

Article

On Admissible Orders on the Set of Discrete Fuzzy Numbers for Application in Decision Making Problems

Juan Vicente Riera ^{1,2,*} , Sebastia Massanet ^{1,2} , Humberto Bustince ³  and Javier Fernandez ³ 

¹ Department of Mathematics and Computer Science, University of the Balearic Islands, Ctra. de Valldemossa, Km.7.5, 07122 Palma de Mallorca, Spain; s.massanet@uib.es

² Health Research Institute of the Balearic Islands (IdISBa), 07010 Palma, Spain

³ Department of Statistics, Computer Science and Mathematics, Campus Arrosadia, Public University of Navarre, 31006 Pamplona, Spain; bustince@unavarra.es (H.B.); fcojavier.fernandez@unavarra.es (J.F.)

* Correspondence: jvicente.riera@uib.es; Tel.: +34-971-17-30-86

Abstract: The study of orders is a constantly evolving topic, not only for its interest from a theoretical point of view, but also for its possible applications. Recently, one of the hot lines of research has been the construction of admissible orders in different frameworks. Following this direction, this paper presents a new representation theorem in the field of discrete fuzzy numbers that enables the construction of two families of admissible orders in the set of discrete fuzzy numbers whose support is a closed interval of a finite chain, leading to the first admissible orders introduced in this framework.

Keywords: discrete fuzzy number; total order; admissible orders; ranking discrete fuzzy numbers



Citation: Riera, J.V.; Massanet, S.; Bustince, H.; Fernandez, J. On Admissible Orders on the Set of Discrete Fuzzy Numbers for Application in Decision Making Problems. *Mathematics* **2021**, *9*, 95. <https://doi.org/10.3390/math9010095>

Received: 4 December 2020

Accepted: 29 December 2020

Published: 4 January 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

It is well-established in the scientific community that the traditional scheme for the solution of a linguistic decision analysis in a Multi-criteria Decision Making Problem (MCDM) consists of three parts [1]:

1. The choice of the linguistic terms and/or their semantics.
2. The choice of the aggregation operator of the linguistic information.
3. The choice of the best alternatives. This step is further divided into two phases:
 - i. Aggregation phase of linguistic information.
 - ii. Exploitation phase, in which a hierarchical order is established among the alternatives according to the value of the collective (or aggregated) linguistic interpretation in order to choose the best alternatives.

From this scheme, the need to study and develop computational linguistic models emerges that allow for representing, aggregating, and ordering such linguistic information as a support tool to help the experts to express their preferences in the most flexible way. In the last few decades, Fuzzy Sets theory [2–4] has shown its potential to design linguistic models which allow for adequately describing the assessments of experts in a decision-making problem. Among the Fuzzy Linguistic Models (FLM), we want to highlight the one based on hesitant fuzzy linguistic term sets [5,6], the one based on Type-2 fuzzy sets [7] and the six categories of fuzzy multi-granular language models established in [8], namely, (1) the traditional multi-granular FLM based on fuzzy membership functions [9,10]; (2) the ordinal multi-granular FLM based on a basic Linguistic Term Set (LTS) [11,12]; (3) the ordinal multi-granular FLM based on 2-tuple FLM [13,14]; (4) the ordinal multi-granular FLM based on hierarchical trees [15]; (5) the multi-granular FLM based on qualitative description spaces [16]; and (6) the ordinal multi-granular FLM based on discrete fuzzy numbers [17,18].

As it has been aforementioned, when making a final decision on a decision-making problem, it is necessary to choose the best alternative or set of alternatives among all the available alternatives and therefore it is necessary to establish a hierarchical ordering of the alternatives based on the preferences given by the experts. The study of orders is not only an important topic from a theoretical point of view, but also for its relevance in many fields such as: decision-making problems, artificial intelligence, optimization problems, etc. It is obvious that the ordering will depend on the fuzzy linguistic computational model and the order considered in each particular decision-making problem. For this reason, the study of orders in the framework of fuzzy sets has been, and still is, a very hot topic. In this direction, many methods have been proposed in the literature to order fuzzy numbers [19,20]. In 2014, Wang and Wang [21] studied total orders in the set of fuzzy numbers using upper dense sequences in the interval $(0, 1]$ and in [22] the concept of admissible order is introduced in the set $\Pi[0, 1] = \{[a, b] \subseteq [0, 1]\}$, understood as a total (linear) order that refines the product order \leq_2 in $\Pi[0, 1]$ where $[a, b] \leq_2 [c, d]$ if and only if $a \leq c$ and $b \leq d$. Moreover, in [22], a method to generate new admissible orders using aggregation functions is presented. The idea of admissible orders has also been adapted to the frameworks of interval-valued Atanassov intuitionistic fuzzy sets being used in decision-making problems [23] and of hesitant sets [24,25].

Discrete fuzzy numbers [26] and, specifically, discrete fuzzy numbers whose support is a closed interval of a finite chain $L_n = \{0, 1, \dots, n\}$ have been thoroughly analyzed in the literature [27–30]. The main reason is that these operators have provided the theoretical foundations of (i) the multigranular linguistic model based on discrete fuzzy numbers [17,18,31] and (ii) the adaptation of the linguistic model based on Z-numbers [32] called mixed-discrete Z-numbers, recently published in [33]. Among the main advantages of this linguistic model, the following properties stand out [8,18]: (i) they allow experts to elicit their preferences in a very flexible way by using different types of granularity, (ii) there is no need to make any transformations to the linguistic expressions before being aggregated, and (iii) there is no loss of information during the aggregation process.

In the framework of discrete fuzzy numbers, as far as we know, orders have been scarcely investigated and the only serious proposals are based on different adaptations of orders among fuzzy numbers. These papers mostly consider ranking indices [34–38], which, from our point of view, present some undesirable behaviors in some applications. Therefore, in this paper, the main goal will be the construction of two different families of admissible orders in the set of discrete fuzzy numbers whose support is a closed interval of a finite chain L_n which are not based on any index function.

After this introduction, in Section 2, the basic concepts and results related to orders and discrete fuzzy numbers are presented to make the work self-contained. Section 3 discusses the problems derived from the use orders based on ranking indices when used in the exploitation phase in a decision-making problem. In particular, this problem is studied in detail with the order presented in [35]. Section 4 constitutes the main core of the paper. A new decomposition theorem in the set of discrete fuzzy numbers is presented that allows for defining two families of admissible orders in $\mathcal{A}_{L_n}^1$. The last section is devoted to some conclusions and possible lines of future work.

2. Preliminaries

In this section, we will present the main concepts related to orders and discrete fuzzy numbers that will be used later.

Definition 1 ([22]). *Given a non-empty set A , a partial order \preceq on the set A is a binary relation on A which is reflexive, antisymmetric, and transitive, i.e., the following properties hold:*

- for each $a \in A$, $a \preceq a$ (reflexivity),
- all $a, b \in L$, if $a \preceq b$ and $b \preceq a$, then $a = b$ (antisymmetry),
- for all $a, b, c \in A$, if $a \preceq b$ and $b \preceq c$, then $a \preceq c$ (transitivity).

We will write $a \prec b$ if $a \preceq b$ but $a \neq b$. A set A with a partial order \preceq is called a partially ordered set (poset, for short) and denoted by (A, \preceq) . If in a poset (A, \preceq) any two elements a, b are comparable, i.e., either $a \preceq b$ or $b \preceq a$ hold, the partial order \preceq is called a linear (or total) order (in this case, A is called a chain).

Let us denote by $L([0, 1])$ the set of all closed subintervals of the unit interval, $L([0, 1]) = \{[a, b] \mid 0 \leq a \leq b \leq 1\}$.

Definition 2 ([22]). Let $(L([0, 1]), \preceq)$ be a poset. The order \preceq is called an admissible order, if

- (i) \preceq is a total (linear) order on $L([0, 1])$,
- (ii) for all $[a, b], [c, d] \in L([0, 1])$, $[a, b] \preceq [c, d]$ whenever $[a, b] \leq_2 [c, d]$ where \leq_2 denotes the classical partial order of intervals $[a, b] \leq_2 [c, d] \iff (a \leq c) \wedge (b \leq d)$.

Three classical examples of admissible orders on $L([0, 1])$ are the lexicographic order, the antilexicographic order and the order proposed by Xu and Yager in [39]. This last order is defined through the following binary relation in the set of all closed subintervals of the unit interval:

$$[a, b] \leq_{XY} [c, d] \iff (a + b < c + d) \vee [(a + b = c + d) \wedge (b - a \leq d - c)]. \tag{1}$$

By a fuzzy subset of \mathbb{R} , we mean a function $A : \mathbb{R} \rightarrow [0, 1]$. For each fuzzy subset A , let $A^\alpha = \{x \in \mathbb{R} : A(x) \geq \alpha\}$ for any $\alpha \in (0, 1]$ be its α -level set (or α -cut). By $\text{supp}(A)$ or A^0 , we mean the support of A , i.e., the set $\{x \in \mathbb{R} : A(x) > 0\}$.

Let us recall the definition of a discrete fuzzy number.

Definition 3 ([26]). A fuzzy subset A of \mathbb{R} with membership mapping $A : \mathbb{R} \rightarrow [0, 1]$ is called a discrete fuzzy number, or dfn for short, if its support is finite, i.e., there exist $x_1, \dots, x_n \in \mathbb{R}$ with $x_1 < x_2 < \dots < x_n$ such that $\text{supp}(A) = \{x_1, \dots, x_n\}$, and there are natural numbers s, t with $1 \leq s \leq t \leq n$ such that:

- 1. $A(x_i) = 1$ for all i with $s \leq i \leq t$. (core)
- 2. $A(x_i) \leq A(x_j)$ for all i, j with $1 \leq i \leq j \leq s$.
- 3. $A(x_i) \geq A(x_j)$ for all i, j with $t \leq i \leq j \leq n$.

A dfn A with $\text{supp}(A) = \{x_1, \dots, x_n\}$ will be denoted for short as $A = \{A(x_1)/x_1, \dots, A(x_n)/x_n\}$.

The following theorem allows us to identify when a discrete fuzzy subset verifies the discrete fuzzy number conditions established in the previous definition.

Theorem 1 ([40]). (Representation of discrete fuzzy numbers) Let A be a discrete fuzzy number. Then, the following statements (1)–(4) hold:

- 1. A^α is a nonempty finite subset of \mathbb{R} , for any $\alpha \in [0, 1]$.
- 2. $A^{\alpha_2} \subseteq A^{\alpha_1}$ for any $\alpha_1, \alpha_2 \in [0, 1]$ with $0 \leq \alpha_1 \leq \alpha_2 \leq 1$
- 3. For any $\alpha_1, \alpha_2 \in [0, 1]$ with $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, if $x \in A^{\alpha_1} \setminus A^{\alpha_2}$, then either $x < y$ for all $y \in A^{\alpha_2}$ or $x > y$ for all $y \in A^{\alpha_2}$.
- 4. For any $\alpha_0 \in (0, 1]$, there exist $\alpha'_0 \in (0, \alpha_0)$ such that $A^\alpha = A^{\alpha_0}$ for any $\alpha \in [\alpha'_0, \alpha_0]$.

Conversely, if for any $\alpha \in [0, 1]$, there exists $A^\alpha \subset \mathbb{R}$ satisfying analogous conditions to the (1)–(4), then there exists a unique dfn A such that its α -cuts are exactly the sets A^α for any $\alpha \in [0, 1]$.

Next, we provide the definition of the so-called relevant α -levels.

Definition 4. Let A be a discrete fuzzy number such that $\text{supp}(A) = \{x_1, \dots, x_n\}$. Then, $\alpha \in (0, 1]$ is called a relevant α -level for A if there exists $x_i \in \text{supp}(A)$ such that $A(x_i) = \alpha$.

Note that 1 is always a relevant α -level for any discrete fuzzy number A . Indeed, from a functional point of view, the set of relevant α -levels for A is given by the image of the membership function of A , excluding 0.

From now on, we will denote by L_n the finite chain $L_n = \{0, 1, \dots, n\}$, by \mathcal{D}_{L_n} the set of all discrete fuzzy numbers whose support is contained in L_n and by $\mathcal{A}_1^{L_n}$ the set of discrete fuzzy numbers whose support is a subinterval of the finite chain L_n . The interest in the study of $\mathcal{A}_1^{L_n}$ lies in the fact that this class of discrete fuzzy sets can be used as linguistic expressions that adequately model the opinions of an expert in a problem of decision-making (for more details, see [17,18]).

Let $A, B \in \mathcal{A}_1^{L_n}$ be two discrete fuzzy numbers. Note that the supports of A and B and their α -cuts are subintervals of L_n . Let $A^\alpha = [x_1^\alpha, x_p^\alpha]$, $B^\alpha = [y_1^\alpha, y_k^\alpha]$ be the α -level cuts for A and B , respectively. The following result holds for $\mathcal{A}_1^{L_n}$, but it is not true for the set of discrete fuzzy numbers in general (see [27]).

Theorem 2 ([27]). *The triplet $(\mathcal{A}_1^{L_n}, \text{MIN}, \text{MAX})$ is a bounded distributive lattice, where $1_n \in \mathcal{A}_1^{L_n}$ (the unique discrete fuzzy number whose support is the singleton $\{n\}$) and $1_0 \in \mathcal{A}_1^{L_n}$ (the unique discrete fuzzy number whose support is the singleton $\{0\}$) are the maximum and the minimum, respectively, and where $\text{MIN}(A, B)$ and $\text{MAX}(A, B)$ are the discrete fuzzy numbers belonging to the set $\mathcal{A}_1^{L_n}$ such that they have the sets*

$$\begin{aligned} \text{MIN}(A, B)^\alpha &= \{z \in L_n \mid \min(x_1^\alpha, y_1^\alpha) \leq z \leq \min(x_p^\alpha, y_k^\alpha)\} \text{ and} \\ \text{MAX}(A, B)^\alpha &= \{z \in L_n \mid \max(x_1^\alpha, y_1^\alpha) \leq z \leq \max(x_p^\alpha, y_k^\alpha)\} \end{aligned} \tag{2}$$

as α -cuts respectively for each $\alpha \in [0, 1]$ and $A, B \in \mathcal{A}_1^{L_n}$.

Remark 1 ([27]). *Using these operations, we can define a partial order on $\mathcal{A}_1^{L_n}$ in the usual way: $A \preceq B$ if and only if $\text{MIN}(A, B) = A$, or equivalently, $A \preceq B$ if and only if $\text{MAX}(A, B) = B$ for any $A, B \in \mathcal{A}_1^{L_n}$. Equivalently, we can also define the partial order in terms of α -cuts:*

$$\begin{aligned} A \preceq B &\text{ if and only if } \min(A^\alpha, B^\alpha) = A^\alpha, \\ A \preceq B &\text{ if and only if } \max(A^\alpha, B^\alpha) = B^\alpha, \end{aligned}$$

where the minimum is defined through the classical partial order of intervals \leq_2 (introduced in Definition 2), and the maximum is defined analogously.

3. Total Orders on the Set $\mathcal{A}_1^{L_n}$

In the literature, few total orders on the set of discrete fuzzy numbers have been proposed. The common denominator of all these orderings is that they are based on the use of the so-called ranking indices, that is, functions $f : \mathcal{A}_1^{L_n} \rightarrow \mathbb{R}$. In other words, the order relies on the standard order of real numbers after mapping the discrete fuzzy numbers to some real numbers in such a way that, given $A, B \in \mathcal{A}_1^{L_n}$, then

$$\begin{aligned} A \succ_f B &\text{ if and only if } f(A) < f(B), \\ A \prec_f B &\text{ if and only if } f(A) > f(B), \\ A \sim_f B &\text{ if and only if } f(A) = f(B), \\ A \preceq_f B &\text{ if and only if } f(A) \leq f(B), \\ A \succeq_f B &\text{ if and only if } f(A) \geq f(B). \end{aligned}$$

Namely, the total orders presented in [34–38] are embedded in this strategy. However, the use of ranking indices has an undesired behavior from our point of view when they are considered in the exploitation phase in a multi-criteria decision-making problem. Since function f is not one-to-one, each discrete fuzzy number A is similar (\sim) to a set of discrete fuzzy numbers B , those satisfying that $f(A) = f(B)$ but not necessarily satisfying

$B = A$. Therefore, depending on the values of the collective (or aggregated) linguistic assessments of the alternatives, it is possible that the total order does not provide a unique best alternative but a subset of the alternatives.

In order to illustrate the previous fact, let us analyze for instance the total order presented in [35]. This order is firstly proposed in the set of fuzzy numbers and then adapted to the set of discrete fuzzy numbers. It relies on the use of the so-called left and right dominance. Let us recall these concepts. For each discrete fuzzy number A , the lower and upper limits of the k^{th} α -cut (with $\alpha > 0$) for A are defined as

$$l_{A,k} = \min_{x \in \mathbb{R}} \{x \mid A(x) \geq \alpha_k\}, \tag{3}$$

$$r_{A,k} = \max_{x \in \mathbb{R}} \{x \mid A(x) \geq \alpha_k\}, \tag{4}$$

respectively. From these values, the left (right) dominance $D_{A,B}^L (D_{A,B}^R)$ of A over B is defined as the average difference of the lower (upper) limits at some α -levels given by

$$D_{A,B}^L = \frac{1}{n+1} \sum_{k=0}^n (l_{A,k} - l_{B,k}), \text{ and } D_{A,B}^R = \frac{1}{n+1} \sum_{k=0}^n (r_{A,k} - r_{B,k})$$

where $n + 1$ α -cuts are used to calculate the dominance. Finally, the total dominance of A over B with the index of optimism $\beta \in [0, 1]$ is defined as the convex combination of $D_{A,B}^L$ and $D_{A,B}^R$ by

$$\begin{aligned} D_{A,B}(\beta) &= \beta D_{A,B}^R + (1 - \beta) D_{A,B}^L \\ &= \beta \left[\frac{1}{n+1} \sum_{k=0}^n (r_{A,k} - r_{B,k}) \right] + (1 - \beta) \left[\frac{1}{n+1} \sum_{k=0}^n (l_{A,k} - l_{B,k}) \right]. \end{aligned} \tag{5}$$

The above ranking index indicates that the total dominance is actually a comparison function. The larger the index of optimism β is, the more important is the right dominance. Herein, the index of optimism is used to reflect a decision maker’s degree of optimism. A more optimistic decision maker generally takes a larger value of the index, for example, a situation in which $\beta = 1$ (or 0) represents an optimistic (pessimistic) decision maker’s perspective, and only right (left) dominance is considered.

According to [35], fixing the parameter value $\beta \in [0, 1]$, a decision maker can rank a pair of fuzzy numbers, A and B , using $D_{A,B}(\beta)$ according to the following rules:

1. If $D_{A,B}(\beta) > 0$ then $A > B$;
2. If $D_{A,B}(\beta) = 0$ then $A = B$;
3. If $D_{A,B}(\beta) < 0$ then $A < B$.

These rules hide the aforementioned problem of the total orders based on ranking indices. Although they infer that, if $D_{A,B}(\beta) = 0$, then $A = B$, this is actually not true since there exist discrete fuzzy numbers $A \neq B$ such that $D_{A,B}(\beta) = 0$. Let us provide some counterexamples.

First, note that, given $A, B \in \mathcal{A}_1^{L_n}$, it holds that $\sum_{k=0}^n (l_{A,k} - l_{B,k})$ and $\sum_{k=0}^n (r_{A,k} - r_{B,k})$ are integer numbers. On the other hand, it is evident that $D_{A,B}(\beta) = 0$ if and only if $\beta D_{A,B}^R = (\beta - 1) D_{A,B}^L$. Three different cases arise depending on the value of β :

- If $\beta = 0$, then

$$D_{A,B}(0) = 0 \text{ if and only if } \sum_{k=0}^n (l_{A,k} - l_{B,k}) = 0$$

but this does not imply that $A = B$ as the following example shows.

Example 1. Let $A, B \in \mathcal{A}_1^{L_4}$ be such that $A = \{0.5/1, 1/2, 0.5/3\}$ and $B = \{0.5/1, 1/2, 1/3, 0.5/4\}$. It is straightforward to check that $A \neq B$, but, when we consider the α -cuts $\{0.5, 1\}$, then $D_{A,B}(0) = 0$. This contradicts the equality rule introduced in [35].

- If $\beta = 1$, then

$$D_{A,B}(1) = 0 \text{ if and only if } \frac{1}{n+1} \sum_{k=0}^n (r_{i,k} - r_{j,k}) = 0$$

but again this does not imply that $A = B$ as the following example shows.

Example 2. Let $A, B \in \mathcal{A}_1^{L_4}$ be such that $A = \{0.5/2, 1/3, 0.5/4\}$ and $B = \{0.5/1, 1/2, 1/3, 0.5/4\}$. It holds that $A \neq B$ but when the α -cuts $\{0.5, 1\}$ are considered, we obtain $D_{A,B}(1) = 0$, obtaining a contradiction with the equality rule.

- Finally, consider $\beta \in (0, 1)$. It holds that

$$D_{A,B}(\beta) = 0 \text{ if and only if } \sum_{k=0}^n (r_{A,k} - r_{B,k}) = \frac{\beta-1}{\beta} \sum_{k=0}^n (l_{A,k} - l_{B,k}).$$

Let us consider different cases depending on the fraction $\frac{\beta-1}{\beta}$:

- If $\frac{\beta-1}{\beta} \in \mathbb{Z}^-$, note that a sufficient condition to guarantee that $D_{A,B}(\beta) = 0$ is $r_{A,k} - r_{B,k} = \frac{\beta-1}{\beta} (l_{A,k} - l_{B,k})$ for each chosen k as the following example shows.

Example 3. Let $\beta = 0.5$ and $A, B \in \mathcal{A}_1^{L_4}$ be such that $A = \{0.5/1, 1/2, 0.5/3\}$ and $B = \{0.5/0, 0.5/1, 1/2, 0.5/3, 0.5/4\}$. Considering the α -cuts $\{0.5, 1\}$, we have that $D_{A,B}(\beta) = 0$ but $A \neq B$.

- If $\frac{\beta-1}{\beta} \notin \mathbb{Z}^-$, then

$$D_{A,B}(\beta) = 0 \text{ if and only if } \sum_{k=0}^n (r_{A,k} - r_{B,k}) = \sum_{k=0}^n (l_{A,k} - l_{B,k}) = 0.$$

Example 4. Let $A, B \in \mathcal{A}_1^{L_6}$ be such that $A = \{(2/3)/1, 1/2, 1/3, (2/3)/4, (1/3)/5, (1/3)/6\}$ and $B = \{(1/3)/0, (1/3)/1, (2/3)/2, 1/3, 1/4, (2/3)/5\}$. It holds that $D_{A,B}(\beta) = 0$ but $A \neq B$.

To sum up, we have proved that, for any $\beta \in [0, 1]$, there exist discrete fuzzy numbers A, B such that $D_{A,B}(\beta) = 0$ but $A \neq B$. It is evident that the second rule must be modified. Indeed, $D_{A,B}(\beta)$ may be used to define an equivalence relation \sim as follows:

$$A \sim B \text{ if } D_{A,B}(\beta) = 0$$

but, of course, this means that we can not properly distinguish those fuzzy numbers which belong to the same equivalence class. Namely, these total orders based on ranking indices are not able to discriminate between discrete fuzzy numbers which are very different in terms of core or support. Indeed, consider the chain L_6 which can be used to represent the linguistic chain

{Very Bad, Bad, Somewhat Bad, Normal, Somewhat Good, Good, Very Good}.

Then, the discrete fuzzy numbers A and B considered in Example 4 can be understood as generalizations of the linguistic evaluations “Between Bad and Somewhat Bad” and “Between Normal and Somewhat Good”, respectively. Consequently, a total order that establishes that these two linguistic evaluations are similar may be questionable.

4. Novel Total Orders on the Set of Discrete Fuzzy Numbers

In this section, two novel families of total orders on the set of discrete fuzzy numbers whose support is a subinterval of the finite chain L_n are introduced. These orders will satisfy the desirable property that given $A, B \in \mathcal{A}_1^{L_n}$, if $A \preceq B$ and $B \preceq A$, then $A = B$. Thus, for each set of different discrete fuzzy numbers, only one discrete fuzzy number will be preferred among the others according to these total orders. The underlying idea of these total orders is similar to the one used in [21] to construct a total order on the set of fuzzy numbers by using upper dense sequences.

Before proposing the total orders, let us prove the following result related to the decomposition of two discrete fuzzy numbers in terms of α -cuts but using those levels belonging to the union of the relevant levels of both discrete fuzzy numbers.

Theorem 3. *Let $A, B \in \mathcal{A}_1^{L_n}$ be two discrete fuzzy numbers whose sets of relevant α -levels are $S_A = \{\alpha_1 < \dots < \alpha_k = 1\}$ with $k \leq n + 1$, $S_B = \{\beta_1 < \dots < \beta_m = 1\}$ with $m \leq n + 1$, respectively, and $S = S_A \cup S_B = \{\gamma_1 < \gamma_2 < \dots < \gamma_t = 1\}$ with $1 \leq t \leq k + m - 1$. Then,*

$$A = \bigcup_{\gamma \in S} \gamma \cdot \chi_{A^\gamma}, \quad B = \bigcup_{\gamma \in S} \gamma \cdot \chi_{B^\gamma}$$

where \cup denotes the standard fuzzy union and χ_X denotes the indicator function given by

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in \text{supp}(X), \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in L_n$.

Proof. We will prove only the first equality, the proof of the second one is analogous. Let us consider $x \in \text{supp}(A)$. Then, on the one hand, there exists $\gamma_i \in S_A$ such that $A(x) = \gamma_i$. On the other hand, we have that

$$\left(\bigcup_{\gamma \in S} \gamma \cdot \chi_{A^\gamma} \right)(x) = \max_{\gamma \in S} \gamma \cdot \chi_{A^\gamma}(x) = \max \left\{ \max_{\gamma \in \{\gamma_1, \dots, \gamma_i\}} \gamma \cdot \chi_{A^\gamma}(x), \max_{\gamma \in \{\gamma_{i+1}, \dots, 1\}} \gamma \cdot \chi_{A^\gamma}(x) \right\}.$$

Since $A(x) = \gamma_i < \gamma$ for all $\gamma \in \{\gamma_{i+1}, \dots, 1\}$, it holds that

$$\max_{\gamma \in \{\gamma_{i+1}, \dots, 1\}} \gamma \cdot \chi_{A^\gamma}(x) = 0.$$

Consequently,

$$\left(\bigcup_{\gamma \in S} \gamma \cdot \chi_{A^\gamma} \right)(x) = \max_{\gamma \in \{\gamma_1, \dots, \gamma_i\}} \gamma \cdot \chi_{A^\gamma}(x) = \max_{\gamma \in \{\gamma_1, \dots, \gamma_i\}} \gamma = \gamma_i.$$

□

The previous theorem allows for expressing two discrete fuzzy numbers belonging to $\mathcal{A}_1^{L_n}$ by using the same number of α -cuts, which is obviously a finite number. This fact will be the key to defining binary relations on the set of discrete fuzzy numbers in $\mathcal{A}_1^{L_n}$ which will define the pursued total orders.

4.1. First Family of Total Orders on $\mathcal{A}_1^{L_n}$

Let us introduce the following binary relation on $\mathcal{A}_1^{L_n}$, which we will later prove that constitutes an admissible order on $\mathcal{A}_1^{L_n}$. This first total order relies on applying an admissible order on the set of all closed intervals of L_n to the α -cuts of A and B from the support to the core.

Definition 5. Let $A, B \in \mathcal{A}_1^{L_n}$ be two discrete fuzzy numbers whose sets of relevant α -levels are $S_A = \{\alpha_1 < \dots < \alpha_k = 1\}$ with $k \leq n + 1$, $S_B = \{\beta_1 < \dots < \beta_m = 1\}$ with $m \leq n + 1$ respectively, and $S_{AB} = S_A \cup S_B = \{\gamma_1 < \gamma_2 < \dots < \gamma_t = 1\}$ with $1 \leq t \leq k + m - 1$. Let us consider an admissible order \prec_δ on $\Pi[L_n] = \{[a, b] : 0 \leq a \leq b \leq n, a, b \in L_n\}$. We will say $A = B$ if and only if all their level sets in S_{AB} are equal, that is, $A^{\gamma_i} = B^{\gamma_i}$ for all $i \in I = \{1, \dots, t\}$. We will say $A \prec_{\Delta_\delta^\uparrow} B$ if and only if $A \neq B$ and there exists a natural number $j \in I$ such that $A^{\gamma_j} \prec_\delta B^{\gamma_j}$ and $A^{\gamma_i} = B^{\gamma_i}$ for all $i < j$. We will say $A \preceq_{\Delta_\delta^\uparrow} B$ if and only if $A = B$ or $A \prec_{\Delta_\delta^\uparrow} B$.

Before proving that this binary relation is a total order, let us show an example to understand the underlying idea. Indeed, this total order scans the two discrete fuzzy numbers from the support to the core comparing the α -cuts of both discrete fuzzy numbers until the α -cuts at some level are different. If all the α -cuts are equal, then both discrete fuzzy numbers are equal. Let us illustrate this with the following example.

Example 5. Let $A, B \in \mathcal{A}_1^{L_6}$ be such that $A = \{(2/3)/1, 1/2, 1/3, (2/3)/4, (1/3)/5, (1/3)/6\}$ and $B = \{(1/3)/0, (1/3)/1, (2/3)/2, 1/3, 1/4, (2/3)/5\}$, the same discrete fuzzy numbers considered in Example 4. We will consider the binary relation $\preceq_{\Delta_\delta^\uparrow}$ with the Xu and Yager admissible order on $\Pi[L_6]$. In this case, the relevant α -levels of A and B coincide and are $\{1/3, 2/3, 1\}$. Let us compare first the 1/3-cuts:

$$B^{\frac{1}{3}} = [0, 5] \prec_{XY} [1, 6] = A^{\frac{1}{3}}.$$

It holds that $B \prec_{\Delta_{XY}^\uparrow} A$. Note that we have established the ordering by comparing only the 1/3-cuts. If the 1/3-cuts had been equal, then it would have been necessary to compare the 2/3-cuts.

Remark 2. Note that this binary relation depends on the choice of the admissible order \prec_δ on $\Pi[L_n] = \{[a, b] : 0 \leq a \leq b \leq n, a, b \in L_n\}$. This fact will lead to a whole family of total orders in $\mathcal{A}_1^{L_n}$. Indeed, consider the Xu and Yager admissible order on $\Pi[L_6]$ and the lexicographic order. Let $A, B \in \mathcal{A}_1^{L_6}$ be such that

$$A = \{0.8/1, 0.8/2, 0.8/3, 1/4, 0.8/5, 0.8/6\}$$

and $B = \{0.7/1, 0.8/2, 1/3, 0.7/4, 0.7/5, 0.7/6\}$. In this case, $S_{AB} = \{0.7, 0.8, 1\}$. While $A^{0.7} = [1, 6] = B^{0.7}$ independently of the chosen admissible order on $\Pi[L_6]$, on the one hand, $B^{0.8} = [2, 3] \prec_{XY} [1, 6] = A^{0.8}$ and, on the other hand, $A^{0.8} = [1, 6] \prec_{lex} [2, 3] = B^{0.8}$. This implies that $B \prec_{\Delta_{XY}^\uparrow} A$ but $A \prec_{\Delta_{lex}^\uparrow} B$.

On the path to prove that $\preceq_{\Delta_\delta^\uparrow}$ is a total order on $\mathcal{A}_1^{L_n}$, let us first prove that this binary relation fulfills the transitivity property.

Proposition 4. Let $A, B, C \in \mathcal{A}_1^{L_n}$. If $A \prec_{\Delta_\delta^\uparrow} B$ and $B \prec_{\Delta_\delta^\uparrow} C$, then $A \prec_{\Delta_\delta^\uparrow} C$.

Moreover, let S_A, S_B , and S_C be the sets of relevant α -levels of A, B, C , respectively and $S_{AB} = S_A \cup S_B, S_{BC} = S_B \cup S_C$ and $S_{AC} = S_A \cup S_C$. Let $\alpha_j \in S_{AB}$ be such that $A^{\alpha_j} \prec_\delta B^{\alpha_j}$ and $A^\alpha = B^\alpha$ for all $\alpha \in S_{AB}$ with $\alpha < \alpha_j$. Let $\beta_l \in S_{BC}$ be such that $B^{\beta_l} \prec_\delta C^{\beta_l}$ and $B^\beta = C^\beta$ for all $\beta \in S_{BC}$ with $\beta < \beta_l$. In this case, there exists $\gamma_s \in S_{AC}$ such that $A^{\gamma_s} \prec_\delta C^{\gamma_s}$ and $A^\gamma = C^\gamma$ for all $\gamma \in S_{AC}$ with $\gamma < \gamma_s$, where

$$\gamma_s = \min\{\gamma \in S_{AC} \mid \gamma \geq \min\{\alpha_j, \beta_l\} \text{ or } (\gamma < \min\{\alpha_j, \beta_l\}, \gamma \in S_A \setminus S_{BC} \text{ and } \min\{\beta \in S_{BC} \mid \beta > \gamma\} = \beta_l) \text{ or } (\gamma < \min\{\alpha_j, \beta_l\}, \gamma \in S_C \setminus S_{AB} \text{ and } \min\{\alpha \in S_{AB} \mid \alpha > \gamma\} = \alpha_j)\}.$$

Proof. First of all, note that the set which defines γ_s is not empty since $1 \in S_{AC}$ and $1 \geq \min\{\alpha_j, \beta_l\}$.

The methodology will be as follows: (i) to prove that $A^{\gamma_s} \prec_{\delta} C^{\gamma_s}$, (ii) to prove that $A^{\gamma} = C^{\gamma}$ for all $\gamma \in S_{AC}$ with $\gamma < \gamma_s$. We will study three different cases under the hypothesis that $\alpha_j \leq \beta_l$ (the case that $\beta_l > \alpha_j$ is analogous):

1. Let us suppose that $\gamma_s \geq \min\{\alpha_j, \beta_l\} = \alpha_j$. In this case, either $\gamma_s = \alpha_j$ or $\gamma_s > \alpha_j$. In any case, by the representation theorem and the definition of γ_s , we have that $A^{\gamma_s} = A^{\alpha_j}$. Then, $A^{\gamma_s} = A^{\alpha_j} \prec_{\delta} B^{\alpha_j}$. Now, two cases arise:

- (a) If $\alpha_j \in S_C$, then, by the definition of γ_s , it must be $\gamma_s = \alpha_j$. Since $\alpha_j \leq \beta_l$, it holds that $B^{\alpha_j} \preceq_{\delta} C^{\alpha_j}$. Finally, $C^{\alpha_j} = C^{\gamma_s}$. Thus, we have proved that $A^{\gamma_s} \prec_{\delta} C^{\gamma_s}$.
- (b) If $\alpha_j \notin S_C$, let us define

$$t = \min\{\beta \in S_{BC} \mid \beta \geq \alpha_j\}.$$

By the representation theorem, it is clear that $B^{\alpha_j} = B^t$. Now,

- i. if $t = \beta_l$, then it is clear that $\alpha_j \in S_A$ and, consequently, by the definition of γ_s , it must be $\gamma_s = \alpha_j$. Now, by the definition of the binary relation, $B^t = B^{\beta_l} \prec_{\delta} C^{\beta_l}$ and by the representation theorem, $C^{\beta_l} = C^{\alpha_j} = C^{\gamma_s}$.
- ii. If $t < \beta_l$, then $B^t = C^t$. By the representation theorem, $C^t = C^{\alpha_j}$. If $\alpha_j = \gamma_s$, we get $C^{\alpha_j} = C^{\gamma_s}$ trivially. Otherwise, if $\alpha_j > \gamma_s$, since $\alpha_j \notin S_C$ it must be $\alpha_j \notin S_B$ and, consequently, $t = \alpha_j$. In this case, by the definition of γ_s , it holds that $C^{\alpha_j} = C^{\gamma_s}$.

In all cases, we have proved that $A^{\gamma_s} \prec_{\delta} C^{\gamma_s}$.

As a second step, we must prove that $A^{\gamma} = C^{\gamma}$ for all $\gamma \in S_{AC}$ with $\gamma < \gamma_s$. For such γ , clearly $\gamma < \min\{\alpha_j, \beta_l\}$. Two cases must be analyzed:

- (a) If $\gamma \in S_A$, the other two cases arise:
 - i. If $\gamma \in S_{BC}$, then since $\gamma < \min\{\alpha_j, \beta_l\}$, it holds that $A^{\gamma} = B^{\gamma} = C^{\gamma}$.
 - ii. If $\gamma \notin S_{BC}$, let us define

$$t = \min\{\beta \in S_{BC} \mid \beta > \gamma\}.$$

First, note that $t < \beta_l$, otherwise this $\gamma < \min\{\alpha_j, \beta_l\}$ would be a candidate to become γ_s , contradicting with the fact that $\gamma_s \geq \min\{\alpha_j, \beta_l\}$, which is the case we are considering. Now, since $\gamma < \alpha_j$ and $\gamma \in S_A$, $A^{\gamma} = B^{\gamma}$. By the representation theorem, $B^{\gamma} = B^t$ and since $t < \beta_l$ and $t \in S_{BC}$, $B^t = C^t$ and again by the representation theorem, $C^t = C^{\gamma}$. To sum up, $A^{\gamma} = C^{\gamma}$.

(b) The case $\gamma \in S_C$ is analogous to the previous case.

2. Let us suppose now that $\gamma_s < \min\{\alpha_j, \beta_l\}$ with $\gamma_s \in S_A \setminus S_{BC}$ and $\min\{\beta \in S_{BC} \mid \beta > \gamma_s\} = \beta_l$. In this case, first, since $\gamma_s \in S_{AB}$ and $\gamma_s < \alpha_j$, then by the definition of the binary relation, $A^{\gamma_s} = B^{\gamma_s}$. Now, since $\gamma_s \notin S_{BC}$ and $\min\{\beta \in S_{BC} \mid \beta > \gamma_s\} = \beta_l$, by the representation theorem, we get that $B^{\gamma_s} = B^{\beta_l}$ and $C^{\gamma_s} = C^{\beta_l}$. Finally, by the definition of the binary relation, $B^{\beta_l} \prec_{\delta} C^{\beta_l} = C^{\gamma_s}$. Thus, we have proved that $A^{\gamma_s} \prec_{\delta} C^{\gamma_s}$.

As a second step, we must prove that $A^{\gamma} = C^{\gamma}$ for all $\gamma \in S_{AC}$ with $\gamma < \gamma_s$. For such γ , clearly $\gamma < \min\{\alpha_j, \beta_l\}$. Two cases must be analyzed:

- (a) If $\gamma \in S_A$, the other two cases arise:
 - i. If $\gamma \in S_{BC}$, then since $\gamma < \min\{\alpha_j, \beta_l\}$, it holds that $A^{\gamma} = B^{\gamma} = C^{\gamma}$.
 - ii. If $\gamma \notin S_{BC}$, let us define

$$t = \min\{\beta \in S_{BC} \mid \beta > \gamma\}.$$

First, note that $t < \beta_l$, otherwise this $\gamma < \min\{\alpha_j, \beta_l\}$ would be a candidate to become γ_s , contradicting with the fact that $\gamma_s > \gamma$. Now, since $\gamma < \alpha_j$ and $\gamma \in S_A$, $A^\gamma = B^\gamma$. By the representation theorem, $B^\gamma = B^t$ and, since $t < \beta_l$ and $t \in S_{BC}$, $B^t = C^t$ and again by the representation theorem, $C^t = C^\gamma$. To sum up, $A^\gamma = C^\gamma$.

- (b) The case $\gamma \in S_C$ is analogous to the previous case.
- 3. The third case which is that $\gamma_s < \min\{\alpha_j, \beta_l\}$ with $\gamma_s \in S_C \setminus S_{AB}$ and $\min\{\alpha \in S_{AB} \mid \alpha > \gamma_s\} = \alpha_j$ is analogous to Case 2.

□

Example 6. Let $A, B, C \in \mathcal{A}_1^{L_6}$ be such that $A = \{1/1, 0.7/2, 0.3/3\}$, $B = \{0.7/1, 1/2, 0.3/3\}$ and $C = \{0.2/3, 0.5/4, 0.8/5, 1/6\}$. Consider the binary relation $\preceq_{\Delta_\delta^\uparrow}$ with the Xu and Yager admissible order on $\Pi[L_6]$. It is straightforward to check that $A \prec_{\Delta_{XY}^\uparrow} B$ with $\alpha_j = 1$ and $B \prec_{\Delta_{XY}^\uparrow} C$, where $\beta_l = 0.2$. Let us compute the value of γ_s which ensures that $A \prec_{\Delta_{XY}^\uparrow} C$. In this case,

$$S_{AC} = \{0.2, 0.3, 0.4, 0.5, 0.7, 0.8, 1\},$$

and

$$\gamma_s = \min\{\gamma \in S_{AC} \mid \gamma \geq \min\{\alpha_j, \beta_l\} = 0.2\} = 0.2.$$

Note that there is no $\gamma \in S_{AC}$ such that $\gamma < 0.2$ and, moreover,

$$A^{0.2} = [1, 3] \prec_{XY} [3, 6] = C^{0.2}$$

and the strict transitivity follows.

Example 7. Let $A, B, C \in \mathcal{A}_1^{L_6}$ be such that $A = \{0.6/1, 1/2, 0.3/3, 0.3/4\}$, $B = \{0.2/1, 0.5/2, 1/3, 0.5/4\}$ and $C = \{0.2/1, 0.4/2, 1/3, 1/4\}$. Consider the binary relation $\preceq_{\Delta_\delta^\uparrow}$ with the Xu and Yager admissible order on $\Pi[L_6]$. It is clear that $A \prec_{\Delta_{XY}^\uparrow} B$ with $\alpha_j = 0.5$ and $B \prec_{\Delta_{XY}^\uparrow} C$ where $\beta_l = 0.5$. In this case,

$$S_{AC} = \{0.2, 0.3, 0.4, 0.6, 1\}, \quad S_{AB} = \{0.2, 0.3, 0.5, 0.6, 1\}, \quad S_{BC} = \{0.2, 0.4, 0.5, 1\},$$

and

$$\begin{aligned} \gamma_s &= \min\{\gamma \in S_{AC} \mid \gamma \geq \min\{\alpha_j, \beta_l\} = 0.5 \text{ or} \\ &(\gamma < \min\{\alpha_j, \beta_l\} = 0.5, \gamma \in S_A \setminus S_{BC} = \{0.3, 0.6\} \text{ and } \min\{\beta \in S_{BC} \mid \beta > \gamma\} = 0.5) \text{ or} \\ &(\gamma < \min\{\alpha_j, \beta_l\} = 0.5, \gamma \in S_C \setminus S_{AB} = \{0.4\} \text{ and } \min\{\alpha \in S_{AB} \mid \alpha > \gamma\} = 0.5)\} \\ &= \min\{0.3, 0.4, 0.6, 1\} = 0.3. \end{aligned}$$

From Proposition 4, which proves the strict transitivity of the binary relation $\preceq_{\Delta_\delta^\uparrow}$, the transitivity property follows.

Corollary 5. Let $A, B, C \in \mathcal{A}_1^{L_n}$. If $A \preceq_{\Delta_\delta^\uparrow} B$ and $B \preceq_{\Delta_\delta^\uparrow} C$, then $A \preceq_{\Delta_\delta^\uparrow} C$.

Proof. The result is straightforward from Proposition 4 and the definition of the binary relation $\preceq_{\Delta_\delta^\uparrow}$. □

At this point, we are able to prove that the binary relation $\preceq_{\Delta_\delta^\uparrow}$ is a total order.

Theorem 6. The binary relation $\preceq_{\Delta_\delta^\uparrow}$ is a total order on $\mathcal{A}_1^{L_n}$.

Proof. We want to prove that the binary relation $\preceq_{\Delta_\delta^\uparrow}$ fulfills the following four properties: reflexivity, antisymmetry, transitivity, and connexity. Let us consider A, B and $C \in \mathcal{A}_1^{L_n}$ discrete fuzzy numbers whose sets of relevant α -levels are $S_A = \{\alpha_1 < \dots < \alpha_k = 1\}$ with $k \leq n + 1$, $S_B = \{\beta_1 < \dots < \beta_m = 1\}$ with $m \leq n + 1$, $S_C = \{\gamma_1 < \dots < \gamma_t = 1\}$ with $t \leq n + 1$, respectively, and $S_{AB} = S_A \cup S_B = \{\zeta_1 < \zeta_2 < \dots < \zeta_p = 1\}$ with $1 \leq p \leq k + m - 1$.

Reflexivity: It follows trivially from Definition 5.

Antisymmetry: Let us consider $A, B \in \mathcal{A}_1^{L_n}$ fulfilling $A \preceq_{\Delta_\delta^\uparrow} B$ and $B \preceq_{\Delta_\delta^\uparrow} A$. We want to see that $A = B$. We will only consider the case $A \prec_{\Delta_\delta^\uparrow} B$ and $B \prec_{\Delta_\delta^\uparrow} A$.

- (i) If $A \prec_{\Delta_\delta^\uparrow} B$, then according to Definition 5 there exists $\zeta_j \in S_{AB}$ such that $A^{\zeta_j} \prec_\delta B^{\zeta_j}$ and $A^\zeta = B^\zeta$ for all $\zeta < \zeta_j$ with $\zeta \in S_{AB}$.
- (ii) If $B \prec_{\Delta_\delta^\uparrow} A$, then, according to Definition 5, there exists $\zeta_{j'} \in S_{AB}$ such that $B^{\zeta_{j'}} \prec_\delta A^{\zeta_{j'}}$ and $B^\zeta = A^\zeta$ for all $\zeta < \zeta_{j'}$ with $\zeta \in S_{AB}$. Now, three cases arise:
 1. If $\zeta_j = \zeta_{j'}$, we get a contradiction from the fact that $A^{\zeta_j} \prec_\delta B^{\zeta_j} = B^{\zeta_{j'}} \prec_\delta A^{\zeta_{j'}} = A^{\zeta_{j'}}$.
 2. If $\zeta_j < \zeta_{j'}$, we also get a contradiction because from i) $A^{\zeta_j} \prec_\delta B^{\zeta_j}$ but from ii), since $\zeta_j < \zeta_{j'}$, $A^{\zeta_j} = B^{\zeta_j}$.
 3. If $\zeta_j > \zeta_{j'}$, it is analogous to the previous case.

Anyway, the antisymmetry follows.

Transitivity: Follows from Corollary 5.

Connexity: Let us consider $A, B \in \mathcal{A}_1^{L_n}$. We wish to show that $A = B$ or $A \prec_{\Delta_\delta^\uparrow} B$ or $B \prec_{\Delta_\delta^\uparrow} A$ that is all the elements of the set $\mathcal{A}_1^{L_n}$ are comparable. When $A \neq B$, there exist $\zeta \in S_{AB}$ such that $A^\zeta \neq B^\zeta$. Let us consider $Z = \{\zeta \in S_{AB} \mid A^\zeta \neq B^\zeta\}$. It is obvious that Z is a finite set and there exists a minimum. Let ζ^* be this minimum. Then, we have that $A^\zeta = B^\zeta$ for all $\zeta < \zeta^*$ and therefore necessarily either $A \prec_{\Delta_\delta^\uparrow} B$ when $A^{\zeta^*} \prec_\delta B^{\zeta^*}$ or $B \prec_{\Delta_\delta^\uparrow} A$ when $B^{\zeta^*} \prec_\delta A^{\zeta^*}$. \square

Once we have proved that $\preceq_{\Delta_\delta^\uparrow}$ is a total order, let us discuss the concept of admissible order in the set of $\mathcal{A}_1^{L_n}$. As it has been recalled in Definition 2, admissible orders on $L([0, 1])$ are those orders that refine the standard order \leq_2 in $L([0, 1])$. In this way, this idea can be translated to the set $\mathcal{A}_1^{L_n}$ as follows. We know that $\mathcal{A}_1^{L_n}$ is a bounded distributive lattice (see Theorem 2) with one of the two following partial orders (which are equivalent):

$$A \preceq B \text{ if and only if } \text{MIN}(A, B) = B \text{ (or equivalently } \text{MAX}(A, B) = A),$$

where $\text{MAX}(A, B)$ and $\text{MIN}(A, B)$ are the discrete fuzzy numbers defined in Theorem 2. It is clear that these orders coincide with \leq_2 when the support of the involved discrete fuzzy numbers coincide with their core. The next definition illustrates this idea.

Definition 6. Let $(\mathcal{A}_1^{L_n}, \preceq^*)$ be a poset. The order \preceq^* is called an admissible order, if

- (i) \preceq^* is a total order on $\mathcal{A}_1^{L_n}$,
- (ii) for all $A, B \in \mathcal{A}_1^{L_n}$, $A \preceq^* B$ whenever $A \preceq B$ where \preceq denotes the partial $\text{MIN}(\text{MAX})$ order defined in Theorem 2.

The following result ensures that $\preceq_{\Delta_\delta^\uparrow}$ is an admissible order in the sense of the previous definition.

Theorem 7. $\prec_{\Delta_\delta^\uparrow}$ is an admissible order on $\mathcal{A}_1^{L_n}$.

Proof. First, by Theorem 6, $\preceq_{\Delta_\delta^\uparrow}$ is a total order. Now, consider $A, B \in \mathcal{A}_1^{L_n}$ such that $A \preceq B$ where \preceq denotes the partial MIN order. In this case, it holds that $\min(A^\alpha, B^\alpha) = A^\alpha$, or, equivalently $A^\alpha \leq_2 B^\alpha$, for all $\alpha \in (0, 1]$. Let S_{AB} be the set of relevant α -levels of A or B . For all $\alpha \in S_{AB}$, since δ is an admissible order in $L([0, 1])$ and $A^\alpha \leq_2 B^\alpha$, it holds that $A^\alpha \preceq_\delta B^\alpha$. Consequently, it is clear that $A \preceq_{\Delta_\delta^\uparrow} B$. \square

4.2. Second Family of Total Orders on $\mathcal{A}_1^{L_n}$

The second total order that we want to introduce follows a similar pattern as the first one, but the comparison of the α -cuts is made from the top (core) to the bottom. Let us define formally this idea.

Definition 7. Let $A, B \in \mathcal{A}_1^{L_n}$ be two discrete fuzzy numbers whose sets of relevant α -levels are $S_A = \{\alpha_1 < \dots < \alpha_k = 1\}$ with $k \leq n + 1$, $S_B = \{\beta_1 < \dots < \beta_m = 1\}$ with $m \leq n + 1$, respectively, and $S_{AB} = S_A \cup S_B = \{\gamma_1 < \gamma_2 < \dots < \gamma_t = 1\}$ with $1 \leq t \leq k + m - 1$. Let us consider an admissible order \prec_δ on $\Pi[L_n] = \{[a, b] : 0 \leq a \leq b \leq n, a, b \in L_n\}$. We will say $A = B$ if and only if all their level sets in S_{AB} are equal, that is, $A^{\gamma_i} = B^{\gamma_i}$ for all $i \in I = \{1, \dots, t\}$. We will say $A \prec_{\Delta_\delta^\downarrow} B$ if and only if $A \neq B$ and there exists a natural number $j \in I$ such that $A^{\gamma_j} \prec_\delta B^{\gamma_j}$ and $A^{\gamma_i} = B^{\gamma_i}$ for all $i > j$. We will say $A \preceq_{\Delta_\delta^\downarrow} B$ if and only if $A = B$ or $A \prec_{\Delta_\delta^\downarrow} B$.

Let us show an example to understand the idea behind this definition. In this case, this binary relation scans the two discrete fuzzy numbers from the core to the support comparing the α -cuts of both discrete fuzzy numbers until the α -cuts at some level are different. If all the α -cuts are equal, then both discrete fuzzy numbers are equal. Let us illustrate this with the following example.

Example 8. Let $A, B \in \mathcal{A}_1^{L_6}$ be such that $A = \{(2/3)/1, 1/2, 1/3, (2/3)/4, (1/3)/5, (1/3)/6\}$ and $B = \{(1/3)/0, (1/3)/1, (2/3)/2, 1/3, 1/4, (2/3)/5\}$, the same discrete fuzzy numbers considered in Example 5. We will consider the binary relation $\preceq_{\Delta_\delta^\downarrow}$ with the Xu and Yager admissible order on $\Pi[L_6]$. In this case, the relevant α -levels of A and B are $\{1/3, 2/3, 1\}$. Let us compare first the 1-cuts (the cores):

$$A^1 = [2, 3] \prec_{XY} [3, 4] = B^1$$

and it holds that $A \prec_{\Delta_{XY}^\downarrow} B$. Note that we have established the ordering by comparing only the 1-cuts. If the 1-cuts had been equal, then it would have been necessary to compare the 2/3-cuts. The result provided by this binary relation $\preceq_{\Delta_\delta^\downarrow}$ is different from the result given by the first admissible order $\preceq_{\Delta_{XY}^\uparrow}$.

Let us prove that $\preceq_{\Delta_\delta^\downarrow}$ is a total order on $\mathcal{A}_1^{L_n}$, let us first prove again that this binary relation $\preceq_{\Delta_\delta^\downarrow}$ fulfills the transitivity property.

Proposition 8. Let $A, B, C \in \mathcal{A}_1^{L_n}$. If $A \prec_{\Delta_\delta^\downarrow} B$ and $B \prec_{\Delta_\delta^\downarrow} C$, then $A \prec_{\Delta_\delta^\downarrow} C$.

Moreover, let S_A, S_B and S_C be the sets of relevant α -levels of A, B, C , respectively and $S_{AB} = S_A \cup S_B, S_{BC} = S_B \cup S_C$ and $S_{AC} = S_A \cup S_C$. Let $\alpha_j \in S_{AB}$ be such that $A^{\alpha_j} \prec_\delta B^{\alpha_j}$ and $A^\alpha = B^\alpha$ for all $\alpha \in S_{AB}$ with $\alpha > \alpha_j$. Let $\beta_l \in S_{BC}$ be such that $B^{\beta_l} \prec_\delta C^{\beta_l}$ and $B^\beta = C^\beta$ for all $\beta \in S_{BC}$ with $\beta > \beta_l$. In this case, there exists $\gamma_s \in S_{AC}$ such that $A^{\gamma_s} \prec_\delta C^{\gamma_s}$ and $A^\gamma = C^\gamma$ for all $\gamma \in S_{AC}$ with $\gamma > \gamma_s$, where

$$\gamma_s = \min\{\gamma \in S_{AC} \mid \gamma \geq \max\{\alpha_j, \beta_l\}\}.$$

Proof. First of all, note that the set which defines γ_s is not empty since again $1 \in S_{AC}$ and $1 \geq \max\{\alpha_j, \beta_l\}$.

The methodology will be as follows: (i) to prove that $A^{\gamma_s} \prec_{\delta} C^{\gamma_s}$, (ii) to prove that $A^{\gamma} = C^{\gamma}$ for all $\gamma \in S_{AC}$ with $\gamma > \gamma_s$.

Let us start with (i). We know that $\gamma_s \geq \max\{\alpha_j, \beta_l\}$, and we will suppose that $\alpha_j \geq \beta_l$; the other case is analogous.

By the representation theorem and the definition of γ_s , we have that $A^{\gamma_s} = A^{\alpha_j}$. Then, by the definition of $\preceq_{\Delta_{\delta}^{\downarrow}}$, $A^{\gamma_s} = A^{\alpha_j} \prec_{\delta} B^{\alpha_j}$. Now, two cases arise:

1. If $\alpha_j \in S_C$, then, by the definition of γ_s , it must be $\gamma_s = \alpha_j$. Since $\alpha_j \geq \beta_l$, it holds that $B^{\alpha_j} \preceq_{\delta} C^{\alpha_j}$. Finally, $C^{\alpha_j} = C^{\gamma_s}$. Thus, we have proved that $A^{\gamma_s} \prec_{\delta} C^{\gamma_s}$.
2. If $\alpha_j \notin S_C$, let us define

$$t = \min\{\beta \in S_{BC} \mid \beta \geq \alpha_j\}.$$

By the representation theorem, it is clear that $B^{\alpha_j} = B^t$. Now,

- (a) If $t = \alpha_j$, then it is clear that $\alpha_j \in S_B$ and, consequently, by the definition of $\preceq_{\Delta_{\delta}^{\downarrow}}$ and the fact that $\alpha_j \geq \beta_l$, it holds that $B^t \preceq_{\delta} C^t = C^{\alpha_j}$ and, by the representation theorem, $C^{\alpha_j} = C^{\gamma_s}$.
- (b) If $t > \alpha_j$, then $\alpha_j \notin S_{BC}$ and $\alpha_j \in S_A$. Now, by the definition of $\preceq_{\Delta_{\delta}^{\downarrow}}$ and the fact that $t > \beta_l$, it holds that $B^t = C^t$ and then, by the representation theorem, $C^t = C^{\alpha_j} = C^{\gamma_s}$.

In all cases, we have proved that $A^{\gamma_s} \prec_{\delta} C^{\gamma_s}$.

As a second step, we must prove that $A^{\gamma} = C^{\gamma}$ for all $\gamma \in S_{AC}$ with $\gamma > \gamma_s$. For such γ , clearly $\gamma > \max\{\alpha_j, \beta_l\}$. Two cases must be analyzed:

1. If $\gamma \in S_B$, then $\gamma \in S_{AB} \cap S_{BC}$ and, by the definition of the binary relation $\preceq_{\Delta_{\delta}^{\downarrow}}$, it holds that $A^{\gamma} = B^{\gamma} = C^{\gamma}$.
2. If $\gamma \notin S_B$, then two cases must be analyzed:
 - (a) If $\gamma \in S_A$, then since $\gamma > \alpha_j$, it holds that $A^{\gamma} = B^{\gamma}$. Let us define

$$t = \min\{\beta \in S_{BC} \mid \beta \geq \gamma\}.$$

By the representation theorem, $B^{\gamma} = B^t$ and since $t > \beta_l$ and $t \in S_{BC}$, $B^t = C^t$ and again by the representation theorem, $C^t = C^{\gamma}$. To sum up, $A^{\gamma} = C^{\gamma}$.

3. The case $\gamma \notin S_A$ is analogous to the previous case. □

The following example illustrates the strict transitivity.

Example 9. Let $A, B, C \in \mathcal{A}_1^{L_6}$ be such that $A = \{0.4/0, 0.9/1, 0.9/2, 0.9/3, 1/4, 0.9/5\}$, $B = \{0.5/1, 0.8/2, 0.8/3, 1/4, 0.8/5, 0.5/6\}$ and $C = \{0.7/1, 0.7/2, 0.7/3, 1/4, 0.7/5, 0.3/6\}$. Consider the binary relation $\preceq_{\Delta_{\delta}^{\downarrow}}$ with the Xu and Yager admissible order on $\Pi[L_6]$. It is straightforward to check that $A \prec_{\Delta_{XY}^{\downarrow}} B$ with $\alpha_j = 0.9$ and $B \prec_{\Delta_{XY}^{\downarrow}} C$ where $\beta_l = 0.8$. Let us compute the value of γ_s which ensures that $A \prec_{\Delta_{XY}^{\downarrow}} C$. In this case,

$$S_{AC} = \{0.3, 0.4, 0.7, 0.9, 1\},$$

and

$$\gamma_s = \min\{\gamma \in S_{AC} \mid \gamma \geq \max\{\alpha_j, \beta_l\} = 0.9\} = 0.9.$$

Note that $A^{0.9} = [1, 5] \prec_{XY} [4, 4] = C^{0.9}$ while $A^1 = [4, 4] = B^1$.

Similarly to the case of the first family of orders presented in Section 4.1, the transitivity property follows.

Corollary 9. Let $A, B, C \in \mathcal{A}_1^{L_n}$. If $A \prec_{\Delta_\delta^\downarrow} B$ and $B \prec_{\Delta_\delta^\downarrow} C$, then $A \prec_{\Delta_\delta^\downarrow} C$.

Proof. Follows directly from Proposition 8 and the definition of the binary relation $\prec_{\Delta_\delta^\downarrow}$. □

Now, we can show that the binary relation $\prec_{\Delta_\delta^\downarrow}$ is a total order.

Theorem 10. The binary relation $\prec_{\Delta_\delta^\downarrow}$ is a total order on $\mathcal{A}_1^{L_n}$.

Proof. We skip the proof that is analogous to the proof of Theorem 6. □

Finally, by the same reasoning considered in Section 4.1, $\prec_{\Delta_\delta^\downarrow}$ is an admissible order.

Theorem 11. $\prec_{\Delta_\delta^\downarrow}$ is an admissible order on $\mathcal{A}_1^{L_n}$.

4.3. Some Reflections about These Two Families of Admissible Orders

In this section, we have introduced two families of admissible orders on $\mathcal{A}_1^{L_n}$. While both orders have a quite similar definition and are based on the sequential comparison of the α -cuts, their interpretation from the decision-making point of view is very different.

In the first family $\prec_{\Delta_\delta^\uparrow}$, the comparison is made from the support to the core and, therefore, the order prioritizes lower α -cuts. If the discrete fuzzy numbers represent linguistic evaluations provided by experts in a decision-making problem, the order takes into account first the support, which includes all the possible linguistic labels considered by the expert. Therefore, this order could be understood as a conservative assessment, any value within the support is considered to take the decision. This is a common strategy to generate orders in the set of fuzzy numbers.

On the other hand, the second family $\prec_{\Delta_\delta^\downarrow}$ decides the ordering comparing the α -cuts from the core to the support prioritizing higher α -cuts. In a decision-making problem, the order takes into account first the core, which includes only the linguistic labels expressed by the expert as those ones with a highest membership value. Therefore, this order could be understood as a more optimistic assessment, excluding first any linguistic labels not included in the core.

Consequently, the choice of one of the families will depend on the attitude of the group of experts.

5. Conclusions and Future Work

In the existing literature, the amount of total orders in the set of discrete fuzzy numbers whose support is a closed interval in the finite chain L_n is scarce, and they always depend on a ranking index. Specifically, the orders in the set of discrete fuzzy numbers proposed in [34–38] follow this strategy. The use of this kind of orders may have undesired consequences when they are applied to decision-making problems. Indeed, these total orders based on ranking indices are not able to distinguish between discrete fuzzy numbers (and, therefore, between experts’ opinions) which are very different in terms of core and support as it has been explained in detail in Section 3. This inability to distinguish between different discrete fuzzy numbers can jeopardize the results in any decision-making application, where a final alternative must be chosen or a consensus between experts must be reached. For this reason, in this paper, we have presented two families of total orders on $\mathcal{A}_1^{L_n}$ designed from an admissible order on $\Pi[L_n] = \{[a, b] : 0 \leq a \leq b \leq n, a, b \in L_n\}$. These two orders, which have different behaviors depending on the chosen admissible order on $\Pi[L_n]$ and the attitude of the group of experts, are also admissible.

As a future work, we want to embed these admissible orders on $\mathcal{A}_1^{L_n}$ in decision-making, consensus-reaching, and image processing algorithms based on discrete fuzzy numbers.

Author Contributions: Conceptualization, J.V.R., S.M., H.B. and J.F.; Formal analysis, J.V.R., S.M., H.B. and J.F.; Funding acquisition, S.M. and H.B. ; Investigation, J.V.R., S.M., H.B. and J.F.; Methodology, J.V.R., S.M., H.B. and J.F.; Writing—original draft, J.V.R., S.M., H.B. and J.F.; Writing—review & editing, J.V.R., S.M., H.B. and J.F. All authors have read and agreed to the published version of the manuscript.

Funding: This work has been partially supported by the Spanish Grants FEDER/Ministerio de Economía, Industria y Competitividad—AEI/TIN2016-75404-P and PID2019-108392GB-I00 (AEI/10.13039/501100011033).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Herrera, F.; Herrera-Viedma, F. Linguistic decision analysis: Steps for solving decision problems under linguistic information. *Fuzzy Sets Syst.* **2000**, *115*, 67–82. [[CrossRef](#)]
2. Zadeh, L. The concept of a linguistic variable and its application to approximate reasoning—Part I. *Inf. Sci.* **1975**, *8*, 199–249. [[CrossRef](#)]
3. Zadeh, L. The concept of a linguistic variable and its application to approximate reasoning—Part II. *Inf. Sci.* **1975**, *8*, 301–357. [[CrossRef](#)]
4. Zadeh, L. The concept of a linguistic variable and its application to approximate reasoning—Part III. *Inf. Sci.* **1975**, *9*, 43–80. [[CrossRef](#)]
5. Liao, H.; Xu, Z.; Herrera-Viedma, E.; Herrera, F. Hesitant Fuzzy Linguistic Term Set and Its Application in Decision Making: A State-of-the-Art Survey. *Int. J. Fuzzy Syst.* **2018**, *20*, 2084–2110. [[CrossRef](#)]
6. Torra, V. Hesitant fuzzy sets. *Int. J. Intell. Syst.* **2010**, *25*, 529–539. [[CrossRef](#)]
7. Mendel, J., J.R. Type-2 Fuzzy Sets Made Simple. *IEEE Trans. Fuzzy Syst.* **2002**, *10*, 117–127. [[CrossRef](#)]
8. Morente-Molinera, J.; Pérez, I.; Ureña, M.; Herrera-Viedma, E. On multi-granular fuzzy linguistic modeling in group decision-making problems: A systematic review and future trends. *Knowl. Based Syst.* **2015**, *74*, 49–60. [[CrossRef](#)]
9. Jiang, Y.P.; Fan, Z.P.; Ma, J. A method for group decision-making with multi-granularity linguistic assessment information. *Inf. Sci.* **2008**, *178*, 1098–1109. [[CrossRef](#)]
10. Zhang, Z.; Guo, C. A method for multi-granularity uncertain linguistic group decision making with incomplete weight information. *Knowl. Based Syst.* **2012**, *26*, 111–119. [[CrossRef](#)]
11. Chen, Z.; Ben-Arieh, D. On the fusion of multi-granularity linguistic label sets in group decision-making. *Comput. Ind. Eng.* **2006**, *51*, 526–541. [[CrossRef](#)]
12. Herrera, F.; Herrera-Viedma, E.; Martínez, L. A fusion approach for managing multi-granularity linguistic term sets in decision-making. *Fuzzy Sets Syst.* **2000**, *114*, 43–58. [[CrossRef](#)]
13. Herrera, F.; Martínez, L. A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Trans. Fuzzy Syst.* **2000**, *8*, 746–752.
14. Herrera, F.; Martínez, L. A model based on linguistic 2-tuples for dealing with multigranular hierarchical linguistic contexts in multi-expert decision-making. *IEEE Trans. Syst. Man. Cybern.* **2001**, *31*, 227–234. [[CrossRef](#)] [[PubMed](#)]
15. Huynh, V.; Nakamori, Y. A satisfactory-oriented approach to multi expert decision-making with linguistic assessments. *IEEE Trans. Syst. Man. Cybern. Part B Cybern.* **2005**, *35*, 184–196. [[CrossRef](#)] [[PubMed](#)]
16. Roselló, L.; Sánchez, M.; Agell, N.; Prats, F.; Mazaira, F.A. Using consensus and distances between generalized multi-attribute linguistic assessments for group decision-making. *Inf. Fusion* **2014**, *17*, 83–92. [[CrossRef](#)]
17. Massanet, S.; Riera, J.V.; Torrens, J.; Herrera-Viedma, E. A new linguistic computational model based on discrete fuzzy numbers for computing with words. *Inf. Sci.* **2014**, *258*, 277–290. [[CrossRef](#)]
18. Riera, J.V.; Massanet, S.; Herrera-Viedma, E.; Torrens, J. Some interesting properties of the fuzzy linguistic model based on discrete fuzzy numbers to manage hesitant fuzzy linguistic information. *Appl. Soft Comput.* **2015**, *36*, 383–391. [[CrossRef](#)]
19. Ban, A.I.; Coroianu, L. Simplifying the Search for Effective Ranking of Fuzzy Numbers. *IEEE Trans. Fuzzy Syst.* **2015**, *23*. [[CrossRef](#)]
20. Dombi, J.; Jónás, T. Ranking trapezoidal fuzzy numbers using a parametric relation pair. *Fuzzy Sets Syst.* **2020**. [[CrossRef](#)]
21. Wang, W.; Wang, Z. Total orderings defined on the set of all fuzzy numbers. *Fuzzy Sets Syst.* **2014**, *243*, 131–141. [[CrossRef](#)]
22. Bustince, H.; Fernandez, J.; Kolesárová, A.; Mesiar, R. Generation of linear orders for intervals by means of aggregation functions. *Fuzzy Sets Syst.* **2013**, *220*, 69–77. [[CrossRef](#)]
23. Miguel, L.D.; Bustince, H.; Fernandez, J.; Induráin, E.; Kolesárová, A.; Mesiar, R. Construction of admissible linear orders for interval-valued Atanassov intuitionistic fuzzy sets with an application to decision-making. *Inf. Fusion* **2016**, *27*, 189–197. [[CrossRef](#)]
24. Wang, H.; Xu, Z. Admissible orders of typical hesitant fuzzy elements and their application in ordered information fusion in multi-criteria decision-making. *Inf. Fusion* **2016**, *29*, 98–104. [[CrossRef](#)]

25. Wang, H.; Xu, Z. Total Orders of Extended Hesitant Fuzzy Linguistic Term Sets: Definitions, Generations and Applications. *Knowl. Based Syst.* **2016**, *107*, 142–154. [[CrossRef](#)]
26. Voxman, W. Canonical representations of discrete fuzzy numbers. *Fuzzy Sets Syst.* **2001**, *118*, 457–466. [[CrossRef](#)]
27. Casanovas, J.; Riera, J.V. Lattice properties of discrete fuzzy numbers under extended min and max. In Proceedings of the IFSA-EUSFLAT2009, Lisbon, Portugal, 20–24 July 2009; pp. 647–652.
28. Casanovas, J.; Riera, J.V. Extension of discrete t-norms and t-conorms to discrete fuzzy numbers. *Fuzzy Sets Syst.* **2011**, *167*, 65–81. [[CrossRef](#)]
29. Casanovas, J.; Riera, J.V. Weighted means of subjective evaluations. In *Soft Computing in Humanities and Social Sciences*; Seising, R., Sanz, V., Eds.; Studies in Fuzziness and Soft Computing; Springer: Berlin/Heidelberg, Germany, 2012; Volume 273, pp. 323–345.
30. Riera, J.V.; Torrens, J. Aggregation of subjective evaluations based on discrete fuzzy numbers. *Fuzzy Sets Syst.* **2012**, *191*, 21–40. [[CrossRef](#)]
31. Riera, J.V.; Torrens, J. Using discrete fuzzy numbers in the aggregation of incomplete qualitative information. *Fuzzy Sets Syst.* **2015**, *264*, 121–137. [[CrossRef](#)]
32. Zadeh, L. A note on Z-numbers. *Inf. Sci.* **2011**, *9*, 43–80. [[CrossRef](#)]
33. Massanet, S.; Riera, J.V.; Torrens, J. A new approach to Zadeh’s Z-numbers: Mixed-discrete Z-numbers. *Inf. Fusion* **2020**, *53*, 35–42. [[CrossRef](#)]
34. Basirzadeh, H.; Farnam, M.; Hakimi, E. An approach for ranking discrete fuzzy set. *J. Math. Comput. Sci.* **2012**, *2*, 584–592.
35. Chen, L.H.; Lu, H.W. An approximate approach for ranking fuzzy numbers based on left and right dominance. *Comput. Math. Appl.* **2001**, *41*, 1589–1602. [[CrossRef](#)]
36. Choobineh, F.; Li, H. An index for ordering fuzzy numbers. *Fuzzy Sets Syst.* **1993**, *54*, 287–294. [[CrossRef](#)]
37. Rosyida, I.; Widodo.; Rini, C.; Indriati, I.D. Fuzzy chromatic number of union of fuzzy graphs: An algorithm, properties and its application. *Fuzzy Sets Syst.* **2020**, *384*, 115–131. [[CrossRef](#)]
38. Zhang, L.; Wang, Z. Links between ranking and total ordering for fuzzy integers. In Proceedings of the Annual Conference of the North American Fuzzy Information Processing Society (NAFIPS), Washington, DC, USA, 17–19 August 2015; pp. 1–4.
39. Xu, Z.; Yager, R. Some geometric aggregation operators based on intuitionistic fuzzy sets. *Int. J. Gen. Syst.* **2006**, *35*, 417–433. [[CrossRef](#)]
40. Wang, G.; Wu, C.; Zhao, C. Representation and operations of discrete fuzzy numbers. *Southeast Asian Bull. Math.* **2005**, *28*, 1003–1010.