

Article

On the Direct Limit from Pseudo Jacobi Polynomials to Hermite Polynomials

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Abstract: In this short communication, we present a new limit relation that reduces pseudo-Jacobi polynomials directly to Hermite polynomials. The proof of this limit relation is based upon ${}_2F_1$ -type hypergeometric transformation formulas, which are applicable to even and odd polynomials separately. This limit opens the way to studying new exactly solvable harmonic oscillator models in quantum mechanics in terms of pseudo-Jacobi polynomials.

Keywords: pseudo-Jacobi polynomials; Hermite polynomials; ${}_2F_1$ -type hypergeometric transformation formulas

1. Introduction

Polynomials with an orthogonality property under some continuous or discrete measure play a major role in the exact solution of a number of phenomena expressed by means of differential or finite-difference equations. Some well-known problems of quantum mechanics and quantum computing, as well as stochastic processes, probability theory, and statistical finance, are among of these phenomena [1]. Hermite, Laguerre, and Jacobi polynomials, as exact solutions of a second-order differential equations of the hypergeometric type, are among the most widely used classical polynomials. The Askey scheme or table of orthogonal polynomials, introduced in the 1980s, exhibited “hidden” properties of these polynomials and their higher-order generalizations (and their q -deformed or basic analogues [2]). This scheme is a way of organizing orthogonal polynomials of the hypergeometric type (or basic hypergeometric type) into a directed graph, with Wilson polynomials and Racah polynomials (or Askey–Wilson polynomials and q -Racah polynomials) on the top level, and other polynomials arranged on lower levels, roughly speaking, according to the number of variables. A beautiful aspect of the scheme is that it is possible to establish connections between almost all polynomials located at the nearest or almost nearest neighbor nodes in the graph via exact limit relations or special cases, which are indicated by arrows in the graph. For example, the Hermite polynomials described by ${}_2F_0$ hypergeometric functions are located at the lowest level of the scheme, and the generalized Laguerre polynomials located on next level of the scheme are connected to the Hermite polynomials both via an exact limit relation [2] (9.12.13) and through a well-known special case that is separately valid for even and odd polynomials [2] (p. 244) (see Equations (16) and (19) in this paper). Some other interesting properties of these polynomials are a general limit relation between these two polynomials in terms of the Srivastava–Singhal polynomials [3] and the appearance as a generic polynomial solution of a differential equation [4]. In general, one easily observes from the Askey table that all orthogonal polynomials within this scheme are special or limiting cases of the Wilson polynomials or Racah polynomials (or Askey–Wilson or q -Racah polynomials).

In the early days of the Askey scheme [5,6], four classes of polynomials appear at the ${}_2F_1$ level. Nowadays, a fifth class, pseudo-Jacobi polynomials, is usually included



Citation: Jafarov, E.I.; Mammadova, A.M.; Van der Jeugt, J. On the Direct Limit from Pseudo Jacobi Polynomials to Hermite Polynomials. *Mathematics* **2021**, *9*, 88. <https://doi.org/10.3390/math9010088>

Received: 9 December 2020

Accepted: 31 December 2020

Published: 4 January 2021

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in the scheme at this level [2] (p. 183). These pseudo-Jacobi polynomials are among the least studied polynomials of the Askey scheme. These polynomials were first obtained by Routh [7], and later, were independently introduced by V. Romanovski as a finite system of Jacobi-like polynomials [8]. They are sometimes referred to as Routh–Romanovski polynomials. Their q -analogues were studied in [9], wherein further references can be found. An attractive property of the pseudo-Jacobi polynomials is their orthogonality relation for a weight function with support over the whole real line (see Equation (4)). Furthermore, it is known that the pseudo-Jacobi polynomials can be recovered from the continuous Hahn polynomials under a certain limit (see Equation (5)); hence, there is an arrow in the Askey scheme from continuous Hahn polynomials to pseudo-Jacobi polynomials. On the other hand, only a special case of the Bessel polynomials can be obtained as a limit of the pseudo-Jacobi polynomials [2,10]. In that sense, there is no arrow in the Askey scheme from pseudo-Jacobi polynomials to a class of polynomials at a lower level.

The main goal of this short communication is to show that there exists a direct limit from pseudo-Jacobi polynomials to Hermite polynomials, and hence, an extra arrow can be drawn in the Askey scheme.

The paper is structured as follows: In Section 2, some basic properties of both Hermite and pseudo-Jacobi polynomials are recalled. These properties include their hypergeometric expressions, orthogonality relations, and differential equations. In Section 3, the limit relation between these two polynomials is presented. Conclusions with some further discussions are presented in Section 4.

2. Basic Properties of Hermite and Pseudo-Jacobi Polynomials

In this section, we give the main formulas for Hermite and pseudo-Jacobi polynomials. All of these can be found in [2], but it is convenient to list them here for further reference.

Hermite polynomials are defined in terms of ${}_2F_0$ hypergeometric functions as follows [2] (9.15.1):

$$H_n(x) = (2x)^n {}_2F_0 \left(\begin{matrix} -n/2, -(n-1)/2 \\ - \end{matrix} ; -\frac{1}{x^2} \right). \tag{1}$$

They are exact solutions of the following second-order differential equation [2] (9.15.5):

$$y''(x) - 2xy'(x) + 2ny(x) = 0,$$

where $y(x) = H_n(x)$. Hermite polynomials satisfy an orthogonality relation [2] (9.15.2) on the interval $(-\infty, \infty)$:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = 2^n n! \delta_{mn}. \tag{2}$$

Pseudo-Jacobi polynomials belong to a higher level in the Askey scheme, and are defined in terms of ${}_2F_1$ hypergeometric functions as follows [2] (9.9.1):

$$P_n(x; \nu, N) = \frac{(-2i)^n (-N + i\nu)_n}{(n - 2N - 1)_n} {}_2F_1 \left(\begin{matrix} -n, n - 2N - 1 \\ -N + i\nu \end{matrix} ; \frac{1 - ix}{2} \right) \tag{3a}$$

$$= (x + i)^n {}_2F_1 \left(\begin{matrix} -n, N + 1 - n - i\nu \\ 2N + 2 - 2n \end{matrix} ; \frac{2}{1 - ix} \right), \quad n = 0, 1, 2, \dots, N. \tag{3b}$$

Herein, ν is an arbitrary real parameter and N is an arbitrary positive integer. The polynomials $P_n(x; \nu, N)$ are real polynomials in x of degree n , and n is restricted by N . They are also the exact solution of a second-order differential equation, namely [2] (9.9.5):

$$(1 + x^2)y''(x) + 2(\nu - Nx)y'(x) - n(n - 2N - 1)y(x) = 0,$$

where $y(x) = P_n(x; \nu, N)$. Pseudo-Jacobi polynomials also satisfy an orthogonality relation [2] (9.9.2) on the interval $(-\infty, \infty)$:

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{\infty} (1+x^2)^{-N-1} e^{2\nu \arctan x} P_m(x; \nu, N) P_n(x; \nu, N) dx \\ &= \frac{\Gamma(2N+1-2n)\Gamma(2N+2-2n)2^{2n-2N-1}n!}{\Gamma(2N+2-n)|\Gamma(N+1-n+iv)|^2} \delta_{mn}. \end{aligned} \tag{4}$$

The pseudo-Jacobi polynomials can be related to Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ in the following way [2] (p. 233):

$$P_n(x; \nu, N) = \frac{(-2i)^n n!}{(n-2N-1)_n} P_n^{(-N-1+iv, -N-1-iv)}(x),$$

but this is only a formal relation, only referring to the ${}_2F_1$ structure. They follow from the continuous Hahn polynomials $p_n(x; a, b, c, d)$ under the limit relation [2] (p. 233):

$$\begin{aligned} & \lim_{t \rightarrow \infty} p_n \left(xt; \frac{1}{2}(-N+iv-2t), \frac{1}{2}(-N-iv+2t), \frac{1}{2}(-N+iv-2t), \frac{1}{2}(-N-iv+2t) \right) \\ &= \frac{(n-2N-1)_n}{n!} P_n(x; \nu, N). \end{aligned} \tag{5}$$

A special case of the Bessel polynomials $y_n(x; a)$ can be obtained from them as follows:

$$\lim_{\nu \rightarrow \infty} \frac{P_n(\nu x; \nu, N)}{\nu^n} = \frac{2^n}{(n-2N-1)_n} y_n(x; -2N-2).$$

3. Direct Limit Relation between Pseudo-Jacobi and Hermite Polynomials

Before presenting the main theorem, we have a lemma with two transformation formulas (which appeared already in [11], but were not derived there).

Lemma 1. For m a non-negative integer, the following transformation formulas hold:

$${}_2F_1 \left(\begin{matrix} -2m, 2m+2\lambda \\ \lambda+1/2 \end{matrix}; \frac{1-\xi}{2} \right) = (-1)^m \frac{(1/2)_m}{(\lambda+1/2)_m} {}_2F_1 \left(\begin{matrix} -m, \lambda+m \\ 1/2 \end{matrix}; \xi^2 \right), \tag{6}$$

$${}_2F_1 \left(\begin{matrix} -2m-1, 2m+2\lambda+1 \\ \lambda+1/2 \end{matrix}; \frac{1-\xi}{2} \right) = (-1)^m \frac{(3/2)_m}{(\lambda+1/2)_m} \xi {}_2F_1 \left(\begin{matrix} -m, \lambda+m+1 \\ 3/2 \end{matrix}; \xi^2 \right). \tag{7}$$

Proof. First, apply the quadratic transformation formula 2.11.2 in [12] on the left-hand side of (6):

$${}_2F_1 \left(\begin{matrix} -2m, 2m+2\lambda \\ \lambda+1/2 \end{matrix}; \frac{1-\xi}{2} \right) = {}_2F_1 \left(\begin{matrix} -m, m+\lambda \\ \lambda+1/2 \end{matrix}; 1-\xi^2 \right).$$

Next, apply the linear transformation formula 15.8.7 of [13]:

$${}_2F_1 \left(\begin{matrix} -m, m+\lambda \\ \lambda+1/2 \end{matrix}; 1-\xi^2 \right) = \frac{(-m+1/2)_m}{(\lambda+1/2)_m} {}_2F_1 \left(\begin{matrix} -m, m+\lambda \\ 1/2 \end{matrix}; \xi^2 \right),$$

yielding (6).

For the second formula, apply the quadratic transformation formula 2.11.3 in [12] on the left-hand side of (7):

$${}_2F_1 \left(\begin{matrix} -2m-1, 2m+2\lambda+1 \\ \lambda+1/2 \end{matrix}; \frac{1-\xi}{2} \right) = \xi \frac{\Gamma(\lambda+1/2)\Gamma(-1/2)}{\Gamma(-m-1/2)\Gamma(m+\lambda+1/2)} {}_2F_1 \left(\begin{matrix} -m, m+\lambda+1 \\ 3/2 \end{matrix}; \xi^2 \right).$$

After simplification of the Gamma functions, this yields (7). \square

The main result of this note is the following:

Theorem 1. (Limit relation from $P_n(x; \nu, N)$ to $H_n(x)$) The Hermite polynomials (1) follow from the pseudo-Jacobi polynomials given by (3a) or (3b) by setting $x \rightarrow x/\sqrt{N}$ and $\nu \rightarrow \nu/N$ and then letting $N \rightarrow \infty$ in the following way:

$$\lim_{N \rightarrow \infty} N^{\frac{n}{2}} P_n\left(\frac{x}{\sqrt{N}}; \frac{\nu}{N}, N\right) = \frac{1}{2^n} H_n(x). \tag{8}$$

Proof. Using (3a), the left-hand side of Equation (8) can be rewritten as follows:

$$\lim_{N \rightarrow \infty} N^{\frac{n}{2}} P_n\left(\frac{x}{\sqrt{N}}; \frac{\nu}{N}, N\right) = (-2i)^n \lim_{N \rightarrow \infty} N^{\frac{n}{2}} \frac{(-N + i\frac{\nu}{N})_n}{(n - 2N - 1)_n} {}_2F_1\left(\begin{matrix} -n, n - 2N - 1 \\ -N + i\frac{\nu}{N} \end{matrix}; \frac{1 - i\frac{x}{\sqrt{N}}}{2}\right). \tag{9}$$

It is directly clear that

$$\lim_{N \rightarrow \infty} \frac{(-N + i\frac{\nu}{N})_n}{(n - 2N - 1)_n} = 2^{-n}. \tag{10}$$

Hence,

$$\lim_{N \rightarrow \infty} N^{\frac{n}{2}} P_n\left(\frac{x}{\sqrt{N}}; \frac{\nu}{N}, N\right) = (-i)^n \lim_{N \rightarrow \infty} N^{\frac{n}{2}} {}_2F_1\left(\begin{matrix} -n, n - 2N - 1 \\ -N + i\frac{\nu}{N} \end{matrix}; \frac{1 - i\frac{x}{\sqrt{N}}}{2}\right). \tag{11}$$

In the denominator of the hypergeometric series, the term $+i\frac{\nu}{N}$ plays no role in the limit, leading to

$$\lim_{N \rightarrow \infty} N^{\frac{n}{2}} P_n\left(\frac{x}{\sqrt{N}}; \frac{\nu}{N}, N\right) = (-i)^n \lim_{N \rightarrow \infty} N^{\frac{n}{2}} {}_2F_1\left(\begin{matrix} -n, n - 2N - 1 \\ -N \end{matrix}; \frac{1 - i\frac{x}{\sqrt{N}}}{2}\right). \tag{12}$$

After these straightforward simplifications, one has to take more care in order to proceed. In particular, it is now necessary to distinguish the cases for even and odd n . For $n = 2m$, we have

$$\lim_{N \rightarrow \infty} N^m P_{2m}\left(\frac{x}{\sqrt{N}}; \frac{\nu}{N}, N\right) = (-1)^m \lim_{N \rightarrow \infty} N^m {}_2F_1\left(\begin{matrix} -2m, 2m - 2N - 1 \\ -N \end{matrix}; \frac{1 - i\frac{x}{\sqrt{N}}}{2}\right). \tag{13}$$

Using (6), this leads to

$$\begin{aligned} \lim_{N \rightarrow \infty} N^m P_{2m}\left(\frac{x}{\sqrt{N}}; \frac{\nu}{N}, N\right) &= (1/2)_m \lim_{N \rightarrow \infty} \frac{N^m}{(-N)_m} {}_2F_1\left(\begin{matrix} -m, m - N - 1/2 \\ 1/2 \end{matrix}; -\frac{x^2}{N}\right) \\ &= (-1)^m (1/2)_m \lim_{N \rightarrow \infty} {}_2F_1\left(\begin{matrix} -m, m - N - 1/2 \\ 1/2 \end{matrix}; -\frac{x^2}{N}\right) \\ &= (-1)^m (1/2)_m {}_1F_1\left(\begin{matrix} -m \\ 1/2 \end{matrix}; x^2\right). \end{aligned} \tag{14}$$

For the last step, one can simply use

$$\lim_{N \rightarrow \infty} \frac{(-m)_k (m - N - 1/2)_k}{(1/2)_k k!} \left(-\frac{x^2}{N}\right)^k = \frac{(-m)_k}{(1/2)_k k!} (x^2)^k$$

in each term of the hypergeometric series. Taking into account the definition of generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ [2] (9.12.1)

$$L_n^{(\alpha)}(x) = \frac{(\alpha + 1)_n}{n!} {}_1F_1\left(\begin{matrix} -n \\ \alpha + 1 \end{matrix}; x\right), \tag{15}$$

and the following connection between Hermite and generalized Laguerre polynomials [2] (p. 244)

$$H_{2m}(x) = (-1)^m m! 2^{2m} L_m^{(-\frac{1}{2})}(x^2), \tag{16}$$

one obtains that

$$\lim_{N \rightarrow \infty} N^m P_{2m}\left(\frac{x}{\sqrt{N}}; \frac{\nu}{N}, N\right) = \frac{1}{2^{2m}} H_{2m}(x).$$

Next, consider the case $n = 2m + 1$ in (12):

$$\lim_{N \rightarrow \infty} N^{m+\frac{1}{2}} P_{2m+1}\left(\frac{x}{\sqrt{N}}; \frac{\nu}{N}, N\right) = i(-1)^{m+1} \lim_{N \rightarrow \infty} N^{m+\frac{1}{2}} {}_2F_1\left(\begin{matrix} -2m-1, 2m-2N \\ -N \end{matrix}; \frac{1-i\frac{x}{\sqrt{N}}}{2}\right). \tag{17}$$

Here, one can use (7), leading to

$$\begin{aligned} \lim_{N \rightarrow \infty} N^{m+\frac{1}{2}} P_{2m+1}\left(\frac{x}{\sqrt{N}}; \frac{\nu}{N}, N\right) &= (3/2)_m x \lim_{N \rightarrow \infty} \frac{N^m}{(-N)_m} {}_2F_1\left(\begin{matrix} -m, m-N-1/2 \\ 3/2 \end{matrix}; -\frac{x^2}{N}\right) \\ &= (-1)^m (3/2)_m x \lim_{N \rightarrow \infty} {}_2F_1\left(\begin{matrix} -m, m-N-1/2 \\ 3/2 \end{matrix}; -\frac{x^2}{N}\right) \\ &= (-1)^m (3/2)_m x {}_1F_1\left(\begin{matrix} -m \\ 3/2 \end{matrix}; x^2\right). \end{aligned} \tag{18}$$

According to (15), the hypergeometric series in the right-hand side of (18) is again a generalized Laguerre polynomial, with $\alpha = \frac{1}{2}$. Taking into account the relation [2] (p. 244)

$$H_{2m+1}(x) = (-1)^m m! 2^{2m+1} x L_m^{(\frac{1}{2})}(x^2), \tag{19}$$

one obtains that

$$\lim_{N \rightarrow \infty} N^{m+\frac{1}{2}} P_{2m+1}\left(\frac{x}{\sqrt{N}}; \frac{\nu}{N}, N\right) = \frac{1}{2^{2m+1}} H_{2m+1}(x).$$

This proves the limit relation (8). \square

It is interesting to observe that under this limit, the orthogonality relation for pseudo-Jacobi polynomials actually reduces to the orthogonality for Hermite polynomials. This is because the weight function for pseudo-Jacobi polynomials behaves under this limit as

$$\lim_{N \rightarrow \infty} \left(1 + \frac{x^2}{N}\right)^{-N-1} e^{2\frac{\nu}{N} \arctan \frac{x}{\sqrt{N}}} = e^{-x^2},$$

by using the classical limit

$$\lim_{n \rightarrow \infty} \left(1 + \frac{t}{n}\right)^n = e^t.$$

For the norm squared of the polynomials under this limit, it is sufficient to use Stirling's approximations

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

and

$$\Gamma(z) \sim \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z$$

in order to relate the right-hand side of (4) to the right-hand side of (2).

4. Discussion and Conclusions

Taking into account the existence of a direct limit from pseudo-Jacobi polynomials $P_n(x; \nu, N)$ to Hermite polynomials $H_n(x)$ by setting $x \rightarrow x/\sqrt{N}$ and $\nu \rightarrow \nu/N$ and then letting $N \rightarrow \infty$, let us explore this relation in more detail. In particular, let us examine the graphs of the corresponding polynomials. In Figure 1, we have plotted Hermite polynomials $H_n(x)$ and pseudo-Jacobi polynomials $2^n N^{\frac{n}{2}} P_n\left(\frac{x}{\sqrt{N}}; \frac{\nu}{N}, N\right)$ for certain values of n and of the variables. Taking into account that the Hermite polynomial $H_0(x)$ is constant and $H_1(x)$ is equal to $2x$, we concentrate on the plots for $n = 2$ and $n = 3$, the simplest cases for even and odd degrees. In addition, taking into account the dependence of the weight function for pseudo-Jacobi polynomials on ν , we have chosen two values of ν with opposite signs, namely $\nu = \pm 10$.

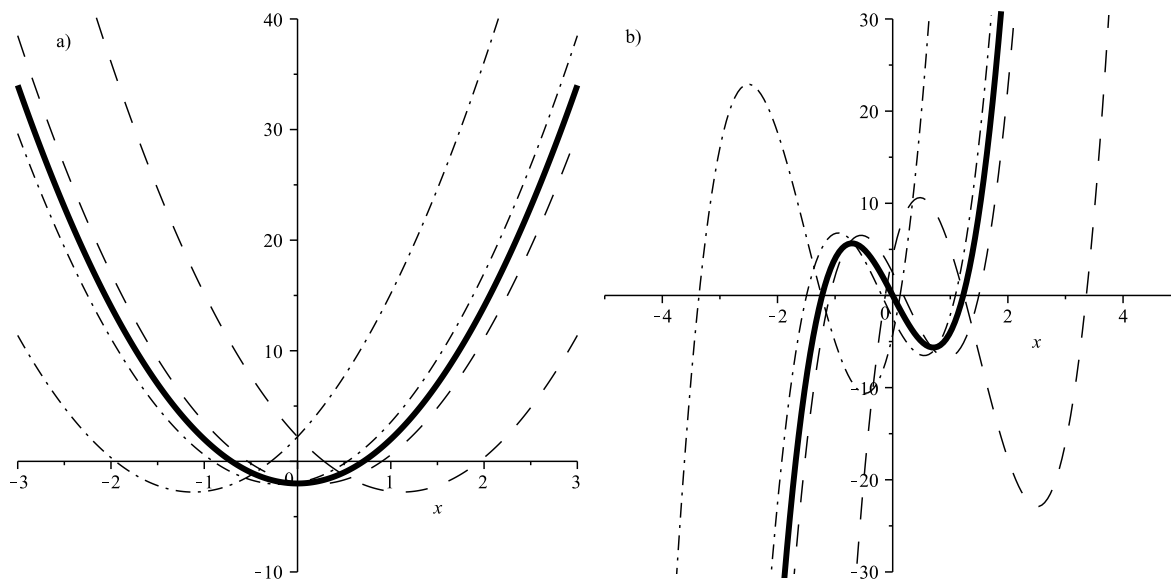


Figure 1. Hermite polynomials $H_n(x)$ vs. pseudo-Jacobi polynomials $2^n N^{\frac{n}{2}} P_n\left(\frac{x}{\sqrt{N}}; \frac{\nu}{N}, N\right)$. Panel (a) depicts $n = 2$ and panel (b) depicts $n = 3$. The Hermite polynomials are plotted by a solid line. The pseudo-Jacobi polynomials with $\nu = -10$ are plotted by a dash-dotted line, and the pseudo-Jacobi polynomials with $\nu = 10$ are plotted by a dashed line. The plots are given for $N = 5$ and $N = 15$, where the case $N = 15$ is closest to the solid line.

In both plots, one observes that the value of ν plays the role of shifting the pseudo-Jacobi polynomial to the left or right of the Hermite polynomial depending on the sign of this parameter. Actually, as the limit relation (8) holds for the parameter $\nu \rightarrow \nu/N$, the plots of pseudo-Jacobi polynomials tend to the plots of the Hermite polynomials as N increases. Both plots clearly demonstrate the limit behavior and how the value of the parameter ν no longer plays a role as it disappears under the limit. Similar plots with similar behavior can be made for the polynomials with a degree n higher than 3.

To conclude, we emphasize again that we have obtained a proper limit relation that reduces the pseudo-Jacobi polynomials to the Hermite polynomials and, moreover, transfers the orthogonality relation from pseudo-Jacobi polynomials to that of Hermite polynomials. To prove the limit, we had to use ${}_2F_1$ -type hypergeometric transformation formulas for even and odd polynomials separately. We think that the consequences of our result can be of major importance in quantum mechanical models. It is well known that Hermite polynomials play a vital role as exact solutions of the quantum mechanical harmonic

oscillator problem. In the context of non-relativistic quantum theory under the canonical approach, where the commutation relation between the one-dimensional momentum and position operators is a c -number, the exact solution of the time-independent Schrödinger equation for the one-dimensional non-relativistic harmonic oscillator with homogeneous effective mass m_0 leads to wave functions of stationary states in terms of the Hermite polynomials $H_n(x)$ [14]. The exact solution of same problem, but with an additional external homogeneous field, leads to wave functions of the stationary states in terms of the Hermite polynomials with a shifted variable as $H_n(x + x_0)$. The solution of the same Schrödinger equation under the non-canonical approach leads to wave functions of stationary states in terms of the generalized Laguerre polynomials $L_n^{(\alpha)}(x^2)$ [15], sometimes referred to as generalized Hermite polynomials [16]. Therefore, in view of the established limit (8), we think that it is interesting to study new exactly solvable harmonic oscillator models in terms of pseudo-Jacobi polynomials $P_n\left(\frac{x}{\sqrt{N}}; \frac{\nu}{N}, N\right)$, where, for example, the parameter ν takes the role of a shifting parameter. Even beyond that, in view of the limit (5), one could examine new quantum oscillator models where the continuous Hahn polynomials $p_n(x; a, b, c, d)$ [17], subject to certain parameter restrictions, play the role of wave functions.

Author Contributions: Conceptualization, E.I.J. and A.M.M.; Methodology, J.V.d.J.; Validation, E.I.J., A.M.M. and J.V.d.J.; Formal Analysis, E.I.J., A.M.M. and J.V.d.J.; Investigation, E.I.J., A.M.M. and J.V.d.J.; Resources, E.I.J., A.M.M. and J.V.d.J.; Writing—Original Draft Preparation, E.I.J., A.M.M. and J.V.d.J.; Writing—Review & Editing, J.V.d.J.; Visualization, E.I.J. All authors have read and agreed to the published version of the manuscript.

Funding: E.I.J. kindly acknowledges that this work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan—Grant No. EIF-KETPL-2-2015-1(25)-56/01/1. J.V.d.J. was supported by the EOS Research Project 30889451.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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