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Kähler–Einstein Metrics on Smooth Fano Symmetric Varieties with Picard Number One

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Abstract: Symmetric varieties are normal equivarient open embeddings of symmetric homogeneous spaces, and they are interesting examples of spherical varieties. We prove that all smooth Fano symmetric varieties with Picard number one admit Kähler–Einstein metrics by using a combinatorial criterion for K-stability of Fano spherical varieties obtained by Delcroix. For this purpose, we present their algebraic moment polytopes and compute the barycenter of each moment polytope with respect to the Duistermaat–Heckman measure.

Keywords: Kähler–Einstein metrics; symmetric varieties; moment polytopes

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1. Introduction

A Kähler metric on a complex manifold is said to be *Kähler–Einstein* if the Riemannian metric defined by its real part has constant Ricci curvature. The existence of Kähler–Einstein metrics on Fano manifolds has become a central topic in complex geometry in recent years. In contrast to Calabi–Yau and general type [1,2], Fano manifolds do not necessarily have a Kähler–Einstein metric in general, and there are obstructions based on the (holomorphic) automorphism group.

The first obstruction was discovered by Matsushima in [3]. He proved that the reductivity of the automorphism group is a necessary condition for the existence of Kähler–Einstein metrics. Later, Futaki [4] proved that the existence of Kähler–Einstein metrics implies that the Futaki invariant, a functional on the Lie algebra of the automorphism group, vanishes. As a generalization of this invariant on test configurations, Tian [5,6] and Donalson [7] introduced a certain algebraic stability condition, which is called the *K-stability*. The famous Yau–Tian–Donaldson conjecture predicts that the existence of a Kähler–Einstein metric on a Fano manifold is equivalent to the K-stability. Eventually, this conjecture was solved by Chen–Donaldson–Sun [8–10] and Tian [11].

Despite of these obstructions, each Fano homogeneous manifold admits a Kähler–Einstein metric [12,13]. Therefore, one can expect the existence of a Kähler–Einstein metric on a Fano manifold if it has large automorphism group. A natural candidate is the *almost-homogeneous* manifold, that is, a manifold with an open dense orbit of a complex Lie group. For the case of toric Fano manifolds, Wang and Zhu [14] proved that the existence of a Kähler–Einstein metric is equivalent to the vanishing of the Futaki invariant. In fact, this was based on the theorem by Mabuchi [15], which says that the Futaki invariant vanishes if and only if the barycenter of the moment polytope is the origin. This gave us a powerful combinatorial criterion for the existence of a Kähler–Einstein metric on a toric Fano manifold, which is much easier to check than the K-stability condition.

An important class of almost-homogeneous varieties is *spherical* varieties including toric varieties, *group compactifications* ([16]), and *symmetric* varieties. A normal variety is called *spherical* if it admits an action of a reductive group of which a Borel subgroup acts with an open orbit on the variety. As a generalization of Wang and Zhu's work, Delcroix [17] extended a combinatorial criterion for K-stability of Fano spherical manifolds, in terms of its moment polytope and spherical data. In particular, this criterion is also applicable to smooth Fano symmetric varieties (see Corollary 5.9 of [17]).

By combining the above criterion and Ruzzi's classification [18] of smooth Fano symmetric varieties with Picard number one, we prove the following.

Theorem 1. *All smooth Fano symmetric varieties with Picard number one admit Kähler–Einstein metrics.*

For this theorem, the condition on the Picard number is crucial because a smooth Fano symmetric variety with higher Picard number may have no Kähler–Einstein metrics. For example, the blow-up of the wonderful compactification of $\mathrm{Sp}(4, \mathbb{C})$ along the closed orbit does not admit any Kähler–Einstein metrics (see Example 5.4 of [16]). Moreover, we note that Delcroix already provided the existence of Kähler–Einstein metrics on smooth Fano embedding of $\mathrm{SL}(3, \mathbb{C}) / \mathrm{SO}(3, \mathbb{C})$, and group compactifications of $\mathrm{SL}(3, \mathbb{C})$ and G_2, respectively (see Example 5.13 of [17]). The above theorem leads us to complete all remaining cases of smooth Fano symmetric varieties with Picard number one also admit Kähler–Einstein metrics.

2. Criterion for Existence of Kähler–Einstein Metrics on Symmetric Varieties

Let G be a connected reductive algebraic group over $\mathbb{C}$.

2.1. Spherical Varieties and Algebraic Moment Polytopes

We review general notions and results about spherical varieties. The normal equivariant embeddings of a given spherical homogeneous space are classified by combinatorial objects called *colored fans*, which generalize the fans appearing in the classification of toric varieties. We refer to the works in [19–21] as references for spherical varieties.

Definition 1. *A normal variety X equipped with an action of G is called* spherical *if a Borel subgroup B of G acts on X with an open and dense orbit.*

Let G/H be an open dense G-orbit of a spherical variety X and T a maximal torus of B. By definition, the *spherical weight lattice* $\mathcal{M}$ of G/H is a subgroup of characters $\chi \in \mathfrak{X}(B) = \mathfrak{X}(T)$ of (nonzero) B-semi-invariant functions in the function field $\mathbb{C}(G/H) = \mathbb{C}(X)$, that is,

$$\mathcal{M} = \{\chi \in \mathfrak{X}(T) : \mathbb{C}(G/H)^{(B)}_{\chi} \neq 0\},$$

where $\mathbb{C}(G/H)^{(B)}_{\chi} = \{f \in \mathbb{C}(G/H) : b \cdot f = \chi(b) f \text{ for all } b \in B\}$. Note that every function f_χ in $\mathbb{C}(G/H)^{(B)}$ is determined by its weight χ up to constant because $\mathbb{C}(G/H)^B = \mathbb{C}$, that is, any B-invariant rational function on X is constant. The spherical weight lattice $\mathcal{M}$ is a free abelian group of finite rank. We define the *rank* of G/H as the rank of the lattice $\mathcal{M}$. Let $\mathcal{N} = \mathrm{Hom}(\mathcal{M}, \mathbb{Z})$ denote its dual lattice together with the natural pairing $\langle \cdot , \cdot \rangle \colon \mathcal{N} \times \mathcal{M} \to \mathbb{Z}$.

As the open B-orbit of a spherical variety X is an affine variety, its complement has pure codimension one and is a finite union of B-stable prime divisors.

Definition 2. *For a spherical variety X, B-stable but not G-stable prime divisors in X are called* colors *of X. A color of X corresponds to a B-stable prime divisor in the open G-orbit G/H of X. We denote by $\mathfrak{D} = \{D_1, \cdots , D_k\}$ the set of colors of X (or G/H).*

As a B-semi-invariant function f_χ in $\mathbb{C}(G/H)^{(B)}_{\chi}$ is unique up to constant, we define the *color map* $\rho \colon \mathfrak{D} \to \mathcal{N}$ by $\langle \rho(D), \chi \rangle = \nu_D(f_\chi)$ for $\chi \in \mathcal{M}$, where ν_D is the discrete valuation associated to a divisor D, that is, $\nu_D(f)$ is the vanishing order of f along D. Unfortunately, the color map is generally not injective. In addition, every discrete $\mathbb{Q}$-valued valuation ν of the function field $\mathbb{C}(G/H)$ induces a homomorphism $\hat{\rho}(\nu)\colon \mathcal{M} \to \mathbb{Q}$ defined by $\langle \hat{\rho}(\nu), \chi \rangle = \nu(f_\chi)$, so that we get a map $\hat{\rho}\colon \{\text{discrete } \mathbb{Q}\text{-valued valuations on } G/H\} \to \mathcal{N} \otimes \mathbb{Q}$. Luna and Vust [22] showed that the restriction of $\hat{\rho}$ to the set of G-invariant discrete valuations on G/H is injective. From now on, we will regard a G-invariant discrete valuation on G/H as an element of $\mathcal{N} \otimes \mathbb{Q}$ via the map $\hat{\rho}$, and in order to simplify the notation $\hat{\rho}(\nu_E)$ will be written as $\hat{\rho}(E)$ for a G-stable divisor E in X.

Let L be a G-linearized ample line bundle on a spherical G-variety X. By the multiplicity-free property of spherical varieties, the algebraic moment polytope $\Delta(X, L)$ encodes the structure of representation of G in the spaces of multi-sections of tensor powers of L.

Definition 3. *The* algebraic moment polytope $\Delta(X, L)$ *of L with respect to B is defined as the closure of $\bigcup_{k \in \mathbb{N}} \Delta_k/k$ in $\mathcal{M} \otimes \mathbb{R}$, where Δ_k is a finite set consisting of (dominant) weights λ such that $H^0(X, L^{\otimes k}) = \bigoplus_{\lambda \in \Delta_k} V_G(\lambda)$. Here, $V_G(\lambda)$ means the irreducible representation of G with highest weight λ.*

For a compact connected Lie group K and a compact connected Hamiltonian K-manifold (M, ω, μ), Kirwan [23] proved that the intersection of the image of M through the moment map μ with the positive Weyl chamber with respect to a Borel subgroup B of G is a convex polytope, where G is the complexification of K. The algebraic moment polytope $\Delta(X, L)$ for a polarized G-variety X was introduced by Brion in [24] as a purely algebraic version of the Kirwan polytope. This is indeed the convex hull of finitely many points in $\mathcal{M} \otimes \mathbb{R}$ (see the work in [24]). Moreover, if X is smooth, then $\Delta(X, L)$ can be interpreted as the Kirwan polytope of (X, ω_L) with respect to the action of a maximal compact subgroup K of G, where ω_L is a K-invariant Kähler form in the first Chern class $c_1(L)$.

Example 1 (Equivariant compactifications of reductive groups)**.** *Any reductive group G is spherical with respect to the action of $G \times G$ by left and right multiplication from the Bruhat decomposition. Let us consider the* wonderful compactification *of a simple algebraic group G of adjoint type constructed by De Concini and Procesi [25]. As a specific example, the wonderful compactification $\mathbb{P}(Mat_{2\times 2}(\mathbb{C})) \cong \mathbb{P}^3$ of the projective general linear group* $\mathrm{PGL}(2, \mathbb{C})$ *has the action of* $\mathrm{PGL}(2, \mathbb{C}) \times \mathrm{PGL}(2, \mathbb{C})$ *induced by the multiplication of matrices on the left and on the right. It is known that the spherical weight lattice $\mathcal{M}$ of the wonderful compactification of a simple algebraic group G of adjoint type coincides with the root lattice of G. As the anticanonical line bundle $K_{\mathbb{P}^3}^{-1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^3}(4)$,*

$$\begin{aligned} H^0(\mathbb{P}^3, K_{\mathbb{P}^3}^{-1}) &= \mathrm{Sym}^4\, \mathbb{C}^4 \\ &\cong \mathrm{End}(V_{\mathrm{PGL}(2,\mathbb{C})}(0)) \oplus \mathrm{End}(V_{\mathrm{PGL}(2,\mathbb{C})}(2\varpi_1)) \oplus \mathrm{End}(V_{\mathrm{PGL}(2,\mathbb{C})}(4\varpi_1)), \end{aligned}$$

where ϖ_1 denotes the fundamental weight of $\mathrm{PGL}(2, \mathbb{C})$*. Repeating this calculation for tensor powers $(K_{\mathbb{P}^3}^{-1})^{\otimes k}$, we obtain*

$$\frac{1}{k}\Delta_k = \left\{0, \frac{2}{k}\varpi_1, \frac{4}{k}\varpi_1, \cdots, \frac{4k-2}{k}\varpi_1, \frac{4k}{k}\varpi_1\right\}.$$

Therefore, the algebraic moment polytope $\Delta(\mathbb{P}^3, K_{\mathbb{P}^3}^{-1})$ of the wonderful compactification of $\mathrm{PGL}(2, \mathbb{C})$ *is a closed interval $[0, 4\varpi_1] = [0, 2\alpha_1]$ in $\mathcal{M} \otimes \mathbb{R} \cong \mathbb{R} \cdot \alpha_1$, where α_1 denotes the simple root of* $\mathrm{PGL}(2, \mathbb{C})$.

2.2. Symmetric Spaces and Symmetric Varieties

For an algebraic group involution θ of a connected reductive algebraic group G, let $G^\theta = \{g \in G : \theta(g) = g\}$ be the subgroup consisting of elements fixed by θ. If H is a closed subgroup of G such that the identity component of H coincides with the identity component of G^θ, then the homogeneous space G/H is called a *symmetric homogeneous space*. By taking a universal cover of G, we can always assume that G is simply connected.

When G is simply connected, by (see Section 8.1 in [26]) G^θ is connected and H is a closed subgroup between G^θ and its normalizer $N_G(G^\theta)$ in G, that is, $G^\theta \subset H \subset N_G(G^\theta)$.

Definition 4. *A normal G-variety X together with an equivariant open embedding $G/H \hookrightarrow X$ of a symmetric homogeneous space G/H is called a* symmetric variety.

Vust proved that a symmetric homogeneous space G/H is spherical (see in [27], Theorem 1 in Section 1.3). By using the Luna–Vust theory on spherical varieties, Ruzzi [18] classified the smooth projective symmetric varieties with Picard number one from the classification of corresponding colored fans. As a result, there are only six nonhomogeneous smooth projective symmetric varieties with Picard number one, and their restricted root systems (see Section 2.3 for the definition) are of either type A_2 or type G_2. Moreover, Ruzzi gave geometric descriptions of them in [28].

In the case that the restricted root system is of type A_2 (Theorem 3 of the work in [28]), the symmetric varieties are the smooth equivariant completions of symmetric homogeneous spaces $\mathrm{SL}(3,\mathbb{C})/\mathrm{SO}(3,\mathbb{C})$, $(\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C}))/\mathrm{SL}(3,\mathbb{C})$, $\mathrm{SL}(6,\mathbb{C})/\mathrm{Sp}(6,\mathbb{C})$, E_6/F_4, and are isomorphic to a general hyperplane section of rational homogeneous manifolds which are in the third row of the *geometric Freudenthal–Tits magic square*.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$v_4(\mathbb{P}^1)$	$\mathbb{P}(T_{\mathbb{P}^2})$	$\mathrm{Gr}_\omega(2,6)$	$\mathbb{OP}^2_0$
$\mathbb{C}$	$v_2(\mathbb{P}^2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$\mathrm{Gr}(2,6)$	$\mathbb{OP}^2$
$\mathbb{H}$	$\mathrm{LGr}(3,6)$	$\mathrm{Gr}(3,6)$	$\mathbb{S}_6$	E_7/P_7
$\mathbb{O}$	F_4^{ad}	E_6^{ad}	E_7^{ad}	E_8^{ad}

Remark 1. *Though all the rational homogeneous manifolds admit Kähler–Einstein metrics, a general hyperplane section of a rational homogeneous manifold is not necessarily the case. For example, a general hyperplane section of the Grassmannian $\mathrm{Gr}(2,2n+1)$, called an odd symplectic Grassmannian of isotropic planes, does not admit Kähler–Einstein metrics by the Matsushima theorem in [3] because the automorphism group of the odd symplectic Grassmannian is not reductive (see Theorem 1.1 in [29]).*

In the case that the restricted root system is of type G_2 (Theorem 2 of [28]), the symmetric varieties are the smooth equivariant completions of either $G_2/(\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))$ or $(G_2 \times G_2)/G_2$. The smooth equivariant completion with Picard number one of the symmetric space $G_2/(\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))$, called the *Cayley Grassmannian*, and the smooth equivariant completion with Picard number one of the symmetric space $(G_2 \times G_2)/G_2$, called the *double Cayley Grassmannian*, have been studied by Manivel [30,31].

Their geometric properties including the dimension, the Fano index, the restricted root system are listed in Table 1. For the deformation rigidity properties of smooth projective symmetric varieties with Picard number one, see in [32].

Table 1. Nonhomogeneous smooth projective symmetric varieties with Picard number one.

X_i	G/G^θ	$\dim X_i$	$\iota(X_i)$	Φ_θ	Multiplicity	Geometric Description
1	$\mathrm{SL}(3,\mathbb{C})/\mathrm{SO}(3,\mathbb{C})$	5	3	A_2	1	hyperplane section of $\mathrm{LGr}(3,6)$
2	$(\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C}))/\mathrm{SL}(3,\mathbb{C})$	8	5	A_2	2	hyperplane section of $\mathrm{Gr}(3,6)$
3	$\mathrm{SL}(6,\mathbb{C})/\mathrm{Sp}(6,\mathbb{C})$	14	9	A_2	4	hyperplane section of $\mathbb{S}_6$
4	E_6/F_4	26	17	A_2	8	hyperplane section of E_7/P_7
5	$G_2/(\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))$	8	4	G_2	1	Cayley Grassmannian
6	$(G_2 \times G_2)/G_2$	14	7	G_2	2	double Cayley Grassmannian

2.3. Existence of Kähler–Einstein Metrics on Symmetric Varieties

We recall Delcroix's criterion for K-stability of smooth Fano symmetric varieties in [17].

For an algebraic group involution θ of G, a torus T in G is *split* if $\theta(t) = t^{-1}$ for any $t \in T$. A torus T is *maximally split* if T is a θ-stable maximal torus in G which contains a split torus T_s of maximal dimension among split tori. Then, θ descends to an involution of $\mathfrak{X}(T)$ for a maximally split torus T, and the rank of a symmetric homogeneous space G/H is equal to the dimension of a maximal split subtorus T_s of T.

Let $\Phi = \Phi(G,T)$ be the root system of G with respect to a maximally split torus T. By Lemma 1.2 of [25], we can take a set of positive roots Φ^+ such that either $\theta(\alpha) = \alpha$ or $\theta(\alpha)$ is a negative root for all $\alpha \in \Phi^+$; then, we denote $2\rho_\theta = \sum_{\alpha \in \Phi^+ \backslash \Phi^\theta} \alpha$, where $\Phi^\theta = \{\alpha \in \Phi : \theta(\alpha) = \alpha\}$. The set $\Phi_\theta = \{\alpha - \theta(\alpha) : \alpha \in \Phi \backslash \Phi^\theta\}$ is a (possibly non-reduced) root system, which is called the *restricted root system*. Let $\mathcal{C}_\theta^+$ denote the cone generated by positive restricted roots in $\Phi_\theta^+ = \{\alpha - \theta(\alpha) : \alpha \in \Phi^+ \backslash \Phi^\theta\}$.

Proposition 1 (Corollary 5.9 of [17])**.** *Let X be a smooth Fano embedding of a symmetric homogeneous space G/H. Then X admits a Kähler–Einstein metric if and only if the barycenter of the moment polytope $\Delta(X, K_X^{-1})$ with respect to the Duistermaat–Heckman measure*

$$\prod_{\alpha \in \Phi^+ \backslash \Phi^\theta} \kappa(\alpha, p)\, dp$$

is in the relative interior of the translated cone $2\rho_\theta + \mathcal{C}_\theta^+$, where κ denotes the Killing form on the Lie algebra $\mathfrak{g}$ of G.

In fact, this result is a direct consequence of a combinatorial criterion for the existence of a Kähler–Ricci soliton on smooth Fano spherical varieties obtained by Delcroix (see in [17], Theorem A). The proof consists of the existence of a special equivariant test configuration with horospherical central fiber and the explicit computation of the modified Futaki invariant on Fano horospherical varieties.

3. Moment Polytopes of Smooth Fano Symmetric Varieties and Their Barycenters

We prove in this section our main result Theorem 1. The proof combines Proposition 1 together with the following result allowing us to compute (algebraic) moment polytopes of Fano symmetric varieties.

Proposition 2. *Let X be a smooth Fano embedding of a symmetric space G/G^θ. Then, there exist integers m_i such that a Weil divisor $-K_X = \sum_{i=1}^k m_i D_i + \sum_{j=1}^\ell E_j$ represents the anticanonical line bundle K_X^{-1} for colors D_i and G-stable divisors E_j in X, and the moment polytope $\Delta(X, K_X^{-1})$ is $2\rho_\theta + Q_X^*$, where the polytope Q_X is the convex hull of the set*

$$\left\{ \frac{\rho(D_i)}{m_i} : i = 1, \cdots, k \right\} \cup \{\hat{\rho}(E_j) : j = 1, \cdots, \ell\}$$

in $\mathcal{N} \otimes \mathbb{R}$ and its dual polytope Q_X^* is defined as $\{m \in \mathcal{M} \otimes \mathbb{R} : \langle n, m \rangle \geq -1$ for every $n \in Q_X\}$.

This statement is a specialization of a result of Gagliardi and Hofscheier ([33], Section 9) in which they studied the anticanonical line bundle on a Gorenstein Fano spherical variety. It is based on the works of Brion [34,35] on algebraic moment polytopes and anticanonical divisors of Fano spherical varieties. For the convenience of the reader, we provide a sketch of the proof.

Proof. Let us recall results about the anticanonical line bundle on a spherical variety from Sections 4.1 and 4.2 in [35]. For a spherical G-variety X, there exists a B-semi-invariant global section $s \in \Gamma(X, K_X^{-1})$ with $\mathrm{div}(s) = \sum_{i=1}^k m_i D_i + \sum_{j=1}^\ell E_j$. Furthermore, the B-weight of this section s is the sum of $\alpha \in \Phi$ such that $\mathfrak{g}_{-\alpha}$ does not stabilize the open B-orbit in X. Thus, when X is a symmetric variety associated to an involution θ of G, the weight of s is equal to $2\rho_\theta = \sum_{\alpha \in \Phi^+ \backslash \Phi^\theta} \alpha$.

For a Gorenstein Fano spherical variety X, Brion obtained the relation between the moment polytope $\Delta(X, K_X^{-1})$ and a polytope Δ_{-K_X} associated to the anticanonical divisor in Proposition 3.3 of [34]. More precisely, if X is a smooth Fano embedding of G/G^θ, then the moment polytope $\Delta(X, K_X^{-1})$ is $2\rho_\theta + \Delta_{-K_X}$ and a polytope Δ_{-K_X} associated to the anticanonical (Cartier) divisor $-K_X$ is the dual polytope Q_X^*. $\square$

Let $\Phi = \Phi(G, T)$ be the root system of G with respect to a maximally split torus T. Fix a set of positive roots Φ^+ such that either $\theta(\alpha) = \alpha$ or $\theta(\alpha)$ is a negative root for all $\alpha \in \Phi^+$. We recall that the *coroot* $\alpha^\vee$ of a root $\alpha \in \Phi$ is defined as the unique element in the Lie algebra $\mathfrak{t}$ of T such that $\alpha(x) = \frac{2\kappa(x,\alpha^\vee)}{\kappa(\alpha^\vee,\alpha^\vee)}$ for all $x \in \mathfrak{t}$. Given a set of simple roots $\{\alpha_1, \alpha_2, \cdots, \alpha_r\} \subset \Phi$, we define the fundamental weights $\{\varpi_1, \varpi_2, \cdots, \varpi_r\}$ dual to the coroots by requiring $\langle \alpha_i^\vee, \varpi_j \rangle = \delta_{i,j}$ for $i, j = 1, 2, \cdots, r = \dim T$.

3.1. Smooth Fano Embedding of $\mathrm{SL}(3,\mathbb{C})/\mathrm{SO}(3,\mathbb{C})$ with Picard Number One

Considering the involution θ of $\mathrm{SL}(n,\mathbb{C})$ defined by sending g to the inverse of its transpose $\theta(g) = (g^t)^{-1}$, which is usually called of Type AI, the subgroup fixed by θ is $\mathrm{SO}(n,\mathbb{C})$. As $\theta(\alpha) = -\alpha$ for $\alpha \in \Phi = \Phi_{\mathrm{SL}_3}$, the set Φ^θ is empty and the restricted root system Φ_θ is the double 2Φ of the root system Φ. The spherical weight lattice $\mathcal{M} = \mathfrak{X}(T/T \cap G^\theta)$ is formed by 2λ for weights $\lambda \in \mathfrak{X}(T)$. Thus, the dual lattice $\mathcal{N}$ is generated by half of the coroots $\frac{1}{2}\alpha_1^\vee, \frac{1}{2}\alpha_2^\vee$ from the relation $\langle \alpha_i^\vee, \varpi_j \rangle = \delta_{i,j}$. In general, Vust [36] proved that when G is semisimple and simply connected, the spherical weight lattice $\mathcal{M}$ of the symmetric space G/G^θ is the lattice of restricted weights determined by the restricted root system, which implies that $\mathcal{N}$ is the lattice generated by *restricted coroots* forming a root system dual to the restricted root system Φ_θ.

Let X_1 be the smooth Fano embedding of $\mathrm{SL}(3,\mathbb{C})/\mathrm{SO}(3,\mathbb{C})$ with Picard number one. Using the description in [28], we know that the two colors D_1, D_2 and the G-stable divisor E in X_1 have the images $\rho(D_1) = \frac{1}{2}\alpha_1^\vee$, $\rho(D_2) = \frac{1}{2}\alpha_2^\vee$ and $\hat{\rho}(E) = -\frac{1}{2}\varpi_1^\vee - \frac{1}{2}\varpi_2^\vee = -\frac{1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee$ in $\mathcal{N}$, respectively. Recall from Theorem 6 of the work in [28] that the maximal colored cones of its colored fan are $(\mathrm{Cone}(\alpha_1^\vee, -\varpi_1^\vee - \varpi_2^\vee), \{D_1\})$ and $(\mathrm{Cone}(\alpha_2^\vee, -\varpi_1^\vee - \varpi_2^\vee), \{D_2\})$. Then, we have two relations $2D_1 - D_2 - E = 0$ and $-D_1 + 2D_2 - E = 0$, so that $D_1 = D_2 = E$ in the Picard group $\mathrm{Pic}(X_1)$.

Proposition 3. *Let X_1 be the smooth Fano embedding of $\mathrm{SL}(3,\mathbb{C})/\mathrm{SO}(3,\mathbb{C})$ with Picard number one. The moment polytope $\Delta_1 = \Delta(X_1, K_{X_1}^{-1})$ is the convex hull of three points 0, $6\varpi_1$, $6\varpi_2$ in $\mathcal{M} \otimes \mathbb{R}$.*

Proof. From the colored data of $\mathrm{SL}(3,\mathbb{C})/\mathrm{SO}(3,\mathbb{C})$ and the G-orbit structure of X_1, we know the relation $-K_{X_1} = D_1 + D_2 + E$ of the anticanonical divisor. Using Proposition 2, $\rho(D_1)$, $\rho(D_2)$, and $\hat{\rho}(E)$ are used as inward-pointing facet normal vectors of the moment polytope $\Delta(X_1, K_{X_1}^{-1})$. First, $\rho(D_1) = \frac{1}{2}\alpha_1^\vee$ gives an inequality

$$\left\langle \frac{1}{2}\alpha_1^\vee, x \cdot 2\varpi_1 + y \cdot 2\varpi_2 - 2\rho_\theta \right\rangle = x - 1 \geq -1$$

because $2\rho_\theta = 2\alpha_1 + 2\alpha_2 = 2\varpi_1 + 2\varpi_2$. Similarly, as $\rho(D_2) = \frac{1}{2}\alpha_2^\vee$ gives a domain $\{x \cdot 2\varpi_1 + y \cdot 2\varpi_2 \in \mathcal{M} \otimes \mathbb{R} : y \geq 0\}$, the images of two colors D_1, D_2 determine the positive Weyl chamber. Last, $\hat{\rho}(E) = -\frac{1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee$ gives a domain $\{x \cdot 2\varpi_1 + y \cdot 2\varpi_2 \in \mathcal{M} \otimes \mathbb{R} : x + y \leq 3\}$. Thus the moment polytope $\Delta(X_1, K_{X_1}^{-1})$ is the intersection of three half-spaces, so that it is the convex hull of three points $0, 6\varpi_1, 6\varpi_2$ in $\mathcal{M} \otimes \mathbb{R}$. $\square$

Corollary 1. *The smooth Fano embedding X_1 of $\mathrm{SL}(3,\mathbb{C})/\mathrm{SO}(3,\mathbb{C})$ with Picard number one admits a Kähler–Einstein metric.*

Proof. Choosing a realization of the root system A_2 in the Euclidean plane $\mathbb{R}^2$ with $\alpha_1 = (1, 0)$ and $\alpha_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, for $p = (x, y)$ we obtain its Duistermaat–Heckman measure

$$\prod_{\alpha\in\Phi^+}\kappa(\alpha,p)\,dp = x\Big(-\frac{x}{2}+\frac{\sqrt{3}}{2}y\Big)\Big(\frac{x}{2}+\frac{\sqrt{3}}{2}y\Big)\,dxdy.$$

From Proposition 3, we can compute the volume

$$\begin{aligned}\mathrm{Vol}_{DH}(\Delta_1) &= \int_0^{\sqrt{3}}\int_0^{\sqrt{3}y} x\Big(-\frac{x}{2}+\frac{\sqrt{3}}{2}y\Big)\Big(\frac{x}{2}+\frac{\sqrt{3}}{2}y\Big)\,dxdy \\ &\quad + \int_{\sqrt{3}}^{2\sqrt{3}}\int_0^{6-\sqrt{3}y} x\Big(-\frac{x}{2}+\frac{\sqrt{3}}{2}y\Big)\Big(\frac{x}{2}+\frac{\sqrt{3}}{2}y\Big)\,dxdy = \frac{27}{5}\sqrt{3}\end{aligned}$$

and the barycenter

$$\mathbf{bar}_{DH}(\Delta_1) = (\bar{x},\bar{y}) = \frac{1}{\mathrm{Vol}_{DH}(\Delta_1)}\int_{\Delta_1} p\prod_{\alpha\in\Phi^+}\kappa(\alpha,p)\,dp = \Big(\frac{5}{4},\frac{5\sqrt{3}}{4}\Big) = \frac{5}{4}\times 2\rho_\theta$$

of the moment polytope Δ_1 with respect to the Duistermaat–Heckman measure. Therefore, $\mathbf{bar}_{DH}(\Delta_1)$ is in the relative interior of the translated cone $2\rho_\theta+\mathcal{C}_\theta^+$ (see Figure 1), so X_1 admits a Kähler–Einstein metric by Proposition 1. $\square$

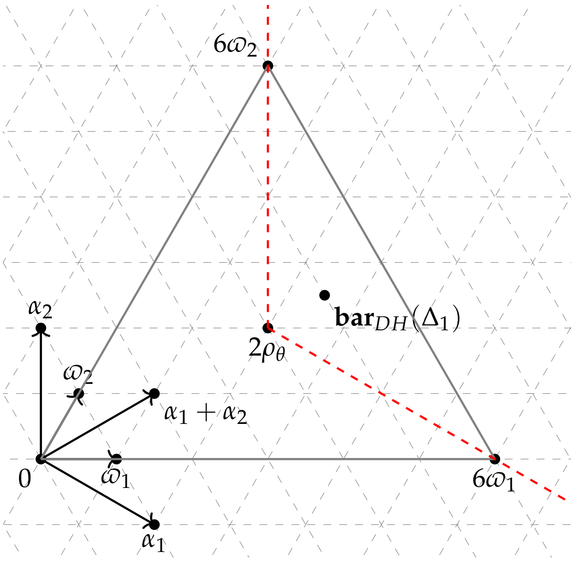

Figure 1. $\Delta_1=\Delta(X_1,K_{X_1}^{-1})$.

3.2. *Smooth Fano Embedding of* $(\mathrm{SL}(3,\mathbb{C})\times\mathrm{SL}(3,\mathbb{C}))/\mathrm{SL}(3,\mathbb{C})$ *with Picard Number One*

Any reductive algebraic group L is a symmetric homogeneous space $(L\times L)/\mathrm{diag}(L)$ under the action of the group $G=L\times L$ for the involution $\theta(g_1,g_2)=(g_2,g_1)$, $g_1,g_2\in L$. If T is a maximal torus of L, then $T\times T$ is a maximal torus of G and we get the spherical weight lattice

$$\mathcal{M}=\mathfrak{X}((T\times T)/\mathrm{diag}(T))=\{(\lambda,-\lambda):\lambda\in\mathfrak{X}(T)\}.$$

Thus, $\mathcal{M}$ can be identified with $\mathfrak{X}(T)$ by the projection to the first coordinate. Under this identification, the restricted root system Φ_θ is identified with the root system Φ_L of L with respect to T, and the dual lattice $\mathcal{N}$ is generated by the coroots $\alpha_1^\vee,\alpha_2^\vee,\cdots,\alpha_r^\vee$, where $r=\dim T$.

Let X_2 be the smooth Fano embedding of $(\mathrm{SL}(3,\mathbb{C})\times\mathrm{SL}(3,\mathbb{C}))/\mathrm{SL}(3,\mathbb{C})$ with Picard number one. Using the description in [28], we know that the two colors D_1,D_2 and the

G-stable divisor E in X_2 have the images $\rho(D_1) = \alpha_1^\vee$, $\rho(D_2) = \alpha_2^\vee$ and $\hat{\rho}(E) = -\alpha_1^\vee - \alpha_2^\vee$ in $\mathcal{N}$, respectively.

Proposition 4. *Let X_2 be the smooth Fano symmetric embedding of $(\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C}))/\mathrm{SL}(3,\mathbb{C})$ with Picard number one. The moment polytope $\Delta_2 = \Delta(X_2, K_{X_2}^{-1})$ is the convex hull of three points $0, 5\varpi_1, 5\varpi_2$ in $\mathcal{M} \otimes \mathbb{R}$.*

Proof. From the colored data of $(\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C}))/\mathrm{SL}(3,\mathbb{C})$ and the G-orbit structure of X_2, we know the relation $-K_{X_2} = 2D_1 + 2D_2 + E$ of the anticanonical divisor. Using Proposition 2, $\frac{1}{2}\rho(D_1)$, $\frac{1}{2}\rho(D_2)$, and $\hat{\rho}(E)$ are used as inward-pointing facet normal vectors of the moment polytope $\Delta(X_2, K_{X_2}^{-1})$. First, $\frac{1}{2}\rho(D_1) = \frac{1}{2}\alpha_1^\vee$ gives an inequality

$$\left\langle \frac{1}{2}\alpha_1^\vee, x \cdot \varpi_1 + y \cdot \varpi_2 - 2\rho_\theta \right\rangle = \frac{1}{2}(x - 2) \geq -1$$

because $2\rho_\theta = 2\alpha_1 + 2\alpha_2 = 2\varpi_1 + 2\varpi_2$. Similarly, as $\frac{1}{2}\rho(D_2) = \frac{1}{2}\alpha_2^\vee$ gives a domain $\{x \cdot \varpi_1 + y \cdot \varpi_2 \in \mathcal{M} \otimes \mathbb{R} : y \geq 0\}$, the images of two colors D_1, D_2 determine the positive Weyl chamber. Lastly, $\hat{\rho}(E) = -\alpha_1^\vee - \alpha_2^\vee$ gives a domain $\{x \cdot \varpi_1 + y \cdot \varpi_2 \in \mathcal{M} \otimes \mathbb{R} : x + y \leq 5\}$. Thus, the moment polytope $\Delta(X_2, K_{X_2}^{-1})$ is the intersection of three half-spaces, so that it is the convex hull of three points $0, 5\varpi_1, 5\varpi_2$ in $\mathcal{M} \otimes \mathbb{R}$. □

Corollary 2. *The smooth Fano embedding X_2 of $(\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C}))/\mathrm{SL}(3,\mathbb{C})$ with Picard number one admits a Kähler–Einstein metric.*

Proof. As in the proof of Corollary 1, we choose a realization of the root system A_2 in the Euclidean plane $\mathbb{R}^2$ with $\alpha_1 = (1,0)$ and $\alpha_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Then, the Duistermaat–Heckman measure on the moment polytope is given as

$$\prod_{\alpha \in \Phi^+} \kappa(\alpha, p)\, dp = \prod_{\beta \in \Phi^+_{\mathrm{SL}(3,\mathbb{C})}} \kappa(\beta, p)^2\, dp = x^2 \left(-\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2 \left(\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2 dxdy.$$

From Proposition 4, we can compute

$$\begin{aligned} \mathrm{Vol}_{DH}(\Delta_2) &= \int_0^{\frac{5}{6}\sqrt{3}} \int_0^{\sqrt{3}y} x^2 \left(-\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2 \left(\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2 dxdy \\ &\quad + \int_{\frac{5}{6}\sqrt{3}}^{\frac{5}{3}\sqrt{3}} \int_0^{5-\sqrt{3}y} x^2 \left(-\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2 \left(\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2 dxdy = \frac{78,125}{18,432}\sqrt{3} \end{aligned}$$

and the barycenter

$$\begin{aligned} \mathbf{bar}_{DH}(\Delta_2) = (\bar{x}, \bar{y}) &= \frac{1}{\mathrm{Vol}_{DH}(\Delta_2)} \left(\int_{\Delta_2} x \prod_{\alpha \in \Phi^+} \kappa(\alpha, p)\, dp, \int_{\Delta_2} y \prod_{\alpha \in \Phi^+} \kappa(\alpha, p)\, dp \right) \\ &= \left(\frac{10}{9}, \frac{10\sqrt{3}}{9} \right) = \frac{10}{9} \times 2\rho_\theta \end{aligned}$$

of the moment polytope Δ_2 with respect to the Duistermaat–Heckman measure. Therefore, $\mathbf{bar}_{DH}(\Delta_2)$ is in the relative interior of the translated cone $2\rho_\theta + \mathcal{C}_\theta^+$ (see Figure 2), so X_2 admits a Kähler–Einstein metric by Proposition 1. □

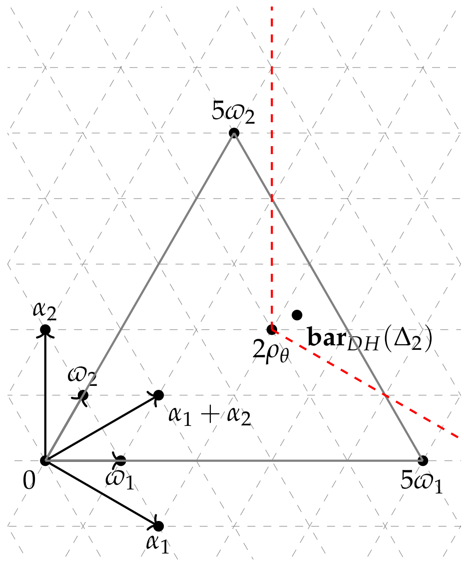

Figure 2. $\Delta_2 = \Delta(X_2, K_{X_2}^{-1})$.

3.3. Smooth Fano Embedding of $\mathrm{SL}(6,\mathbb{C})/\mathrm{Sp}(6,\mathbb{C})$ with Picard Number One

Recall the involution of Type AII. Let θ be an involution of $\mathrm{SL}(2m,\mathbb{C})$ defined by $\theta(g) = J_m(g^t)^{-1}J_m^t$, where J_m is the $2m\times 2m$ block diagonal matrix formed by $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. Then, $G^\theta = \mathrm{Sp}(2m,\mathbb{C})$ is the group of elements that preserve a nondegenerate skew-symmetric bilinear form $\omega(v,w) = v^t J_m w$. We can check that the restricted root system Φ_θ is the root system of type A_2 with multiplicity four, and the spherical weight lattice $\mathcal{M} = \mathfrak{X}(T/T\cap G^\theta)$ is generated by 2λ for weights $\lambda\in\mathfrak{X}(T_s)$, where T_s denotes a split subtorus of dimension two in a maximal torus $T\subset \mathrm{SL}(6,\mathbb{C})$. In fact, if we choose the torus of diagonal matrices as T, then the maximal split torus T_s consists of diagonal matrices of the form $\mathrm{diag}(a_1,a_1,a_2,a_2,a_3,a_3)$ with $a_1,a_2,a_3\in\mathbb{C}^*$ and $a_1^2a_2^2a_3^2 = 1$. Denoting by $\alpha_k\colon T_s\to\mathbb{C}^*$ for $k=1,2$ the characters defined by

$$\alpha_k(\mathrm{diag}(a_1,a_1,a_2,a_2,a_3,a_3)) = \frac{a_k}{a_{k+1}},$$

we have the restricted root system $\Phi_\theta = \{\pm 2\alpha_1, \pm 2\alpha_2, \pm(2\alpha_1+2\alpha_2)\}$ of type A_2. Then, the dual lattice $\mathcal{N}$ is generated by the coroots $\frac{1}{2}\alpha_1^\vee, \frac{1}{2}\alpha_2^\vee$.

Let X_3 be the smooth Fano embedding of $\mathrm{SL}(6,\mathbb{C})/\mathrm{Sp}(6,\mathbb{C})$ with Picard number one. Using the description in [28], we know that the two colors D_1, D_2 and the G-stable divisor E in X_3 have the images $\rho(D_1) = \frac{1}{2}\alpha_1^\vee$, $\rho(D_2) = \frac{1}{2}\alpha_2^\vee$ and $\hat\rho(E) = -\frac{1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee$ in $\mathcal{N}$, respectively.

Proposition 5. *Let X_3 be the smooth Fano symmetric embedding of $\mathrm{SL}(6,\mathbb{C})/\mathrm{Sp}(6,\mathbb{C})$ with Picard number one. The moment polytope $\Delta_3 = \Delta(X_3, K_{X_3}^{-1})$ is the convex hull of three points 0, $18\varpi_1$, $18\varpi_2$ in $\mathcal{M}\otimes\mathbb{R}$.*

Proof. From the colored data of $\mathrm{SL}(6,\mathbb{C})/\mathrm{Sp}(6,\mathbb{C})$ and the G-orbit structure of X_3, we know the relation $-K_{X_3} = 4D_1 + 4D_2 + E$ of the anticanonical divisor. Using Proposition 2, $\frac{1}{4}\rho(D_1)$, $\frac{1}{4}\rho(D_2)$ and $\hat\rho(E)$ are used as inward-pointing facet normal vectors of the moment polytope $\Delta(X_3, K_{X_3}^{-1})$. Like the previous computations, $\frac{1}{4}\rho(D_1)$ and $\frac{1}{4}\rho(D_2)$ determine the positive restricted Weyl chamber. Indeed, $\frac{1}{4}\rho(D_1) = \frac{1}{8}\alpha_1^\vee$ gives an inequality

$$\left\langle \frac{1}{8}\alpha_1^\vee, x\cdot 2\varpi_1 + y\cdot 2\varpi_2 - 2\rho_\theta \right\rangle = \frac{1}{8}(2x-8) \geq -1$$

because $2\rho_\theta = 8\alpha_1 + 8\alpha_2 = 8\varpi_1 + 8\varpi_2$. As $\hat\rho(E) = -\frac{1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee$ gives a domain $\{x\cdot 2\varpi_1 + y\cdot 2\varpi_2 \in \mathcal{M}\otimes\mathbb{R} : x+y\leq 9\}$, the moment polytope $\Delta(X_3, K_{X_3}^{-1})$ is the intersection of this half-space with the positive restricted Weyl chamber. Thus $\Delta(X_3, K_{X_3}^{-1})$ is the convex hull of three points $0, 18\varpi_1, 18\varpi_2$ in $\mathcal{M} \otimes \mathbb{R}$. $\square$

Corollary 3. *The smooth Fano embedding X_3 of $\mathrm{SL}(6,\mathbb{C})/\mathrm{Sp}(6,\mathbb{C})$ with Picard number one admits a Kähler–Einstein metric.*

Proof. As the multiplicity of each restricted root in the restricted root system Φ_θ is four, the Duistermaat–Heckman measure on $\mathcal{M} \otimes \mathbb{R}$ is given as

$$\prod_{\alpha \in \Phi^+ \setminus \Phi^\theta} \kappa(\alpha, p)\, dp = x^4 \Big(-\frac{x}{2} + \frac{\sqrt{3}}{2} y \Big)^4 \Big(\frac{x}{2} + \frac{\sqrt{3}}{2} y \Big)^4 dxdy.$$

Then, the barycenter

$$\mathbf{bar}_{DH}(\Delta_3) = (\bar{x}, \bar{y}) = \left(\frac{21}{5}, \frac{21\sqrt{3}}{5} \right) = \frac{21}{20} \times 2\rho_\theta$$

is in the relative interior of the translated cone $2\rho_\theta + \mathcal{C}_\theta^+$ (see Figure 3). Therefore, X_3 admits a Kähler–Einstein metric by Proposition 1. $\square$

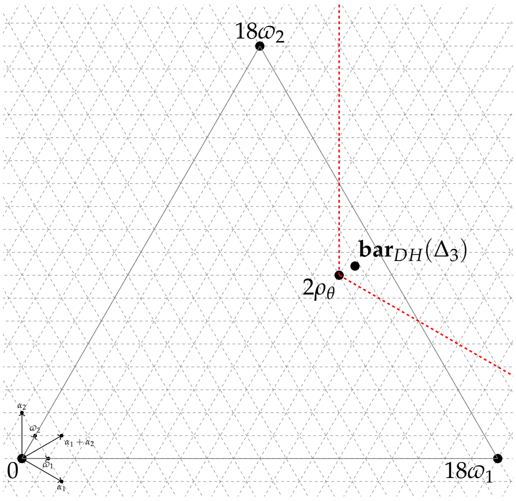

Figure 3. $\Delta_3 = \Delta(X_3, K_{X_3}^{-1})$.

3.4. Smooth Fano Embedding of E_6/F_4 with Picard Number One

Let θ be the involution on the simple algebraic group E_6 of Type EIV. Then, G^θ is isomorphic to the simple algebraic group F_4, and the restricted root system Φ_θ is the root system of type A_2 generated by the simple restricted roots $2\alpha_1, 2\alpha_2$ with multiplicity eight. The spherical weight lattice $\mathcal{M} = \mathfrak{X}(T/T \cap G^\theta)$ is generated by 2λ for weights $\lambda \in \mathfrak{X}(T_s)$, where T_s denotes a split subtorus of dimension two in a maximal torus $T \subset E_6$, so that the dual lattice $\mathcal{N}$ is generated by the coroots $\frac{1}{2}\alpha_1^\vee, \frac{1}{2}\alpha_2^\vee$.

Let X_4 be the smooth Fano embedding of E_6/F_4 with Picard number one. Using the description in [28], we know that the two colors D_1, D_2 and the G-stable divisor E in X_4 have the images $\rho(D_1) = \frac{1}{2}\alpha_1^\vee, \rho(D_2) = \frac{1}{2}\alpha_2^\vee$ and $\hat{\rho}(E) = -\frac{1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee$ in $\mathcal{N}$, respectively.

Proposition 6. *Let X_4 be the smooth Fano symmetric embedding of E_6/F_4 with Picard number one. The moment polytope $\Delta_4 = \Delta(X_4, K_{X_4}^{-1})$ is the convex hull of three points $0, 34\varpi_1, 34\varpi_2$ in $\mathcal{M} \otimes \mathbb{R}$.*

Proof. From the colored data of E_6/F_4 and the G-orbit structure of X_4, we know the relation $-K_{X_4} = 8D_1 + 8D_2 + E$ of the anticanonical divisor. Using Proposition 2, $\frac{1}{8}\rho(D_1)$, $\frac{1}{8}\rho(D_2)$ and $\hat{\rho}(E)$ are used as inward-pointing facet normal vectors of the moment polytope

$\Delta(X_4, K_{X_4}^{-1})$. In particular, $\frac{1}{8}\rho(D_1)$ and $\frac{1}{8}\rho(D_2)$ determine the positive restricted Weyl chamber. Indeed, $\frac{1}{8}\rho(D_1) = \frac{1}{16}\alpha_1^\vee$ gives an inequality

$$\left\langle \frac{1}{16}\alpha_1^\vee, x \cdot 2\varpi_1 + y \cdot 2\varpi_2 - 2\rho_\theta \right\rangle = \frac{1}{16}(2x - 16) \geq -1$$

because $2\rho_\theta = 16\alpha_1 + 16\alpha_2 = 16\varpi_1 + 16\varpi_2$. As $\hat{\rho}(E) = -\frac{1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee$ gives a domain $\{x \cdot 2\varpi_1 + y \cdot 2\varpi_2 \in \mathcal{M} \otimes \mathbb{R} : x + y \leq 17\}$, the moment polytope $\Delta(X_4, K_{X_4}^{-1})$ is the intersection of this half-space with the positive restricted Weyl chamber. Thus $\Delta(X_4, K_{X_4}^{-1})$ is the convex hull of three points $0, 34\varpi_1, 34\varpi_2$ in $\mathcal{M} \otimes \mathbb{R}$. $\square$

Corollary 4. *The smooth Fano embedding X_4 of E_6/F_4 with Picard number one admits a Kähler–Einstein metric.*

Proof. As the multiplicity of each restricted root in the restricted root system Φ_θ is eight, the Duistermaat–Heckman measure on $\mathcal{M} \otimes \mathbb{R}$ is given as

$$\prod_{\alpha \in \Phi^+ \setminus \Phi^\theta} \kappa(\alpha, p)\, dp = x^8 \left(-\frac{x}{2} + \frac{\sqrt{3}}{2}y \right)^8 \left(\frac{x}{2} + \frac{\sqrt{3}}{2}y \right)^8 dxdy.$$

Then, the barycenter

$$\mathbf{bar}_{DH}(\Delta_4) = (\bar{x}, \bar{y}) = \left(\frac{221}{27}, \frac{221\sqrt{3}}{27} \right) = \frac{221}{216} \times 2\rho_\theta$$

is in the relative interior of the translated cone $2\rho_\theta + \mathcal{C}_\theta^+$ (see Figure 4). Therefore, X_4 admits a Kähler–Einstein metric by Proposition 1. $\square$

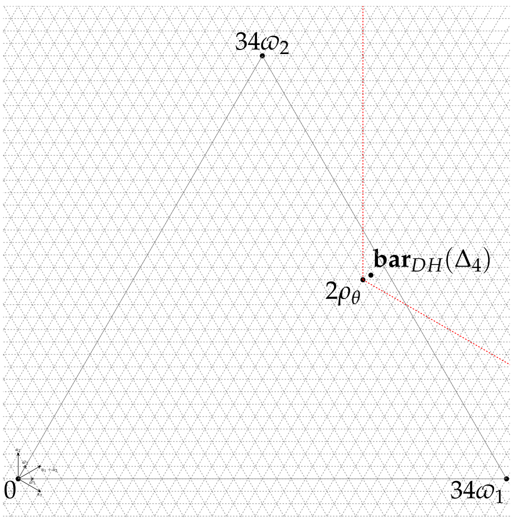

Figure 4. $\Delta_4 = \Delta(X_4, K_{X_4}^{-1})$.

3.5. Smooth Fano Embedding of $G_2/(\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))$ with Picard Number One

Let θ be the unique nontrivial involution on the simple algebraic group G_2. Then, G^θ is isomorphic to $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$, but Φ^θ is empty and the restricted root system Φ_θ is the root system of type G_2. The spherical weight lattice $\mathcal{M} = \mathfrak{X}(T/T \cap G^\theta)$ is generated by 2λ for weights $\lambda \in \mathfrak{X}(T)$ of a maximal torus $T \subset G_2$, so that the dual lattice $\mathcal{N}$ is generated by the coroots $\frac{1}{2}\alpha_1^\vee, \frac{1}{2}\alpha_2^\vee$.

Let X_5 be the smooth Fano embedding of $G_2/(\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))$ with Picard number one. Using the description in [28], we know that the two colors D_1, D_2 and the G-stable divisor E in X_5 have the images $\frac{1}{2}\alpha_1^\vee$, $\frac{1}{2}\alpha_2^\vee$ and $-\frac{1}{2}\varpi_2^\vee$ in $\mathcal{N}$, respectively. Recall that the maximal colored cone of its colored fan is $(\mathrm{Cone}(\alpha_2^\vee, -\varpi_2^\vee), \{D_2\})$ from Theorem 6 of [28]. Then we have two relations $2D_1 - D_2 = 0$ and $-3D_1 + 2D_2 - E = 0$, so that $D_2 = 2D_1 = 2E$ in $\mathrm{Pic}(X_5)$.

Choose a realization of the root system G_2 in the Euclidean plane $\mathbb{R}^2$ with $\alpha_1 = (1,0)$ and $\alpha_2 = \left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$. Then, the complex Lie group G_2 has six positive roots:

$$\begin{aligned}\Phi^+ &= \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\} \\ &= \left\{(1,0), \left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right), (0, \sqrt{3})\right\}.\end{aligned}$$

From the relation $(\alpha_i^\vee, \varpi_j) = \delta_{ij}$, the fundamental weights corresponding to the system of simple roots are $\varpi_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\varpi_2 = (0, \sqrt{3})$.

Proposition 7. *Let X_5 be the smooth Fano symmetric embedding of $G_2/(\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))$ with Picard number one. The moment polytope $\Delta_5 = \Delta(X_5, K_{X_5}^{-1})$ is the convex hull of three points $0, 8\varpi_1, 4\varpi_2$ in $\mathcal{M} \otimes \mathbb{R}$.*

Proof. From the colored data of $G_2/(\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))$ and the G-orbit structure of X_5, we know the relation $-K_{X_5} = D_1 + D_2 + E$ of the anticanonical divisor. Using Proposition 2, $\rho(D_1)$, $\rho(D_2)$ and $\hat{\rho}(E)$ are used as inward-pointing facet normal vectors of the moment polytope $\Delta(X_5, K_{X_5}^{-1})$. As before, $\rho(D_1)$ and $\rho(D_2)$ determine the positive Weyl chamber. Indeed, $\rho(D_1) = \frac{1}{2}\alpha_1^\vee$ gives an inequality

$$\left\langle \frac{1}{2}\alpha_1^\vee, x \cdot 2\varpi_1 + y \cdot 2\varpi_2 - 2\rho_\theta \right\rangle = \frac{1}{2}(2x - 2) \geq -1$$

because $2\rho_\theta = 10\alpha_1 + 6\alpha_2 = 2\varpi_1 + 2\varpi_2$. As $\hat{\rho}(E) = -\frac{1}{2}\varpi_2^\vee = \left(0, -\frac{1}{\sqrt{3}}\right)$ gives a domain $\{x \cdot 2\varpi_1 + y \cdot 2\varpi_2 \in \mathcal{M} \otimes \mathbb{R} : x + 2y \leq 4\}$ from $\langle \varpi_2^\vee, \varpi_1 \rangle = 1$ and $\langle \varpi_2^\vee, \varpi_2 \rangle = 2$, the moment polytope $\Delta(X_5, K_{X_5}^{-1})$ is the intersection of this half-space with the positive Weyl chamber. Thus, $\Delta(X_5, K_{X_5}^{-1})$ is the convex hull of three points 0, $8\varpi_1 = (4, 4\sqrt{3})$, $4\varpi_2 = (0, 4\sqrt{3})$ in $\mathcal{M} \otimes \mathbb{R}$. □

Corollary 5. *The smooth Fano embedding X_5 of $G_2/(\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))$ with Picard number one admits a Kähler–Einstein metric.*

Proof. For $p = (x, y)$, the Duistermaat–Heckman measure on $\mathcal{M} \otimes \mathbb{R}$ is given as

$$\prod_{\alpha \in \Phi^+} \kappa(\alpha, p)\, dp = x\left(-\frac{3}{2}x + \frac{\sqrt{3}}{2}y\right)\left(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)\left(\frac{3}{2}x + \frac{\sqrt{3}}{2}y\right)(\sqrt{3}y)\, dxdy.$$

From Proposition 7, we can compute the volume

$$\begin{aligned}\mathrm{Vol}_{DH}(\Delta_5) &= \int_0^{4\sqrt{3}} \int_0^{\frac{y}{\sqrt{3}}} x\left(-\frac{3}{2}x + \frac{\sqrt{3}}{2}y\right)\left(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)\left(\frac{3}{2}x + \frac{\sqrt{3}}{2}y\right)(\sqrt{3}y)\, dxdy \\ &= 29{,}952\sqrt{3},\end{aligned}$$

and the barycenter

$$\mathbf{bar}_{DH}(\Delta_5) = (\bar{x}, \bar{y}) = \left(\frac{512}{273}, \frac{32\sqrt{3}}{9}\right) \approx (1.875, 3.556 \times \sqrt{3})$$

of the moment polytope Δ_5 with respect to the Duistermaat–Heckman measure. Therefore, $\mathbf{bar}_{DH}(\Delta_5)$ is in the relative interior of the translated cone $2\rho_\theta + \mathcal{C}_\theta^+$ (see Figure 5), so X_5 admits a Kähler–Einstein metric by Proposition 1. □

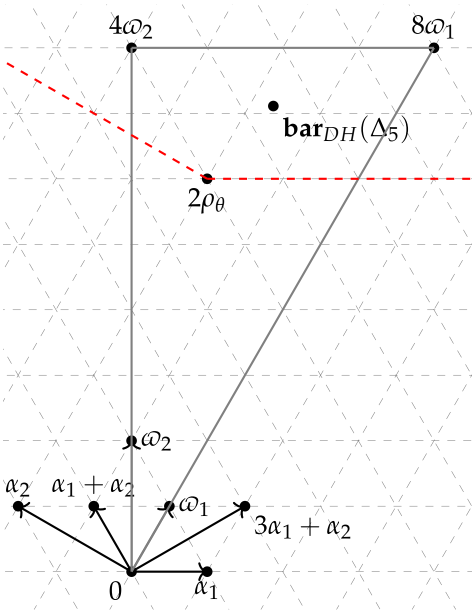

Figure 5. $\Delta_5 = \Delta(X_5, K_{X_5}^{-1})$.

3.6. Smooth Fano Embedding of $(G_2 \times G_2)/G_2$ with Picard Number One

As explained in Section 3.2, the simple algebraic group G_2 can be considered as a symmetric homogeneous space $(G_2 \times G_2)/\mathrm{diag}(G_2)$ under the action of the group $G_2 \times G_2$ for the involution $\theta(g_1, g_2) = (g_2, g_1)$, $g_1, g_2 \in G_2$. The spherical weight lattice $\mathcal{M}$ can be identified with the character group $\mathfrak{X}(T)$ of a maximal torus $T \subset G_2$ by the projection to the first coordinate, and the dual lattice $\mathcal{N}$ is generated by the coroots $\alpha_1^\vee, \alpha_2^\vee$.

Let X_6 be the smooth Fano embedding of $(G_2 \times G_2)/G_2$ with Picard number one. Using the description in [28], we know that the two colors D_1, D_2 and the G-stable divisor E in X_6 have the images $\rho(D_1) = \alpha_1^\vee$, $\rho(D_2) = \alpha_2^\vee$, and $\hat{\rho}(E) = -\varpi_2^\vee$ in $\mathcal{N}$, respectively.

Proposition 8. *Let X_6 be the smooth Fano symmetric embedding of $(G_2 \times G_2)/G_2$ with Picard number one. The moment polytope $\Delta_6 = \Delta(X_6, K_{X_6}^{-1})$ is the convex hull of three points 0, $7\varpi_1$, $\frac{7}{2}\varpi_2$ in $\mathcal{M} \otimes \mathbb{R}$.*

Proof. From the colored data of $(G_2 \times G_2)/G_2$ and the G-orbit structure of X_6, we know the relation $-K_{X_6} = 2D_1 + 2D_2 + E$ of the anticanonical divisor. Using Proposition 2, $\frac{1}{2}\rho(D_1)$, $\frac{1}{2}\rho(D_2)$, and $\hat{\rho}(E)$ are used as inward-pointing facet normal vectors of the moment polytope $\Delta(X_6, K_{X_6}^{-1})$. As $2\rho_\theta = 10\alpha_1 + 6\alpha_2 = 2\varpi_1 + 2\varpi_2$, $\frac{1}{2}\rho(D_1) = \frac{1}{2}\alpha_1^\vee$ gives an inequality

$$\left\langle \frac{1}{2}\alpha_1^\vee, x \cdot \varpi_1 + y \cdot \varpi_2 - 2\rho_\theta \right\rangle = \frac{1}{2}(x-2) \geq -1.$$

Therefore, $\frac{1}{2}\rho(D_1)$ and $\frac{1}{2}\rho(D_2)$ determine the positive Weyl chamber. In the same way, as $\hat{\rho}(E) = -\varpi_2^\vee$ gives a domain $\{x \cdot \varpi_1 + y \cdot \varpi_2 \in \mathcal{M} \otimes \mathbb{R} : x + 2y \leq 7\}$, the moment polytope $\Delta(X_6, K_{X_6}^{-1})$ is the intersection of this half-space with the positive Weyl chamber. Thus $\Delta(X_6, K_{X_6}^{-1})$ is the convex hull of three points 0, $7\varpi_1 = \left(\frac{7}{2}, \frac{7\sqrt{3}}{2}\right)$, $\frac{7}{2}\varpi_2 = \left(0, \frac{7\sqrt{3}}{2}\right)$ in $\mathcal{M} \otimes \mathbb{R}$. □

Corollary 6. *The smooth Fano embedding X_6 of $(G_2 \times G_2)/G_2$ with Picard number one admits a Kähler–Einstein metric.*

Proof. We can compute the barycenter of Δ_6 with respect to the Duistermaat–Heckman measure

$$\mathbf{bar}_{DH}(\Delta_6) = (\bar{x}, \bar{y}) = \left(\frac{139601}{79360}, \frac{49\sqrt{3}}{15}\right) \approx (1.759, 3.267 \times \sqrt{3})$$

from

$$\mathrm{Vol}_{DH}(\Delta_6) = \int_0^{\frac{7\sqrt{3}}{2}} \int_0^{\frac{y}{\sqrt{3}}} x^2 \Big(-\frac{3}{2}x + \frac{\sqrt{3}}{2}y\Big)^2 \Big(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y\Big)^2 \Big(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\Big)^2 \Big(\frac{3}{2}x + \frac{\sqrt{3}}{2}y\Big)^2 (\sqrt{3}y)^2 \, dxdy.$$

As $2\rho_\theta = (1, 3\sqrt{3})$ and the cone $\mathcal{C}_\theta^+$ is generated by the vectors $(1,0)$ and $(-3, \sqrt{3})$, the barycenter $\mathbf{bar}_{DH}(\Delta_6)$ is in the relative interior of the translated cone $2\rho_\theta + \mathcal{C}_\theta^+$ (see Figure 6). Therefore, X_6 admits a Kähler–Einstein metric by Proposition 1. □

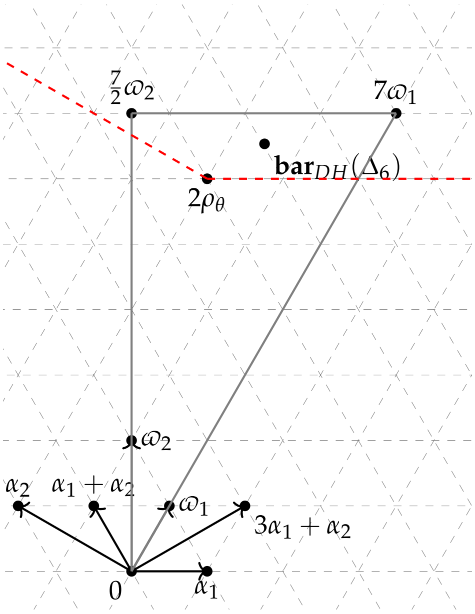

Figure 6. $\Delta_6 = \Delta(X_6, K_{X_6}^{-1})$.

Finally, by Ruzzi's classification of the smooth projective symmetric varieties with Picard number one in [18], Corollaries 1–6 imply the following statement. Therefore, we conclude Theorem 1.

Theorem 2. *All smooth Fano symmetric varieties with Picard number one admit Kähler–Einstein metrics.*

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