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Extreme Values and Reliability Measures of Continuous Erlang Mixtures

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

This study delves into Extreme Value Theory, focusing on the limiting distributions of continuous Erlang mixtures and their mixing distributions. It categorizes Erlang mixtures based on their extreme value types, demonstrating that the limiting distributions of the Erlang mixtures can be either Type I (Gumbel) or Type II (Fréchet), contingent on the mixing distribution. Further, this study derives the mean residual lifetime and equilibrium distributions of these mixed distributions.

Keywords: Extreme value theory; limiting distribution; Erlang mixture; mean residual lifetime distribution; equilibrium distribution; mixing distribution; mixed distribution.

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1 Introduction

Extreme value theory dates back to 1902 when Tippet, aided by Fisher, derived three asymptotic limiting distributions which he used to model the strength of a cotton thread. Extreme value theory has since been studied and applied in various fields.

Escalante-Sandoval [1] compared the General Extreme Value distribution to the mixed Gumbel and mixed General Extreme Value distributions when he applied them in estimating floods in regions with heterogeneous data. He proved that there was a lower standard error of fit when using the former. Ahn et al. [2] compared two tail estimation methods: the extreme value theory (EVT) and a class of log phase-type (LogPH) distributions. He noted that LogPH was preferable to the EVT because, unlike the EVT, the LogPH fits the whole data range and not only the tail part. The LogPH could also fit heavy-tailed loss data without separate modeling to the tail side due to its denseness.

Cramer[3] studied extreme value theory of progressively Type-II censored order statistics and their connection and application to limit laws for upper, lower, central, and intermediate progressively Type-II censored order statistics. Reynkens [4] noted that maximum domains of attraction (MDAs) could be described in two ways; one was through convergence to the generalized Pareto distribution and the other way was based on the regular variation of the tail quantile function. An application of extreme value theory to finance and insurance was made and in particular, a recent financial crisis was investigated as being a Black Swan event.

Gwak et al. [5] studied extreme value distributions of finite mixed distributions. They noted that a slow convergence rate of extreme values to a General Extreme Value distribution could cause huge bias in quantile estimation. They also observed that the use of the nonparametric quantile estimator of the extreme value to reduce bias problems in quantile estimation led to a high variance problem, thus the need to combine both methods.

Kang et al. [6] argued that the limiting distribution of finite mixed distributions or mixtures of convolutions of independent identically distributed Erlang random variables was determined by one or more distributions in the mixtures or convolutions and were of the Gumbel extreme-value type. They also showed that the limiting distribution of independent phase-type distributed random variables was the Gumbel extreme-value type. An application of these results was made to the completion times of jobs with parallel tasks and serial subtasks with Erlang or normal distributions and also to the completion times of jobs with parallel tasks that were Markovian PERT networks or task graphs.

Mladenović [7] presented extreme value distributions of finite mixtures of normal distributions, Cauchy distributions, and uniform and truncated exponential distributions. He noted that the extreme value distributions of component distributions were similar to those of the mixed distributions. Kang et al.[8], using exponential and Erlang mixtures, established that the asymptotic distribution of a finite mixture of continuous distributions was similar to that of the dominant component distribution in the mixture, and the norming constants were also related to the component distributions.

MacDonald et al. [9] proposed an extreme value mixture model that combined a non-parametric kernel density estimator with a generalized Pareto distribution tail model. They noted that the model did not need a pre-specification of a parametric form due to the flexibility of the non-parametric component. They also observed that computation was simplified because the model had just one extra parameter.

Extreme value theory can thus be applied in many fields, such as finance in modeling extreme profits or losses of valuable assets, in assessing and quantifying the risk of extreme natural calamities such as floods, hurricanes, and droughts, in engineering to construct structures that will not be affected by extreme conditions such as bridges, dams, and aircraft, in insurance in forecasting extreme events that could result in large insurance claims such as earthquakes and major accidents, and in predicting the probability of rare events such as the COVID 19 and extreme natural disasters [10].

The focus of this work is on limiting distributions of continuous Erlang mixtures. Kang et al.[8] studied Extreme value distributions of the Erlang uniform and Erlang exponential mixtures, and illustrated that Extreme Value distributions of continuous Erlang mixtures depended on their mixing distributions, and were either Type I (Gumbel) or Type II (Fretchet). He also noted that extreme value distributions of continuous Erlang mixtures were not always as those for the mixing distributions, unlike in the case of finite mixtures. The extreme value distribution of the Erlang-uniform distribution was proven to be Type I (Gumbel) when the minimum value parameter $l = 0$, and to be Type II (Fretchét) when the minimum value parameter $l > 0$. The extreme value distribution of the Erlang-exponential distribution was presented as Type I (Gumbel).

The rest of the paper is structured as follows: Section 2 defines extreme value theory, while Section 3 explains its categories in mathematical terms. The mathematical tools used in the research are detailed in Section 4. Section 5 demonstrates that Erlang mixed distributions with the Fréchet limiting distribution have Gumbel extreme value type mixing distributions. Section 6 illustrates that mixtures with the Gumbel limiting distribution possess mixing distributions that fall into any of the three limiting distribution types. Finally, Section 7 provides the conclusion of the paper.

2 Extreme Value Theory

The idea of extreme value distribution is borrowed from the central limit theory for sums of n random variables,

$$S_n = X_1 + X_2 + ... + X_n \tag{2.1}$$

which states that appropriately normalized sums $(S_n - a_n)/b_n$ converge to the standard normal distribution as n goes to infinity, that is;

$$\lim_{n\to\infty} P\left(\frac{S_n - a_n}{b_n} \leq x\right) = \Phi(x) \tag{2.2}$$

where $a_n = nE(X)$ and $b_n = \sqrt{Var(X)}$ are normalizing constants. [11])

Extreme value theory (EVT) deals with the convergence of maxima (highest order statistic), $M_n = \max(X_1, X_2, ..., X_n)$, of a given distribution, that is;

$$\lim_{n\to\infty} P\left(\frac{M_n - d_n}{C_n} \leq x\right) = \lim F^n(C_n x + d_n) = H(x), \quad x \in \mathbb{R} \tag{2.3}$$

where $H(x)$ is a non-degenerate distribution function (a limiting distribution that is not concentrated on a single point), and $F(x)$, the cumulative distribution function of x, is said to be in the maximum domain of attraction of the extreme value distribution of H, written as $F \in MDA(H)$ where $C_n > 0$, $d_n \in \mathbb{R}$ are normalizing constants. ([12, 13, 14])

The equation (2.3) above holds $\leftrightarrow$

$$\lim_{n\to\infty} n\left(1 - F(C_n x + d_n)\right) \to -logH(x), \quad x \in \mathbb{R} \tag{2.4}$$

for each x such that $H(x) > 0$. When $H(x) = 0$ the limit is interpreted as ∞.

The limiting distribution $H(x)$ is of three extreme value types, namely; Type I (Gumbel), Type II (Fretchét), and Type III (Weibull), their differences being their right endpoints and norming constants, among other factors.

3 Extreme Value Distributions

According to [12] and [13], extreme value distributions fall under three categories;

$$TypeI(Gumbel) : \Lambda(x) = exp(-exp[-x]), \quad -\infty < x < \infty \tag{3.1}$$

$$TypeII(Fretch\acute{e}t) : \Phi_\alpha(x) = \begin{cases} 0, & if\ x \leq 0 \\ exp(-x^{-\alpha}), & if\ x > 0, for\ some\ \alpha > 0 \end{cases} \tag{3.2}$$

$$TypeIII(Weibull) : \Psi_\alpha(x) = \begin{cases} exp(-(-x)^{\alpha}), & if\ x \leq 0, for\ some\ \alpha > 0 \\ 1, & if\ x > 0 \end{cases} \tag{3.3}$$

1. **Fretchet distribution $\Phi_\alpha(x)$**

   A distribution function F is said to be in the maximum domain of attraction of the Fretchét distribution $\Phi_\alpha(x) \leftrightarrow$ its tail function is of the form $\bar{F}(x) = x^{-\alpha}L(x)$, where $\bar{F}(x) = 1 - F(x)$ is the tail function of the distribution, $L(x)$ is a positive, Lebesgue-measurable function on $(0, \infty)$ which is slowly varying at ∞, and $\alpha > 0$ is the tail index of the distribution.

   All $F \in MDA(\Phi_\alpha)$ have an infinite right endpoint $x_F = \infty$, and the normalizing constants are $C_n = F^{\leftarrow}(1 - n^{-1}) = n^{\frac{1}{\alpha}} L_1(n)$ and $d_n = 0$ where $L_1(n)$ is a slowly varying positive, Lebesgue-measurable function.

2. **Weibull distribution $\Psi_\alpha(x)$**

   A distribution function F is in the maximum domain of attraction of the Weibull distribution $\Psi_\alpha(x)$ $\leftrightarrow$ its tail function is of the form $\bar{F}(x_F - x^{-1}) = x^{-\alpha}L(x)$, where $\bar{F}(x) = 1 - F(x)$ is the tail function of the distribution, $L(x)$ is a slowly varying positive, Lebesgue-measurable function and x_F is the right endpoint.
   All $F \in MDA(\Psi_\alpha)$ have a finite right endpoint $x_F < \infty$ and the norming constants are $C_n = x_F - F^{\leftarrow}(1 - n^{-1})$ and $d_n = x_F$.

   Remark: $\Psi_\alpha(-x^{-1}) = \Phi_\alpha(x), \quad x > 0.$

3. **Gumbel distribution $\Lambda(x)$**

   A distribution function F with right endpoint $x_F \leq \infty$ belongs to the maximum domain of the Gumbel distribution $\Lambda(x) \leftrightarrow$ it is tail equivalent to a Von Mises function, that is, $\bar{F}(x) = c(x)exp\left[-\int_z^x \frac{1}{a(t)}dt\right] \quad z < x < x_F$, where $\bar{F}(x) = 1 - F(x)$ is the tail function of the distribution, c is some positive constant and $a(x)$, the auxiliary function of F, is a positive and continuous function concerning Lebesgue measure with density $a'(x)$ and $\lim_{x\to x_F} a'(x) = 0$.

F is a Von Mises function with auxiliary function $a(x) = \frac{\bar{F}(x)}{f(x)} \leftrightarrow \lim_{x \to x_F} \frac{\bar{F}(x)F''(x)}{f^2(x)} = -1$.

$F \in MDA(\Lambda)$ can have both infinite and finite right endpoints $x_F \leq \infty$ and the norming constants are $d_n = F^{\leftarrow}(1 - n^{-1})$ and $C_n = a(d_n)$, where $a(x)$ is the auxiliary function of the distribution.

Remark: Continuous mixed Erlang distributions will be of either the Type I (Gumbel) or Type II (Fretchét) extreme value types, since their right endpoints are infinite, that is, $x_F = \infty$.

4 Special Functions

- **Regular and slow variation**

  i. A positive, Lebesgue-measurable function L on $(0, \infty)$ is slowly varying at ∞, denoted by $L \in R_0$, if

  $$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1, \quad t > 0 \tag{4.1}$$

  ii. A positive, Lebesgue-measurable function L on $(0, \infty)$ is regularly varying at ∞ of index $\alpha \in \mathbb{R}$, denoted by $L \in R_\alpha$, if

  $$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = t^\alpha, \quad t > 0 \tag{4.2}$$

  (see [15])

- **Karamata's theorem**

  It states that, if $L \in R_0$ is locally bound in $[x_0, \infty)$ for some $x_0 \geq 0$, then

  i. for $j > -1$,

  $$\int_{x_0}^{x} t^j L(t)dt \sim (j+1)^{-1} x^{j+1} L(x), \quad x \to \infty \tag{4.3}$$

  ii. for $j < -1$,

  $$\int_{x}^{\infty} t^j L(t)dt \sim -(j+1)^{-1} x^{j+1} L(x), \quad x \to \infty \tag{4.4}$$

  (see [14])

- **L'Hopital's rule**

  It states that

  $$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \tag{4.5}$$

- **Reliability Measures**

  Survival analysis is a branch of statistics that deals with the time of an event. It deals with both uncensored or complete and censored or incomplete data. The time to an event, also called the survival time, is a random variable that can be denoted by $T \geq 0$. Thus the survival function $S(t)$ is given by;

$$\begin{aligned} S(t) &= Prob(T > t) \\ &= 1 - Prob(T \leq t) \\ &= 1 - F(t) \end{aligned} \tag{4.6}$$

  where $F(t)$ is the cumulative distribution function of T. $S(t)$ is a non-increasing function of time. $S(0) = 1$ and $\lim_{t\to\infty} S(t) = 0$. Other functions of survival time are the probability density function $f(t)$ and the hazard function $h(t)$. The hazard function is the rate at which events occur, given no previous events, and it is given by;

$$h(x) = \frac{f(x)}{S(x)} \tag{4.7}$$

  i. The mean residual lifetime (mean excess loss) of a random variable X, denoted by $m(x)$, is the average remaining lifetime of a process that has survived beyond time x. It is given by;

$$\begin{aligned} m(x) &= E(T - x|T > x) \\ &= \int_x^\infty \frac{1 - F(t)}{1 - F(x)} dt, \quad t > x \end{aligned} \tag{4.8}$$

  where $F(x)$ is the cumulative distribution function of X.

  ii. The equilibrium distribution of a random variable X, denoted by $f_e(x)$, is used to determine whether a lifetime process preceding X, at a given time T, equals the mean of the distribution. It is given by;

$$f_e(x) = \frac{S(x)}{E(X)}, \quad x > 0 \tag{4.9}$$

  where $S(x)$ and $E(X)$ are the survival function and arithmetic mean of X respectively.

- **Confluent hypergeometric functions and their properties**

  1. **Kummer's confluent hypergeometric function**

  The Kummer's confluent hypergeometric function (Kummer's series) is defined as;

$${}_1F_1(a, c; x) = \sum_{n=0}^{\infty} \frac{a_{(n)}}{c_{(n)}} \frac{x^n}{n!} \tag{4.10}$$

  where $a_{(n)} = a(a+1)(a+2)...(a+n-1), c_{(n)} = c(c+1)(c+2)...(c+n-1), c \neq 0, -1, -2, ..., a_{(0)} = c_{(0)} = 1$ and its integral representation is given by;

$${}_1F_1(a, c; x) = \frac{1}{B(a, c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{xt} dt \tag{4.11}$$

where a and c are the parameters and x is the argument of the function.

Properties:

$$\begin{aligned} {}_1F_1(a,c;x) &= e^x {}_1F_1(c-a,c;-x) \\ {}_1F_1(a,a;x) &= e^x \\ \frac{x^a}{a} {}_1F_1(a,a+1;-x) &= \int_0^t t^{a-1}e^{-t}dt = \gamma(a,x) \end{aligned}$$

where $\gamma(a,x)$ is the lower incomplete gamma function.

The n^{th} derivative of Kummer's confluent hypergeometric function is given by;

$$\frac{d^n}{dx^n} {}_1F_1(a,c;x) = \frac{a_{(n)}}{c_{(n)}} {}_1F_1(a+n,c+n;x) \tag{4.12}$$

$$\text{where} \quad x_{(n)} = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} x(x+1)...(x+n-1), & if\ n>1 \\ 1, & if\ n=0, \end{cases}$$

$$\text{Hence} \quad \frac{d}{dx} {}_1F_1(a,c;x) = \frac{a}{c} {}_1F_1(a+1,c+1;x) \tag{4.13}$$

$$\text{and} \quad \frac{d^2}{dx^2} {}_1F_1(a,c;x) = \frac{a(a+1)}{c(c+1)} {}_1F_1(a+2,c+2;x) \tag{4.14}$$

and from Kummer's differential equation $x\frac{d^2\omega}{dx^2} + (c-x)\frac{d\omega}{dx} - a\omega = 0$,

$$\begin{aligned} \frac{a}{c} {}_1F_1(a+1,c+1;x) &= \frac{a}{c-x} {}_1F_1(a,c;x) - \frac{xa(a+1)}{c(c+1)(c-x)} {}_1F_1(a+2,c+2;x) \\ &\sim \frac{a}{c-x} {}_1F_1(a,c;x) \end{aligned} \tag{4.15}$$

where $\omega = {}_1F_1(a,c;x)$.

2. **Tricomi confluent hypergeometric function**

The Tricomi confluent hypergeometric function is defined as

$$\Psi(a,c;x) = \frac{1}{\Gamma a} \int_0^\infty t^{a-1}(1+t)^{c-a-1}e^{-xt}dt \tag{4.16}$$

where a and c are the parameters and x is the argument of the function.

The Tricomi function is defined for all values of a, c and x. It is generally complex for $x < 0$ and singular at $x = 0$.

Properties:

$$\Psi(a,c;x) = x^{1-c}\Psi(a-c+1, 2-c; x)$$
$$\Psi(a,a;x) = e^x \int_x^\infty t^{-a} e^{-t} dt$$
$$\Psi(1-a,1-a;x) = e^x \int_x^\infty t^{a-1} e^{-t} dt = e^x \Gamma(a,x)$$
$$\Psi(1,1+a;x) = x^{-a} e^x \Gamma(a,x)$$

where $\Gamma(a,x)$ is the upper incomplete gamma function.

The n^{th} derivative of the Tricomi confluent hypergeometric function is given by;

$$\frac{d^n}{dx^n}\Psi(a,c;x) = (-1)^n a_{(n)} \Psi(a+n, c+n; x) \tag{4.17}$$

$$\text{where} \quad x_{(n)} = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} x(x+1)...(x+n-1), & \textit{if } n > 1 \\ 1, & \textit{if } n = 0, \end{cases}$$

$$\text{Hence} \quad \frac{d}{dx}\Psi(a,c;x) = -a\Psi(a+1, c+1; x) \tag{4.18}$$

$$\text{and} \quad \frac{d^2}{dx^2}\Psi(a,c;x) = a(a+1)\Psi(a+2, c+2; x) \tag{4.19}$$

and from Kummer's differential equation $x\frac{d^2\omega}{dx^2} + (c-x)\frac{d\omega}{dx} - a\omega = 0$,

$$\begin{aligned} -a\Psi(a+1,c+1;x) &= \frac{-a}{x-c}\Psi(a,c;x) + \frac{xa(a+1)}{x-c}\Psi(a+2,c+2;x) \\ &\sim \frac{-a}{x-c}\Psi(a,c;x) \end{aligned} \tag{4.20}$$

where $\omega = \Psi(a,c;x)$.

The Tricomi and Kummer's confluent hypergeometric functions are related by;

$$\Psi(a,c;x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} {}_1F_1(a,c;x) + \frac{\Gamma(c-1)x^{1-c}}{\Gamma a} {}_1F_1(a-c+1, 2-c; x)$$
$$c \neq 0, -1, -2, ... \tag{4.21}$$

5 Frechét Limiting Distribution

In this section, the mixed distributions are Pareto-like and thus the tail functions are of the form $\bar{F}(x) \sim Kx^{-\alpha}, \quad x \to \infty$ for some K, $\alpha > 0$, and they are therefore $F \in MDA(\phi_\alpha)$ and the norming constants are of the form $C_n = (Kn)^{\frac{1}{\alpha}}$ and $d_n = 0$.

Of importance to note also is that the mixing distributions of the mixtures are $F \in MDA(\Lambda)$.

Erlang-Type II Gamma distribution

1. Mixing distribution

The type II Gamma distribution is

$$f(y) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} e^{-\frac{y}{\beta}} y^{\alpha-1}, \quad y > 0; \beta > 0, \alpha > 0 \tag{5.1}$$

Using L'Hopital's rule, the tail of the mixing distribution can be obtained as follows;

$$\lim_{y\to\infty} \frac{\bar{F}(y)}{f(y)} = \lim_{y\to\infty} \frac{1}{\frac{1}{\beta} - \frac{(\alpha-1)}{y}} = \beta \tag{5.2}$$

$$\text{and therefore} \quad \bar{F}(y) \sim \frac{1}{\beta^{\alpha-1}\Gamma(\alpha)} y^{\alpha-1} exp\left\{ -\int_0^y \frac{1}{\beta} dt \right\} \tag{5.3}$$

and thus $F \in MDA(\Lambda)$ which is a Von Mises function with auxiliary function $a(y) = \beta$.

The norming constants are obtained by solving the below equation

$$\bar{F}(d_n) = \frac{1}{\beta^{\alpha-1}\Gamma(\alpha)} e^{-\frac{d_n}{\beta}} d_n^{\alpha-1} = n^{-1} \tag{5.4}$$

to obtain $d_n = \beta(lnn - ln\Gamma(\alpha) + (\alpha-1)lnlnn)$ and $C_n = a(d_n) = \beta^{-1}$.

2. Mixed distribution

The mixed distribution is the generalized Pareto distribution with parameters $n, \alpha, \frac{1}{\beta}$, and has pdf;

$$\begin{aligned} f_n(t) &= \frac{\left(\frac{1}{\beta}\right)^{\alpha} t^{n-1}}{B(\alpha,n)(t+\frac{1}{\beta})^{\alpha+n}} \\ &= \frac{\left(\frac{1}{\beta}\right)^{\alpha}}{B(\alpha,n)} t^{-(\alpha+1)} \left(1 + \frac{1}{\beta t}\right)^{-(\alpha+n)} \end{aligned} \tag{5.8}$$

(see [16])

By Karamata's theorem, the tail function is given by;

$$\begin{aligned} \bar{F}(t) &\sim \frac{\left(\frac{1}{\beta}\right)^{\alpha}}{\alpha B(\alpha,n)} t^{-\alpha} \left(1 + \frac{1}{\beta t}\right)^{-(n+\alpha)} \\ &\sim \frac{\left(\frac{1}{\beta}\right)^{\alpha}}{\alpha B(\alpha,n)} t^{-\alpha} \end{aligned} \tag{5.9}$$

which is of the form $t^{-\alpha}L(t)$ and is regularly varying since $\lim_{x\to\infty}(1+\frac{1}{\beta t})^{-(n+\alpha)} = 1$ and thus $F \in MDA(\Phi_\alpha)$.

The norming constants are thus $C_n = \left(\frac{n}{\alpha\beta^{\alpha}B(\alpha,n)}\right)^{\frac{1}{\alpha}}$.

3. Reliability measures

The mean excess loss function of the mixture is

$$m(t) = \frac{t}{\alpha - 1} \tag{5.10}$$

and the equilibrium distribution is

$$f_e(t) = \frac{\left(\frac{1}{\beta}\right)^{\alpha-1}}{\alpha t^{\alpha} B(\alpha - 1, n+1)} \tag{5.11}$$

Erlang-Type I Two-Parameter Lindley Distribution

1. Mixing distribution

The Type I Two-Parameter Lindley distribution is

$$f(y) = \frac{\theta^{\alpha+1}}{\theta+1}\frac{(\alpha+y)}{\Gamma(\alpha+1)}e^{-\theta y}y^{\alpha-1}, \quad y > 0; \theta > 0, \alpha > 0 \tag{5.12}$$

Its tail function can be obtained using L'Hopital's rule as

$$\lim_{y\to\infty}\frac{\bar{F}(y)}{f(y)} = \lim_{y\to\infty}\frac{1}{\left(\theta - \frac{1}{\alpha+y} - \frac{(\alpha-1)}{y}\right)} = \theta^{-1} \tag{5.13}$$

$$\text{and therefore} \quad \bar{F}(y) \sim \frac{\theta^{\alpha}}{\theta+1}\frac{(\alpha+y)}{\Gamma(\alpha+1)}y^{\alpha-1}exp\left\{-\int_0^y \frac{1}{\frac{1}{\theta}}dt\right\} \tag{5.14}$$

Thus, $F \in MDA(\Lambda)$ and is a Von Mises function with auxiliary function $a(y) = \theta^{-1}$ and norming constants $d_n = \overleftarrow{F}(1-n^{-1}) = \theta^{-1}[lnn + (\alpha-1)(-ln\theta + lnlnn) + \alpha ln\theta - ln(\theta+1) - ln\Gamma(\alpha+1) + ln(\alpha + \theta^{-1}lnn)]$ and $C_n = a(d_n) = \theta^{-1}$.

2. Mixed distribution

The Erlang-Type I Two-Parameter Lindley Distribution has probability density function

$$\begin{aligned} f_n(t) &= \frac{t^{n-1}\theta^{\alpha+1}}{B(n,\alpha+1)(t+\theta)^{n+\alpha+1}}\left(\frac{1+\alpha(t+\theta)(n+\alpha)^{-1}}{\theta+1}\right) \\ &= \frac{\alpha\theta^{\alpha+1}}{(\theta+1)}\frac{t^{-(\alpha+1)}}{(\alpha+n)B(n,\alpha+1)}\left(1+\frac{\theta}{t}\right)^{-(n+\alpha+1)}\left(\frac{\alpha(\theta+1)+n}{\alpha t}+1\right) \end{aligned} \tag{5.15}$$

(see [16])

Using Karamata's theorem, the tail function can be obtained as

$$\begin{aligned}\bar{F}(t) &\sim \frac{\theta^{\alpha+1}}{(\theta+1)}\frac{t^{-\alpha}}{(\alpha+n)B(n,\alpha+1)}\left(1+\frac{\theta}{t}\right)^{-(n+\alpha+1)}\left(\frac{\alpha(\theta+1)+n}{\alpha t}+1\right)\\ &\sim \frac{\theta^{\alpha+1}}{(\theta+1)}\frac{t^{-\alpha}}{(\alpha+n)B(n,\alpha+1)}\end{aligned} \tag{5.16}$$

which is of the form $t^{-\alpha}L(t)$ and is a regularly varying function since $\lim_{t\to\infty}\left(1+\frac{\theta}{t}\right)^{-(n+\alpha+1)}\left(\frac{\alpha(\theta+1)+n}{\alpha t}+1\right)=1$ and therefore $F\in MDA(\Phi_\alpha)$.
The norming constants are $C_n=\left(\frac{n\theta^{\alpha+1}}{(\theta+1)(\alpha+n)B(n,\alpha+1)}\right)^{\frac{1}{\alpha}}$.

3.Reliability measures

The mean excess loss function of the mixed distribution is

$$m(t)=\frac{t}{\alpha-1} \tag{5.17}$$

and the equilibrium distribution is

$$f_e(t)=\frac{\theta^\alpha}{t^\alpha B(\alpha-1,n+1)[\alpha(\theta+1)-1]} \tag{5.18}$$

6 Gumbel Limiting Distribution

Mixed distributions are in the form of confluent hypergeometric functions and are $F\in MDA(\Lambda)$ and the mixing distributions are $F\in MDA(\Psi_\alpha)$, $F\in MDA(\Phi_\alpha)$ and $F\in MDA(\Lambda)$.

Erlang-Scaled Beta distribution

1. **Mixing distribution**
   The mixing (Scaled Beta) distribution,

$$f(y)=\frac{(\frac{y}{\mu})^{\alpha-1}(1-\frac{y}{\mu})^{\beta-1}\frac{1}{\mu}}{B(\alpha,\beta)},\quad 0<y<\mu;\alpha>0,\beta>0,\mu>0 \tag{6.13}$$

has a finite right endpoint $y_F=\mu$ and $\overline{F}(\mu-y^{-1})$ is of the form $y^{-\beta}L(y)$ and is a regularly varying function since $\lim_{y\to\infty}\left(1-\frac{1}{\mu y}\right)^{\alpha-1}=1$, that is, by Karamata's theorem;

$$\overline{F}(\mu-y^{-1})\sim\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)}\frac{y^{-\beta}}{\mu^\beta}\left(1-\frac{1}{\mu y}\right)^{\alpha-1} \tag{6.14}$$

Hence $F\in MDA(\psi_\beta)$ with norming constants $d_n=y_F=\mu$ and $C_n=\mu-\overleftarrow{F}(1-n^{-1})=\mu\left(\frac{n\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)}\right)^{\frac{-1}{\beta}}$.

2. **Mixed distribution**

The pdf of the mixture is given by;

$$\begin{aligned} f_n(t) &= \frac{\mu^n t^{n-1}}{\Gamma(n)} \frac{B(n+\alpha, n+\alpha+\beta)}{B(\alpha,\beta)} {}_1F_1(n+\alpha, n+\alpha+\beta; -\mu t) \\ &= \frac{\mu^n e^{-\mu t} t^{n-1}}{\Gamma(n)} \frac{B(n+\alpha, n+\alpha+\beta)}{B(\alpha,\beta)} {}_1F_1(\beta, n+\alpha+\beta; \mu t) \end{aligned} \tag{6.15}$$

(see [16])

An application of L'Hopital's rule on $\overline{F}(t)/f(t)$ yields;

$$\lim_{t\to\infty} \frac{\bar{F}(t)}{f(t)} = \lim_{t\to\infty} \frac{1}{\mu - \frac{(n-1)}{t} - \frac{\beta}{n+\alpha+\beta-\mu t}} = \mu^{-1} \tag{6.16}$$

$$\text{and thus} \quad \overline{F}(t) \sim \frac{(\mu t)^{n-1}}{\Gamma(n)} \frac{B(n+\alpha, n+\alpha+\beta)}{B(\alpha,\beta)} {}_1F_1(\beta, n+\alpha+\beta; \mu t) exp\left\{ -\int_0^t \frac{1}{\frac{1}{\mu}} dy \right\} \tag{6.17}$$

Therefore, $F \in MDA(\Lambda)$ and is a von Mises function with auxiliary function $a(t) = \mu^{-1}$ and norming constants $d_n = \overleftarrow{F}(1-n^{-1}) = \mu^{-1}[(n-1)lnlnn - ln\Gamma(n) + lnB(n+\alpha, n+\alpha+\beta) - lnB(\alpha,\beta) + ln_1F_1(\beta, n+\alpha+\beta; lnn)]$ and $C_n = a(d_n) = \mu^{-1}$.

3. **Reliability measures**

The equilibrium distribution of the mixed distribution is

$$f_e(t) = \frac{\mu^n}{n} e^{-\mu t} t^{n-1} \frac{B(n+\alpha, n+\alpha+\beta)}{B(\alpha-1,\beta)} {}_1F_1(\beta, n+\alpha+\beta; \mu t) \tag{6.18}$$

Erlang-Pareto I distribution

1. **Mixing distribution**

The Pareto I distribution is

$$f(y) = \alpha\beta^\alpha y^{-(\alpha+1)}, \quad y > \beta; \beta > 0, \alpha > 0 \tag{6.19}$$

By Karamata's theorem, the tail function can be obtained as

$$\bar{F}(y) \sim \beta^\alpha y^{-\alpha} \tag{6.20}$$

and is of the form $y^{-\alpha}L(y)$, which is regularly varying since $\lim_{y\to\infty} 1 = 1$. Thus $F \in MDA(\phi_\alpha)$ with norming constants $C_n = \beta n^{\frac{1}{\alpha}}$.

2. **Mixed distribution**

The Erlang-Pareto I mixture has pdf;

$$f_n(t) = \frac{\alpha\beta^n}{\Gamma(n)} e^{-\beta t} t^{n-1} \psi(1, n-\alpha+1; \beta t) \tag{6.21}$$

(see [16])

and a tail function, obtained using L'Hopital's rule, as;

$$\lim_{t\to\infty} \frac{\bar{F}(t)}{f(t)} = \lim_{t\to\infty} \frac{1}{\beta - \frac{(n-1)}{t} + \frac{1}{\beta t-(n-\alpha+1)}} = \beta^{-1} \tag{6.22}$$

$$\text{and thus} \quad \bar{F}(t) \sim \frac{\alpha}{\Gamma(n)} (\beta t)^{n-1} \psi(1, n-\alpha+1; \beta t) exp\left\{ -\int_0^t \frac{1}{\frac{1}{\beta}} dy \right\} \tag{6.23}$$

and hence $F \in MDA(\Lambda)$ and is a Von Mises function with auxiliary function $a(t) = \beta^{-1}$ and norming constants $d_n = \overleftarrow{F}(1-n^{-1}) = \beta^{-1}[lnn + (n-1)lnlnn + ln\alpha - ln\Gamma(n) + ln\psi(1, n-\alpha+1; lnn)]$ and $C_n = a(d_n) = \beta^{-1}$.

3. **Reliability measures**

The mean excess loss function of the mixed distribution is

$$m(t) = \left(\frac{1}{\beta} - 1\right) \frac{e^{\beta t}\Gamma(n, \beta t)}{\beta^n t^{n-1}} \tag{6.24}$$

and the equilibrium distribution is

$$f_e(t) = \frac{\alpha+1}{n} \frac{\beta^n}{\Gamma(n)} e^{-\beta t} t^{n-1} \psi(1, n-\alpha+1; \beta t) \tag{6.25}$$

Erlang-Shifted 25mma(Pearson Type III) distribution

1. **Mixing distribution**

The Shifted gamma(Pearson Type III) distribution,

$$f(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta(y-\mu)} (y-\mu)^{\alpha-1}, \quad y > \mu; \alpha > 0, \beta > 0, \mu > 0 \tag{6.26}$$

$$\text{has a tail} \quad \bar{F}(y) \sim \frac{e^{\beta\mu}(\beta(y-\mu))^{\alpha-1}}{\Gamma(\alpha)} exp\left\{ -\int_0^y \frac{1}{\frac{1}{\beta}} dt \right\} \tag{6.27}$$

which is obtained by L'Hopital's rule as illustrated below.

$$\lim_{y\to\infty}\frac{\bar{F}(y)}{f(y)}=\lim_{y\to\infty}\frac{1}{\beta-\frac{(\alpha-1)}{y-\mu}}=\beta^{-1} \tag{6.28}$$

and thus $F \in MDA(\Lambda)$, and is a Von Mises function with auxiliary function $a(y) = \beta^{-1}$. The norming constants are $d_n = \overleftarrow{F}(1-n^{-1}) = \beta^{-1}[lnn + \beta\mu + (\alpha-1)ln(lnn - \beta\mu) - ln\Gamma(\alpha)]$ and $C_n = a(d_n) = \beta^{-1}$.

2. **Mixed distribution**

   The mixed distribution has a probability density function

$$f_n(t)=\frac{\mu^{n+\alpha}}{\Gamma(n)}t^{n-1}\beta^{\alpha}e^{-\mu t}\psi(\alpha,\alpha+n+1;(t+\beta)\mu) \tag{6.29}$$

   (see [16])

   L'Hopital's rule can be applied in obtaining the tail of the distribution as shown below.

$$\lim_{t\to\infty}\frac{\bar{F}(t)}{f(t)}=\lim_{t\to\infty}\frac{1}{\mu-\frac{(n-1)}{t}+\frac{\alpha}{(t+\beta)\mu-(\alpha+n+1)}}=\mu^{-1} \tag{6.30}$$

$$\text{hence}\quad \bar{F}(t)\sim\frac{(\beta\mu)^{\alpha}(\mu t)^{n-1}}{\Gamma(n)}\psi(\alpha,\alpha+n+1;(t+\beta)\mu)exp\left\{-\int_0^t\frac{1}{\frac{1}{\mu}}dy\right\} \tag{6.31}$$

   Thus $F \in MDA(\Lambda)$, and is a Von Mises function with auxiliary function $a(t) = \mu^{-1}$ and norming constants $d_n = \overleftarrow{F}(1-n^{-1}) = \mu^{-1}[lnn+(n-1)lnlnn+\alpha ln(\beta\mu)-ln\Gamma(n)+ln\psi(\alpha,\alpha+n+1;(lnn+\beta\mu))]$ and $C_n = a(d_n) = \mu^{-1}$.

3. **Reliability measures**

   The mean excess loss function of the mixed distribution is

$$m(t)=\left(\frac{1}{\mu\Gamma(\alpha)}-1\right)\frac{\Gamma(n,\mu t)e^{\mu t}}{\mu^n t^{n-1}} \tag{6.32}$$

   and the equilibrium distribution is

$$f_e(t)=\frac{1}{n}\frac{\mu^n}{\Gamma(n)}e^{-\mu t}t^{n-1}\frac{\Psi(\alpha,\alpha+n+1;(t+\beta)\mu)}{\Psi(\alpha,\alpha;\beta\mu)} \tag{6.33}$$

7 Conclusion

This study derived extreme value distributions for continuous Erlang mixtures and their mixing distributions. It determined that the limiting distributions of the Erlang mixtures are either Type I (Gumbel) or Type II (Fréchet), as their right endpoints are infinite, classifying the mixed distributions into these extreme value types. Notably, all Erlang mixed distributions with the Fréchet limiting distribution have mixing distributions of the Gumbel extreme value type. Mixtures with the Gumbel limiting distribution have mixing distributions that can be any of the three limiting distribution types. Additionally, the study presented several reliability measures of the Erlang mixtures, including mean excess loss and equilibrium distributions.

This study recommends further research on the applications of extreme value theory, especially in fields such as risk management, finance, environmental science, and engineering. These areas can greatly benefit from the insights provided by extreme value theory in predicting and managing rare and extreme events.

Competing Interests

Authors have declared that no competing interests exist.

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