{Q}_3)\mathcal{Q}_1 - g(\mathcal{Q}_1,\mathcal{Q}_3)\mathcal{Q}_2,\xi\right)\right.\\ &\left.+\mathring{S}\left(\mathcal{Q}_3,\eta(\mathcal{Q}_2)\mathcal{Q}_1-\eta(\mathcal{Q}_1)\mathcal{Q}_2\right)\right\}.\end{aligned} \tag{47}$$

If we make use of (23) and (45) in (47), we have

$$\left[\mathring{L}_4 + \frac{\mathring{r}}{2n+1}\left(\frac{a_0}{2n}+a_1\right)\right]\mathring{S}\left(\mathcal{Q}_3,\eta(\mathcal{Q}_1)\mathcal{Q}_2-\eta(\mathcal{Q}_2)\mathcal{Q}_1\right)=0. \tag{48}$$

If we use (28) in (48), we get

$$\mathring{\lambda}\left[\mathring{L}_4 + \frac{\mathring{r}}{2n+1}\left(\frac{a_0}{2n}+a_1\right)\right]g\left(\mathcal{Q}_3,\eta(\mathcal{Q}_1)\mathcal{Q}_2-\eta(\mathcal{Q}_2)\mathcal{Q}_1\right)=0.$$

It is clear from the last equality that the proof of the theorem is completed. □

Corollary 3. *Let Φ^{2n+1} be Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Φ^{2n+1}. If Φ^{2n+1} is a pseudo-psrojective Ricci semisymmetric, then Φ^{2n+1} is either manifold with scalar curvature $r=0$ provided $a_0+2na_1\neq 0$ or steady.*

For an $(2n+1)$ −dimensional Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection, the $\mathring{\mathcal{M}}$−projective curvature tensor is defined as

$$\mathring{\mathcal{M}}(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_3 = \mathring{R}(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_3 - \tfrac{1}{2n}\Big[\mathring{S}(\mathcal{Q}_2,\mathcal{Q}_3)\mathcal{Q}_1 - \mathring{S}(\mathcal{Q}_1,\mathcal{Q}_3)\mathcal{Q}_2 + g(\mathcal{Q}_2,\mathcal{Q}_3)\mathring{Q}\mathcal{Q}_1 - g(\mathcal{Q}_1,\mathcal{Q}_3)\mathring{Q}\mathcal{Q}_2\Big] \tag{49}$$

If we choose $\mathcal{Q}_3=\xi$ in (49) , we obtain

$$\mathring{\mathcal{M}}(\mathcal{Q}_1,\mathcal{Q}_2)\xi = \frac{1}{2n}\left[\eta(\mathcal{Q}_1)\mathring{Q}\mathcal{Q}_2 - \eta(\mathcal{Q}_2)\mathring{Q}\mathcal{Q}_1\right]. \tag{50}$$

Let us now investigate the $\mathring{\mathcal{M}}$−projective Ricci pseudosymmetric case of the Sasakian manifold.

Theorem 6. *Let Φ^{2n+1} be Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection and (g,ξ,λ,μ) be almost η−Ricci soliton on Φ^{2n+1}. If Φ^{2n+1} is a $\mathring{\mathcal{M}}$−projective Ricci pseudosymmetric, then Φ^{2n+1} is either steady or*

$$\mathring{L}_5 = \frac{-\mathring{\lambda}}{2n}.$$

Proof. Let's assume that Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection be projective $\mathring{\mathcal{M}}$−projective Ricci pseudosymmetric and (g,ξ,λ,μ) be almost η−Ricci soliton on Φ^{2n+1}. That's mean

$$\left(\mathring{\mathcal{M}}(\mathcal{Q}_1,\mathcal{Q}_2)\cdot\mathring{S}\right)(\mathcal{Q}_3,\mathcal{Q}_4) = \mathring{L}_5\mathring{Q}\left(g,\mathring{S}\right)(\mathcal{Q}_3,\mathcal{Q}_4;\mathcal{Q}_1,\mathcal{Q}_2),$$

for all $\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{Q}_3,\mathcal{Q}_4\in\Gamma\left(T\Phi^{2n+1}\right)$. From the last equation, we can easily write

$$\mathring{S}\left(\mathring{\mathcal{M}}(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_3,\mathcal{Q}_4\right) + \mathring{S}\left(\mathcal{Q}_3,\mathring{\mathcal{M}}(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_4\right) = \mathring{L}_5\left\{\mathring{S}((\mathcal{Q}_1\wedge_g\mathcal{Q}_2)\mathcal{Q}_3,\mathcal{Q}_4) + \mathring{S}(\mathcal{Q}_3,(\mathcal{Q}_1\wedge_g\mathcal{Q}_2)\mathcal{Q}_4)\right\}. \tag{51}$$

If we choose $\mathcal{Q}_4=\xi$ in (51) , we get

$$\mathring{S}\left(\mathring{\mathcal{M}}(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_3,\xi\right) + \mathring{S}\left(\mathcal{Q}_3,\mathring{\mathcal{M}}(\mathcal{Q}_1,\mathcal{Q}_2)\xi\right) = \mathring{L}_5\left\{\mathring{S}(g(\mathcal{Q}_2,\mathcal{Q}_3)\mathcal{Q}_1 - g(\mathcal{Q}_1,\mathcal{Q}_3)\mathcal{Q}_2,\xi) + \mathring{S}(\mathcal{Q}_3,\eta(\mathcal{Q}_1)\mathcal{Q}_2 - \eta(\mathcal{Q}_2)\mathcal{Q}_1)\right\}. \tag{52}$$

If we make use of (23) and (50) in (52) , we have

$$\frac{1}{2n}\mathring{S}\left(\eta(\mathcal{Q}_1)\mathring{Q}\mathcal{Q}_2 - \eta(\mathcal{Q}_2)\mathring{Q}\mathcal{Q}_1,\mathcal{Q}_3\right) = \mathring{L}_5 S(\eta(\mathcal{Q}_2)\mathcal{Q}_1 - \eta(\mathcal{Q}_1)\mathcal{Q}_2,\mathcal{Q}_3). \tag{53}$$

If we put (28) in (53), we can write

$$-\frac{\mathring{\lambda}}{2n}\mathring{S}(\mathcal{Q}_3,\eta(\mathcal{Q}_1)\mathcal{Q}_2 - \eta(\mathcal{Q}_2)\mathcal{Q}_1) - \mathring{\lambda}\mathring{L}_5 g(\mathcal{Q}_3,\eta(\mathcal{Q}_1)\mathcal{Q}_2 - \eta(\mathcal{Q}_2)\mathcal{Q}_1) = 0. \tag{54}$$

Again, if we use (28) in (54), we obtain

$$\left[\frac{\mathring{\lambda}^2}{2n} + \mathring{\lambda}\mathring{L}_5\right] g(\mathcal{Q}_3,\eta(\mathcal{Q}_1)\mathcal{Q}_2 - \eta(\mathcal{Q}_2)\mathcal{Q}_1) = 0.$$

It is clear from the last equality that the proof of the theorem is completed. □

Corollary 4. *Let Φ^{2n+1} be Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Φ^{2n+1}. If Φ^{2n+1} is a $\mathring{\mathcal{M}}-$projective Ricci semisymmetric, then Φ^{2n+1} is steady.*

For an $(2n+1)-$dimensional Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection, the $\mathring{W}_2-$curvature tensor is defined as

$$\mathring{W}_2(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_3=\mathring{R}(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_3-\frac{1}{2n}\left[g(\mathcal{Q}_2,\mathcal{Q}_3)\mathring{Q}\mathcal{Q}_1-g(\mathcal{Q}_1,\mathcal{Q}_3)\mathring{Q}\mathcal{Q}_2\right]. \tag{55}$$

If we choose $\mathcal{Q}_3=\xi$ in (55), we can write

$$\mathring{W}_2(\mathcal{Q}_1,\mathcal{Q}_2)\xi=-\frac{1}{2n}\left[\eta(\mathcal{Q}_2)\mathring{Q}\mathcal{Q}_1-\eta(\mathcal{Q}_1)\mathring{Q}\mathcal{Q}_2\right]. \tag{56}$$

Theorem 7. *Let Φ^{2n+1} be Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Φ^{2n+1}. If Φ^{2n+1} is a $\mathring{W}_2-$Ricci pseudosymmetric, then Φ^{2n+1} is either steady or*

$$\mathring{L}_6=\frac{\mathring{\lambda}}{2n}.$$

Proof. Let's assume that Sasakian manifold according to $\mathcal{S}-\mathcal{VK}$ connection be $\mathring{W}_2-$Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Φ^{2n+1}. That's mean

$$\left(\mathring{W}_2(\mathcal{Q}_1,\mathcal{Q}_2)\cdot\mathring{S}\right)(\mathcal{Q}_3,\mathcal{Q}_4)=\mathring{L}_6\mathring{Q}\left(g,\mathring{S}\right)(\mathcal{Q}_3,\mathcal{Q}_4;\mathcal{Q}_1,\mathcal{Q}_2),$$

for all $\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{Q}_3,\mathcal{Q}_4\in\Gamma\left(T\Phi^{2n+1}\right)$. From the last equation, we can easily write

$$\begin{aligned}&\mathring{S}\left(\mathring{W}_2(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_3,\mathcal{Q}_4\right)+\mathring{S}\left(\mathcal{Q}_3,\mathring{W}_2(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_4\right)\\&=\mathring{L}_6\left\{\mathring{S}((\mathcal{Q}_1\wedge_g\mathcal{Q}_2)\mathcal{Q}_3,\mathcal{Q}_4)+\mathring{S}(\mathcal{Q}_3,(\mathcal{Q}_1\wedge_g\mathcal{Q}_2)\mathcal{Q}_4)\right\}.\end{aligned} \tag{57}$$

If we choose $\mathcal{Q}_4=\xi$ in (57), we get

$$\begin{aligned}&\mathring{S}\left(\mathring{W}_2(\mathcal{Q}_1,\mathcal{Q}_2)\mathcal{Q}_3,\xi\right)+\mathring{S}\left(\mathcal{Q}_3,\mathring{W}_2(\mathcal{Q}_1,\mathcal{Q}_2)\xi\right)\\&=\mathring{L}_6\left\{\mathring{S}(g(\mathcal{Q}_2,\mathcal{Q}_3)\mathcal{Q}_1-g(\mathcal{Q}_1,\mathcal{Q}_3)\mathcal{Q}_2,\xi)\right.\\&\left.+\mathring{S}(\mathcal{Q}_3,\eta(\mathcal{Q}_2)\mathcal{Q}_1-\eta(\mathcal{Q}_1)\mathcal{Q}_2)\right\}.\end{aligned} \tag{58}$$

If we make use of (22) and (56) in (58), we have

$$-\frac{1}{2n}\mathring{S}\left(\mathcal{Q}_3,\eta(\mathcal{Q}_2)\mathring{Q}\mathcal{Q}_1-\eta(\mathcal{Q}_1)\mathring{Q}\mathcal{Q}_2\right)=\mathring{L}_6\mathring{S}(\mathcal{Q}_3,\eta(\mathcal{Q}_2)\mathcal{Q}_1-\eta(\mathcal{Q}_1)\mathcal{Q}_2)=0. \tag{59}$$

If we use (28) in the (59), we get

$$\frac{\mathring{\lambda}}{2n}\mathring{S}(\mathcal{Q}_3,\eta(\mathcal{Q}_2)\mathcal{Q}_1-\eta(\mathcal{Q}_1)\mathcal{Q}_2)+\mathring{\lambda}\mathring{L}_6g(\mathcal{Q}_3,\eta(\mathcal{Q}_2)\mathcal{Q}_1-\eta(\mathcal{Q}_1)\mathcal{Q}_2)=0. \tag{60}$$

Again, if we use (28) in (60), we obtain

$$\left[-\frac{\mathring{\lambda}^2}{2n}+\mathring{\lambda}\mathring{L}_6\right]g(\mathcal{Q}_3,\eta(\mathcal{Q}_2)\mathcal{Q}_1-\eta(\mathcal{Q}_1)\mathcal{Q}_2).$$

It is clear from the last equality that the proof of the theorem is completed. □

Corollary 5. *Let Φ^{2n+1} be Sasakian manifold according to $\mathcal{S}$-van-$\mathcal{K}$ connection and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Φ^{2n+1}. If Φ^{2n+1} is a $\mathring{W}_2-$Ricci semisymmetric, then Φ^{2n+1} is steady.*

5 Conclusion

In this paper, we investigate Sasakian manifolds that admit almost η-Ricci solitons with respect to the $\mathcal{S}-\mathcal{VK}$ connection using certain curvature tensors. Concepts of Ricci pseudosymmetry for Sasakian manifolds admitting η-Ricci solitons are introduced based on the selection of specific curvature tensors such as Riemann, concircular, projective, pseudo-projective, $\mathcal{M}$-projective, and W_2 tensors. Subsequently, necessary conditions are established for a Sasakian manifold admitting η-Ricci soliton with respect to the $\mathcal{S}-\mathcal{VK}$ connection to be Ricci semisymmetric, based on the choice of curvature tensors. Characterizations are then derived, and classifications are made under certain conditions.

Competing Interests

Authors have declared that no competing interests exist.

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