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Wiener Complexity versus the Eccentric Complexity

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Abstract: Let $w_G(u)$ be the sum of distances from u to all the other vertices of G. The Wiener complexity, $C_W(G)$, is the number of different values of $w_G(u)$ in G, and the eccentric complexity, $C_{ec}(G)$, is the number of different eccentricities in G. In this paper, we prove that for every integer c there are infinitely many graphs G such that $C_W(G) - C_{ec}(G) = c$. Moreover, we prove this statement using graphs with the smallest possible cyclomatic number. That is, if $c \geq 0$ we prove this statement using trees, and if $c < 0$ we prove it using unicyclic graphs. Further, we prove that $C_{ec}(G) \leq 2C_W(G) - 1$ if G is a unicyclic graph. In our proofs we use that the function $w_G(u)$ is convex on paths consisting of bridges. This property also promptly implies the already known bound for trees $C_{ec}(G) \leq C_W(G)$. Finally, we answer in positive an open question by finding infinitely many graphs G with diameter 3 such that $C_{ec}(G) < C_W(G)$.

Keywords: graph; diameter; wiener index; transmission; eccentricity

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1. Introduction

Let G be a graph. Denote by $V(G)$ and $E(G)$ its vertex and edge sets, respectively. If $u \in V(G)$, then $\deg_G(u)$ denotes the degree of u in G, and if $S \subseteq V(G)$ then $N(S)$ denotes the set S together with the vertices which have a neighbour in S. Obviously, $|N(u)| = \deg_G(u) + 1$. If $R \subseteq E(G)$ then $G - R$ denotes a graph obtained when we remove all the edges of R from G. Similarly, if $S \subseteq V(G)$ then $G - S$ denotes a graph obtained when we remove all the vertices of S and all edges incident with a vertex of S from G. An edge $e \in E(G)$ is a bridge if $G - \{e\}$ has more components than G.

If $u, v \in V(G)$ then $\mathrm{dist}_G(u, v)$ is the length of a shortest path from u to v in G. The longest distance from a vertex u is its eccentricity $e_G(u)$. Hence, $e_G(u) = \max\{\mathrm{dist}_G(u, v);\ v \in V(G)\}$. Using the eccentricity we define the radius $\mathrm{rad}(G) = \min\{e_G(u);\ u \in V(G)\}$, and the diameter $\mathrm{diam}(G) = \max\{e_G(u);\ u \in V(G)\} = \max\{\mathrm{dist}_G(u, v);\ u, v \in V(G)\}$. The *eccentric complexity* of G is defined as

$$C_{ec}(G) = |\{e_G(u);\ u \in V(G)\}|.$$

Observe that $C_{ec}(G) = \mathrm{diam}(G) - \mathrm{rad}(G) + 1$. The eccentric complexity has been introduced in [1]. Also see [2] for related connective eccentric complexity.

On the other hand the *Wiener complexity* of G is

$$C_W(G) = |\{w_G(u);\ u \in V(G)\}|,$$

where $w_G(u) = \sum_{v \in V(G)} \mathrm{dist}(u, v)$ is the transmission of u in G. The parameter $\frac{1}{2} \sum_{u \in V(G)} w_G(u)$ is known as the Wiener index $W(G)$. Hence, $W(G) = \sum_{u,v \in V(G)} \mathrm{dist}(u, v)$. The Wiener complexity $C_W(G)$ of a graph G was introduced in [3]. Further research on $C_W(G)$ can be found in [4–6]. For results on Wiener index see, e.g., [7].

In [8] the authors study the relation between $C_{ec}(G)$ and $C_W(G)$. They prove the following statement.

Theorem 1. *If T is a tree then $C_{ec}(T) \leq C_W(T)$.*

Next, using cartesian products they prove that for every $c \geq 0$ there are graphs G with $C_W(G) - C_{ec}(G) = c$ and for every $k > 0$ there are graphs G with $C_{ec}(G) - C_W(G) = 2^k$. Here we continue in their research. We prove that for every $c \geq 0$ there are infinitely many trees T such that $C_W(T) - C_{ec}(T) = c$. By Theorem 1 to construct graphs G with $C_{ec}(G) > C_W(G)$ we must abandon the class of trees. So we concentrate on graphs with cyclomatic number 1. We prove that for every $c > 0$ there are infinitely many unicyclic graphs G such that $C_{ec}(G) - C_W(G) = c$.

All graphs G with $C_{ec}(G) < C_W(G)$ found in [8] have diameter at least 4, and it was shown that there are no such graphs of diameter at most 2. So the authors posed in [8] the following problem.

Problem 1. *Does there exist a graph G with diameter 3 and $C_{ec}(G) > C_W(G)$?*

We answer Problem 1 affirmatively and we find infinitely many graphs satisfying its requirements.

The outline of the paper is as follows. In Section 2 we characterize all pairs c_1 and c_2 such that there is a tree T with $C_{ec}(T) = c_1$ and $C_W(T) = c_2$. Analogously, in Section 3 we characterize all pairs c_1 and c_2 such that $c_1 < c_2$ and there is a unicyclic graph G with $C_W(G) = c_1$ and $C_{ec}(G) = c_2$. Finally, in Section 4 we deal with Problem 1.

2. Trees

In this section we characterize pairs c_1 and c_2 such that there are (infinitely many) trees T with $C_{ec}(T) = c_1$ and $C_W(T) = c_2$. To do this, first we show that w_T is a strictly convex function on paths consisting of bridges; observe that in a tree, every edge is a bridge. However, firstly we state the following easy lemma.

Lemma 1. *Let G be a connected graph with a bridge u_1u_2. Let G_1 and G_2 be the two components of $G - u_1u_2$, such that $u_i \in V(G_i)$, $1 \leq i \leq 2$, and let n_i be the number of vertices in G_i. Then $w_G(u_1) - w_G(u_2) = n_2 - n_1$.*

Proof. Let w_i be the transmission of u_i in G_i, $1 \leq i \leq 2$. Then

$$w_G(u_1) = w_1 + n_2 + w_2 \qquad \text{and} \qquad w_G(u_2) = n_1 + w_1 + w_2.$$

Hence, $w_G(u_1) - w_G(u_2) = n_2 - n_1$. □

Recall that a function $f(i)$ defined on $\{0, 1, \ldots, t\}$ is *strictly convex*, if for every $i \in \{1, \ldots, t-1\}$, we have $2f(i) < f(i-1) + f(i+1)$, or equivalently $f(i) - f(i-1) < f(i+1) - f(i)$. We have the following statement.

Lemma 2. *Let G be a graph. Further, let $P = v_0v_1 \cdots v_t$ be a path in G such that every edge of P is a bridge. Then $f(i) = w_G(v_i)$ is a strictly convex function on $\{0, 1, \ldots, t\}$.*

Proof. Let $u_1u_2u_3$ be a subpath of P. Then $G - \{u_1u_2, u_2u_3\}$ has three components. Denote by G_i the component of $G - \{u_1u_2, u_2u_3\}$ which contains u_i, for each $i \in \{1, 2, 3\}$. Moreover, denote $s_i = |V(G_i)|$. By Lemma 1, we have

$$w_G(u_1) + w_G(u_3) - 2w_G(u_2) = (s_2 + s_3 - s_1) + (s_1 + s_2 - s_3) = 2s_2 > 0.$$

Consequently, $f(i) = w_G(v_i)$ is strictly convex on $\{0, 1, \ldots, t\}$. □

Observe that considering a diametric path, Lemma 2 directly implies Theorem 1. However, we use it in the following statement which characterizes all possible pairs $C_{\rm ec}(T)$, $C_W(T)$ for trees.

Theorem 2. *It holds:*

(i) *If* $3 \le c_1 \le c_2$ *then there are infinitely many trees* T *with* $C_{\rm ec}(T) = c_1$ *and* $C_W(T) = c_2$.
(ii) *If* $c_1 = 2$ *and* $c_2 \in \{2, 4\}$ *then there are infinitely many trees* T *with* $C_{\rm ec}(T) = c_1$ *and* $C_W(T) = c_2$ *and no trees with* $C_{\rm ec}(T) = c_1$ *and* $C_W(T) \notin \{2, 4\}$.
(iii) *If* $c_1 = 1$ *then there are only two trees* T *with* $C_{\rm ec}(T) = 1$ *and in this case* $C_W(T) = 1$ *as well.*

Proof. Consider (i). Here $c_1 \ge 3$. First suppose that $c_2 > c_1$. Let $k = \lceil \frac{c_2-1}{c_1-1} \rceil$. Take k paths $P_0, P_1, \ldots, P_{k-1}$ of length $c_1 - 1$ and denote their vertices so that $P_i = v_{i,0}v_{i,1} \cdots v_{i,c_1-1}$, where $0 \le i \le k - 1$. Denote

$$\ell = c_2 - 1 - (\lceil \frac{c_2 - 1}{c_1 - 1} \rceil - 1)(c_1 - 1).$$

Let P_k be a path of length ℓ so that $P_k = v_{k,0}v_{k,1} \cdots v_{k,\ell}$. Since $\lceil \frac{c_2-1}{c_1-1} \rceil (c_1 - 1) \ge (c_2 - 1)$, we have $\ell \le c_1 - 1$, and since $\lceil \frac{c_2-1}{c_1-1} \rceil (c_1 - 1) < (c_2 - 1) + (c_1 - 1)$, we have $\ell > 0$. Thus, $1 \le \ell \le c_1 - 1$. Now attach to v_{i,c_1-2} exactly $i - 1$ new pendant vertices, $2 \le i \le k - 1$, and attach to $v_{k,\ell-1}$ exactly $q - 1$ new vertices. We expect that q is a big number. Finally, identify the vertices $v_{0,0}, v_{1,0}, \ldots, v_{k,0}$ into a single vertex, which we denote by c, and denote the resulting tree by T, see Figure 1. Observe that there is 1 pendant vertex attached to v_{0,c_1-2} in T and exactly i pendant vertices are attached to v_{i,c_1-2}, $1 \le i \le k - 1$. Further, since $k \ge 2$ ($k \ge 1$ would suffice since there is also P_0) we have $\mathrm{rad}(T) = c_1 - 1$ and $\mathrm{diam}(T) = 2(c_1 - 1)$, so that $C_{\rm ec}(T) = c_1$. Obviously, if u and v are pendant vertices attached to the same vertex in T then $w_T(u) = w_T(v)$. Also, $w_T(u) = w_T(v)$ if $u = v_{0,i}$ and $v = v_{1,i}$, $1 \le i \le c_1 - 1$. In all other cases we show that $w_T(u) \ne w_T(v)$. Hence, we show that $c, v_{1,1}, v_{1,2}, \ldots, v_{k,\ell}$ have different transmissions.

Denote $r = c_1 - 1$. Let P be a path in T consisting of vertices of P_a and P_b, $1 \le a < b \le k - 1$. Then $P = v_{a,r} \cdots v_{a,1}\, c\, v_{b,1} \cdots v_{b,r}$. Let T' be the nontrivial component of $T - \{v_{a,1}, \ldots, v_{a,r}, v_{b,1}, \ldots, v_{b,r}\}$. Denote by w' the transmission of c in T' and denote by z the number of vertices of T'. Then

$$\begin{aligned}
w_T(v_{a,r}) &= 2(a-1) + \tbinom{2r+1}{2} + 2r(b-1) + w' + (z-1)r;\\
w_T(v_{a,i}) &= (r-i)(a-1) + \tbinom{r-i+1}{2} + \tbinom{r+i+1}{2} + (r+i)(b-1) + w' + (z-1)i,\ 1 \le i \le r-1;\\
w_T(c) &= r(a-1) + \tbinom{r+1}{2} + \tbinom{r+1}{2} + r(b-1) + w';\\
w_T(v_{b,i}) &= (r+i)(a-1) + \tbinom{r+i+1}{2} + \tbinom{r-i+1}{2} + (r-i)(b-1) + w' + (z-1)i,\ 1 \le i \le r-1;\\
w_T(v_{b,r}) &= 2r(a-1) + \tbinom{2r+1}{2} + 2(b-1) + w' + (z-1)r.
\end{aligned}$$

Since $w_T(v_{a,i}) - w_T(v_{b,i}) = (r - i)(a - b) + (r + i)(b - a) = 2i(b - a) > 0$ if $1 \le i \le r - 1$ and $w_T(v_{a,r}) - w_T(v_{b,r}) = (2r - 2)(b - a) > 0$, we have $w_T(v_{a,j}) > w_T(v_{b,j})$, $1 \le j \le r$. And since q and consequently also z are big, the terms containing z in the above expressions are crucial. Therefore $w_T(v_{a,1}) < w_T(v_{b,2})$ and in general $w_T(v_{a,i}) < w_T(v_{b,i+1})$, where $1 \le i < r$. So we conclude that

$$w_T(c) < w_T(v_{b,1}) < w_T(v_{a,1}) < w_T(v_{b,2}) < w_T(v_{a,2}) < w_T(v_{b,3}) < \cdots < w_T(v_{a,r}).$$

Now let P be a path consisting of P_a and P_k, $1 \le a \le k - 1$. Then $P = v_{a,r} \cdots v_{a,1}\, c\, v_{k,1} \cdots v_{k,\ell} = u_{a+\ell}u_{a+\ell-1} \cdots u_0$. We remark that u_j are just different labels for vertices of P which will be used later. Similarly as above, let T' be the nontrivial component of

$T - \{v_{a,1}, \ldots .v_{a,r}, v_{k,1}, \ldots, v_{k,\ell}\}$. Denote by w' the transmission of c in T' and denote by z the number of vertices of T'. Then

$$\begin{aligned} w_T(v_{a,r}) &= 2(a-1) + \binom{r+\ell+1}{2} + (r+\ell)(q-1) + w' + (z-1)r; \\ w_T(v_{a,i}) &= (r-i)(a-1) + \binom{r-i+1}{2} + \binom{\ell+i+1}{2} + (\ell+i)(q-1) + w' + (z-1)i,\ 1 \le i \le r-1; \\ w_T(c) &= r(a-1) + \binom{r+1}{2} + \binom{\ell+1}{2} + \ell(q-1) + w'; \\ w_T(v_{k,i}) &= (r+i)(a-1) + \binom{r+i+1}{2} + \binom{\ell-i+1}{2} + (\ell-i)(q-1) + w' + (z-1)i,\ 1 \le i \le \ell-1; \\ w_T(v_{k,\ell}) &= (r+\ell)(a-1) + \binom{r+\ell+1}{2} + 2(q-1) + w' + (z-1)\ell. \end{aligned}$$

Observe that $u_2 = v_{k,\ell-2}$ if $\ell \ge 3$, $u_2 = c$ if $\ell = 2$ and $u_2 = v_{a,1}$ if $\ell = 1$. In any case, we have $w_T(v_{k,l}) - w_T(u_2) \ge \binom{r+\ell+1}{2} - \binom{r+\ell-1}{2} - \binom{2}{2} > 0$, and so $w_T(u_0) > w_T(u_2)$. And since q is big, analogously as above we conclude that

$$w_T(u_1) < w_T(u_2) < w_T(u_0) < w_T(u_3) < w_T(u_4) < \cdots < w_T(u_{r+\ell}).$$

Let $S = \{c, v_{1,1}, \ldots, v_{k,\ell}\}$. As shown above, vertices in S have pairwise different transmissions, while the vertices outside S have transmissions as some vertices in S. Since

$$|S| = 1 + (\lceil \frac{c_2-1}{c_1-1} \rceil - 1)(c_1 - 1) + [c_2 - 1 - (\lceil \frac{c_2-1}{c_1-1} \rceil - 1)(c_1 - 1)] = c_2,$$

we have $C_W(T) = c_2$.

Now suppose $c_2 = c_1$. Let $P = v_{0,c_1-1} \cdots v_{0,1}\, c\, v_{1,1} \cdots v_{1,c_1-1}$ be a path of length $2(c_1 - 1)$. We attach to both v_{0,c_1-2} and v_{1,c_1-2} exactly q pendant vertices and we denote by T the resulting tree, see Figure 2. Then T has $2c_1 - 1 + 2q$ vertices, $\mathrm{rad}(T) = c_1 - 1$ and $\mathrm{diam}(T) = 2(c_1 - 1)$, so that $C_{\mathrm{ec}}(T) = c_1$. Denote $r = c_1 - 1$. By symmetry, we have $w_T(v_{0,i}) = w_T(v_{1,i})$, $1 \le i \le r$, and $w_T(u) = w_T(v)$ if u and v are pendant vertices of T. So it remains to show that the vertices $v_{0,r}, \ldots, v_{0,1}, c$ have different transmissions. However, since $w_T(v_{0,1}) = v_T(v_{1,1})$, by Lemma 2 we get

$$w_T(c) < w_T(v_{0,1}) < \cdots < v_T(v_{0,r})$$

and so $C_W(T) = r + 1 = c_1$.

Now, consider (ii). So, let $c_1 = 2$. If T is a tree with $\mathrm{rad}(T) \ge 3$, then $\mathrm{diam}(T) \ge 5$ and consequently $C_{\mathrm{ec}}(T) \ge 3$, a contradiction. Hence, either $\mathrm{rad}(T) = 1$ and $\mathrm{diam}(T) = 2$, in which case T is a star $K_{1,t}$, where $t \ge 2$, or $\mathrm{rad}(T) = 2$ and $\mathrm{diam}(T) = 3$, in which case T is a double star $D_{a,b}$, i.e., a graph on $a + b + 2$ vertices obtained by attaching a pendant vertices to one vertex of K_2 and b pendant vertices to the other vertex of K_2, where $1 \le a \le b$. If T is a star $K_{1,t}$, $t \ge 2$, then $C_W(T) = 2$ since the central vertex has transmission smaller than is the transmission of pendant vertices. This establishes the case $c_1 = c_2 = 2$. On the other hand if T is a double star then since pendant vertices adjacent to a common vertex have the same transmission, we have $C_W(T) \le 4$. In the next we consider $T = D_{a,b}$, where $a < b$, since $C_W(D_{a,a}) = 2$, a case already solved by stars. Let v_0, v_1, v_2, v_3 be a path in $D_{a,b}$ such that $\deg_T(v_1) = a + 1$ and $\deg_T(v_2) = b + 1$. Then

$$\begin{aligned} w_T(v_0) &= 2(a-1) + 3 + 3b; \\ w_T(v_1) &= a + 1 + 2b; \\ w_T(v_2) &= 2a + 1 + b; \\ w_T(v_3) &= 3a + 3 + 2(b-1). \end{aligned}$$

Since $0 < a < b$, it is obvious that $2a + 1 + b < a + 1 + 2b < 3a + 1 + 2b < 2a + 1 + 3b$. Thus. $w_T(v_2) < w_T(v_1) < w_T(v_3) < w_T(v_0)$, and so $C_W(T) = 4$.

Finally, consider (iii). Since there are only two trees T such that $\mathrm{rad}(T) = \mathrm{diam}(T)$, namely the complete graphs K_1 and K_2, this part of Theorem 2 is trivial. □

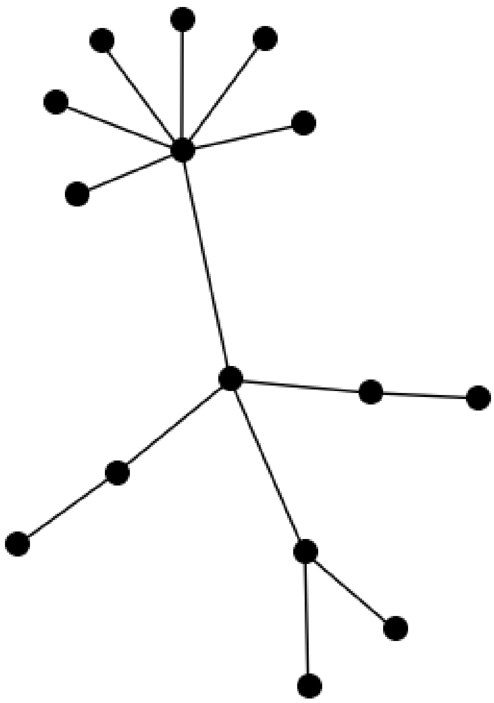

Figure 1. The construction for $c_1 = 3$, $c_2 = 7$, and $q = 6$.

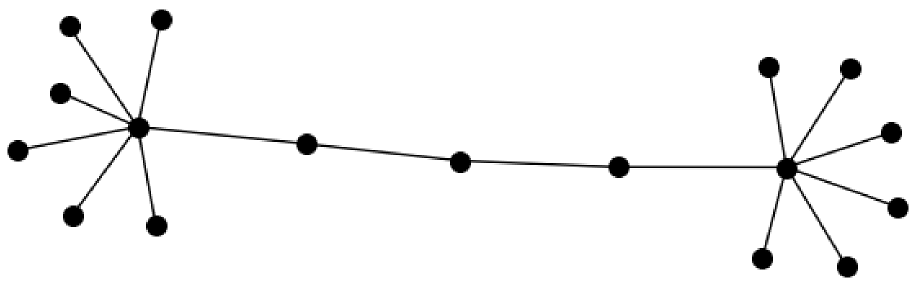

Figure 2. The construction for $c_1 = c_2 = 4$ and $q = 5$.

Theorem 2 has the following consequence.

Corollary 1. *For every $c \geq 0$ there are infinitely many trees T such that $C_W(T) - C_{ec}(T) = c$.*

3. Unicyclic Graphs

In this section, we give counterparts of the previous results for unicyclic graphs. We also bound the eccentric complexity in term of Wiener complexity and characterize the pairs c_1, c_2 such that $c_1 < c_2$ and there are (infinitely many) unicyclic graphs G with $C_W(G) = c_1$ and $C_{ec}(G) = c_2$. We start with the following lemma.

Lemma 3. *Let G be a unicyclic graph with a cycle C. Further, let $u_2, v \in V(C)$ and let u_1 be a neighbour of u_2 which is not in C. If $w_G(u_1) \leq w_G(u_2)$ then $w_G(u_2) < w_G(v)$.*

Proof. Observe that u_1u_2 is a bridge in G. Hence, $G - u_1u_2$ has two components, say G_1 and G_2. Assume that $u_i \in V(G_i)$ and $n_i = |V(G_i)|$, $1 \leq i \leq 2$. By the assumptions and by Lemma 1, $w_G(u_1) - w_G(u_2) = n_2 - n_1 \leq 0$.

Let T be a tree obtained from G by removing an edge of C which is opposite (i.e., antipodal) to v. Observe that if C has odd length, then there is a unique edge opposite to v, while if C has even length, then there are two edges opposite to v. Obviously, $w_G(v) = w_T(v)$ and $w_G(u_2) \leq w_T(u_2)$. Observe also that

$$0 \geq w_G(u_1) - w_G(u_2) = w_T(u_1) - w_T(u_2).$$

Now, consider a path from u_1 to v in T. Assume that the length of this path is $k-1$ and denote their vertices by $u_1u_2u_3\cdots u_k(=v)$. Since u_1u_2 is a bridge in T, we have $w_T(u_1) - w_T(u_2) = n_2 - n_1$ again. And by Lemma 2 we get $w_T(u_1) + w_T(u_3) > 2w_T(u_2)$ or equivalently $w_T(u_1) - w_T(u_2) > w_T(u_2) - w_T(u_3)$. Applying Lemma 2 several times we get

$$0 \geq w_T(u_1) - w_T(u_2) > w_T(u_2) - w_T(u_3) > \cdots > w_T(u_{k-1}) - w_T(v)$$

which implies $w_T(u_1) \leq w_T(u_2) < w_T(u_3) < \cdots < w_T(v)$ and consequently $w_G(u_2) \leq w_T(u_2) < w_T(v) = w_G(v)$. □

The following statement characterizes all possible pairs $C_W(G)$, $C_{ec}(G)$ for unicyclic graphs, provided that $C_W(G) < C_{ec}(G)$.

Theorem 3. *Every unicyclic graph G satisfies*

$$C_{ec}(G) \leq 2C_W(G) - 1.$$

Moreover, for any positive integers c_1 and c_2 with $c_1 < c_2 \leq 2c_1 - 1$ there are infinitely many unicyclic graphs G such that $C_W(G) = c_1$ and $C_{ec}(G) = c_2$.

Proof. Let G be a unicyclic graph with a cycle C of length k. Further, let P_1 and P_2 be two longest paths starting in different vertices of C and which contain only edges which are not in C. Observe that if the length of P_1 is positive, then the path terminates in a pendant vertex of G. Similar statement holds for P_2. Let ℓ_i be the length of P_i, $1 \leq i \leq 2$, and let $P_i = v_{i,0}v_{i,1}\cdots v_{i,\ell_i}$, where $v_{i,0} \in V(C)$. Observe that in each of P_1 and P_2, there are at most two vertices with the same transmission, by Lemma 2. If there are three vertices, say u_1, u_2 and u_3, in $V(P_1) \cup V(P_2)$ which have the same transmission in G, then two of them are in one of the paths P_1 and P_2 while the third one is in the other. Without loss of generality we may assume that $u_1, u_2 \in V(P_1)$ and $u_3 \in V(P_3)$. Then $w_G(v_{1,1}) \leq w_G(v_{1,0})$ by Lemma 2, and so $w_G(v_{1,0}) < w_G(v_{2,0})$ by Lemma 3. If $w_G(v_{2,1}) \leq w_G(v_{2,0})$ then $w_G(v_{2,0}) < w_G(v_{1,0})$ by Lemma 3, a contradiction. Hence $w_G(v_{2,0}) < w_G(v_{2,1})$, and by Lemma 2 $w_G(v_{2,0}) < w_G(v_{2,i})$ for every i, $1 \leq i \leq \ell_2$. Consequently $w_G(u_1) = w_G(u_2) \leq w_G(v_{1,0}) < w_G(v_{2,0}) \leq w_G(u_3)$. Hence, there are not three vertices in $V(P_1) \cup V(P_2)$ which have the same transmission in G. Therefore $C_W(G) \geq \lceil \frac{\ell_1+\ell_2}{2} \rceil + 1$.

On the other hand $\mathrm{diam}(G) \leq \ell_1 + \ell_2 + \lfloor k/2 \rfloor$ and $\mathrm{rad}(G) \geq \lfloor k/2 \rfloor$. So $C_{ec}(G) = \mathrm{diam}(G) - \mathrm{rad}(G) + 1 \leq \ell_1 + \ell_2 + 1$, and hence

$$2C_W(G) - C_{ec}(G) \geq 2\left\lceil \frac{\ell_1+\ell_2}{2} \right\rceil + 2 - l_1 - l_2 - 1 \geq 1.$$

Now we prove the second result. Let c_1 and c_2 satisfy $c_1 < c_2 \leq 2c_1 - 1$. Denote $\Delta = c_2 - c_1$. Let C be a cycle of length 4Δ and let u_1 and v_1 be opposite vertices on C. Attach to u_1 (resp. v_1) a path of length $c_1 - 1$ $u_1u_2\cdots u_{c_1}$ (resp. $v_1v_2\cdots v_{c_1}$). Finally, attach to both u_{c_1-1} and v_{c_1-1} exactly $q \geq 0$ pendant vertices, and denote the resulting graph by G, see Figure 3.

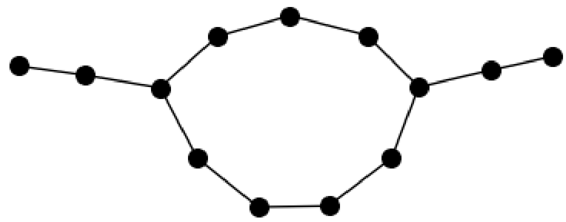

Figure 3. The unicyclic graph on 13 vertices and with odd cycle that has Wiener complexity smaller than eccentric complexity.

Obviously, $\mathrm{diam}(G) = 2(c_1 - 1) + 2\Delta = 2c_2 - 2$. Since $c_2 \leq 2c_1 - 1$, we have $c_2 - c_1 \leq c_1 - 1$, and so $2\Delta \leq \Delta + (c_1 - 1)$. Thus, $\mathrm{rad}(G) = \max\{\mathrm{rad}(C), \lceil \mathrm{diam}(G)/2 \rceil\} = \Delta + (c_1 - 1)$, which means that $C_{\mathrm{ec}}(G) = \mathrm{diam}(G) - \mathrm{rad}(G) + 1 = c_2$.

On the other hand, denote by w^T the transmission of u_1 in the tree attached to C and denote by w^C the transmission of u_1 in C. Then $w_G(u_1) = w^T + w^C + 2\Delta(c_1 - 1 + q) + w^T$, and similarly for every vertex v of C we have $w_G(v) = 2w^T + w^C + 2\Delta(c_1 - 1 + q)$ as well. By Lemmas 2 and 3 it holds $w_G(u_1) < w_G(u_2) < \cdots < w_G(u_{c_1})$ and by symmetry $w_G(u_i) = w_G(v_i)$, $1 \leq i \leq c_1$. Thus $C_W(G) = c_1$, and so G satisfies the assumptions of the theorem. □

Theorem 3 has the following consequence.

Corollary 2. *For every integer $c > 0$ there are infinitely many unicyclic graphs G such that*

$$C_{\mathrm{ec}}(G) - C_W(G) = c.$$

We remark that the attachment vertices u_1 and u_2 do not need to be opposite on C if c_1 is big enough (compared to $\Delta = c_2 - c_1$). We can use also even cycles of length $\not\equiv 0 \pmod 4$ and odd cycles, but again c_1 must be big enough. Though for small order graphs, one with even cycle are quite abundant, the smallest unicyclic graph G with a cycle of odd length satisfying $C_W(G) < C_{\mathrm{ec}}(G)$ has 13 vertices, and its cycle has length 9.

4. Graphs with Diameter 3

In this section we solve Problem 1. Observe that if $\mathrm{diam}(G) = 3$ and $C_{\mathrm{ec}}(G) > C_W(G)$ then $\mathrm{rad}(G) = 2$, $C_{\mathrm{ec}}(G) = 2$ and $C_W(G) = 1$. Hence, there is no unicyclic graph G satisfying the requirements of Problem 1, by Theorem 3.

Let G be a graph with diameter 3. For every vertex $u \in V(G)$, by $d_G^i(u)$ we denote the number of vertices of G which are at distance i from u. Denote $\sigma_G(u) = d_G^1(u) - d_G^3(u)$. We have the following statement.

Lemma 4. *Let G be a graph with diameter 3. Then $C_W(G) = 1$ if and only if all vertices of G have the same value of σ.*

Proof. Let $u \in V(G)$. Then $w_G(u) = d_G^1(u) + 2d_G^2(u) + 3d_G^3(u)$. Since $d_G^1(u) + d_G^2(u) + d_G^3(u) = n - 1$, where $n = |V(G)|$, we have $d_G^2(u) = n - 1 - d_G^1(u) - d_G^3(u)$, and consequently $w_G(u) = 2n - 2 - d_G^1(u) + d_G^3(u)$. Hence, if $v \in V(G)$ with $v \neq u$, then $w_G(v) = w_G(u)$ is equivalent with $\sigma_G(u) = \sigma_G(v)$. □

By Lemma 4, in graphs G of diameter 3 with $C_{\mathrm{ec}}(G) = 2$ and $C_W(G) = 1$, the vertices of eccentricity 3 must have degree greater than is the degree of vertices of eccentricity 2. This looks surprising, nevertheless, there exist such graphs.

Let G be a graph on $2r$ vertices and let $S \subseteq V(G)$ such that $|S| = r$. By $A(G, S)$ we denote the graph obtained from G by adding two vertices, v and v', where v is connected to all vertices of S and v' is connected to all vertices of $V(G) \setminus S$. We have:

Proposition 1. *Let G be a k-regular graph of diameter 2 on $2(k+2)$ vertices. Moreover, let $S \subseteq V(G)$, $|S| = k+2$, such that every vertex of S has a neighbour in $V(G) \setminus S$ and every vertex of $V(G) \setminus S$ has a neighbour in S. Then $\mathrm{diam}(A(G,S)) = 3$, $C_{\mathrm{ec}}(A(G,S)) = 2$ and $C_W(A(G,S)) = 1$.*

Proof. First observe that $N_G(S) = V(G) = N_G(V(G) \setminus S)$. Since every vertex of S has a neighbour in $V(G) \setminus S$ and every vertex of $V(G) \setminus S$ has a neighbour in S, we have $e_{A(G,S)}(u) = 2$ for every $u \in V(G)$. Since $\mathrm{dist}_{A(G,S)}(v,v') = 3$, we have $\mathrm{diam}(A(G,S)) = 3$ and $C_{\mathrm{ec}}(A(G,S)) = 2$.

If $u \in V(G)$ then $\deg_{A(G,S)}(u) = \sigma_{A(G,S)}(u) = k+1$. On the other hand $\deg_{A(G,S)}(v) = \deg_{A(G,S)}(v') = k+2$. Moreover, since in G holds $N(S) = V(G) = N(V(G) \setminus S)$, we have $d^3_{A(G,S)}(v) = d^3_{A(G,S)}(v') = 1$. Thus $\sigma_{A(G,S)}(v) = \sigma_{A(G,S)}(v') = k+1$ and $C_W(A(G,S)) = 1$, by Lemma 4. □

Since there is no 2-regular graph on 8 vertices with diameter 2, the smallest graph G satisfying assumptions of Proposition 1 is the Petersen graph in which S is the set of vertices of one of its 5-cycles. If G is the Petersen graph and S is the set of vertices of one of its 5-cycles, then $A(G,S)$ has 12 vertices.

However, there are also other graphs satisfying the assumptions of Proposition 1.

Lemma 5. *Let $k \geq 6$ be an even number, and let $D = (\{1,4,7,\dots\} \cap \{1,2,\dots,k+1\}) \cup \{k+1,k,k-1,\dots,i\}$ with $|D| = k/2$. Let G be the Cayley graph with $V(G) = \mathbb{Z}_{2k+4}$ and $E(G) = \{ij; i-j \in D \cup -D\}$. Finally, let $S = \{0,2,\dots,2k+2\}$. Then G and S satisfy the assumptions of Proposition 1.*

Proof. Obviously, G is k-regular. Since $1 \in D$ and $S = \{0,2,\dots,2k+2\}$, S satisfies the assumptions of Proposition 1. Hence, it remains to prove that $\mathrm{diam}(G) = 2$.

We only show that $e_G(0) = 2$, and since G is vertex-transitive, we conclude that $\mathrm{diam}(G) = 2$. So it is enough to show that if $1 \leq r \leq k+2$, then either $0r \in E(G)$, or $0(r-1) \in E(G)$ or $0(r+1) \in E(G)$, because $\alpha : u \to 2k+4-u$ is an isomorphism of G. Let $t = k/2$ and let D' be a set of t numbers starting with 1 and continuing with difference 3. Then $D' = \{1,4,7,\dots,3t-2\}$. Since $k \geq 6$, we have $t \geq 3$ and $3t-2 \geq k+1$. Hence, it follows that $k+1 \in D$ which means that $\mathrm{dist}_G(0,k+2) = 2$. And since $D \supseteq D' \cap \{1,2,\dots,k+1\}$, we have $0r \in E(G)$ or $0(r-1) \in E(G)$ or $0(r+1) \in E(G)$ for every r with $1 \leq r \leq k+1$. Thus, $e_G(0) = 2$. □

Let G be the Petersen graph or a graph from Lemma 5 and let S be as described. Then $A(G,S)$ has diameter 3 and $C_{\mathrm{ec}}(A(G,S)) > C_W(A(G,S))$. However, all these graphs have exactly 2 vertices with eccentricity 3. Next statement shows that there are required graphs with $2t$ vertices with eccentricity 3 for arbitrary $t \geq 1$.

Let H be a graph. By $B_t(H)$ we denote a graph on $t|V(H)|$ vertices obtained from H by replacing every vertex by K_t. Moreover, vertices from different copies of K_t are adjacent in $B_t(H)$ if and only if these copies of K_t are obtained from adjacent vertices in H.

Theorem 4. *Let G be a graph and $S \subseteq V(G)$ such that G and S satisfy the assumptions of Proposition 1. Moreover, let $t \geq 1$. Then $\mathrm{diam}(B_t(A(G,S))) = 3$, $C_{\mathrm{ec}}(B_t(A(G,S))) = 2$ and $C_W(B_t(A(G,S))) = 1$.*

Proof. For $t = 1$ the statement reduces to Proposition 1. Therefore, in the following we assume $t \geq 2$. Denote $H = B_t(A(G,S))$. Let u be a vertex of H obtained from a vertex of G. Then $e_H(u) = 2$ and $\deg_H(u) = d^1_H(u) = (t-1) + kt + t$, and so $\sigma_H(u) = kt + 2t - 1$.

Now let u' be a vertex of H obtained from v or v' (i.e., from the vertices of $A(G,S)$ which are not in G). Then $e_H(u') = 3$, $\deg_H(u') = d^1_H(u') = (t-1) + (k+2)t$ and $d^3_H(u') = t$. Hence $\sigma(u') = kt + 2t - 1$ as well. Thus, $\mathrm{diam}(H) = 3$, $C_{\mathrm{ec}}(H) = 2$ and by Lemma 4 we have $C_W(H) = 1$. □

In [8] the authors checked all graphs on at most 10 vertices and none of them had $C_W < C_{ec}$ and diameter 3. We checked the same for graphs on 11 vertices. Thus, the smallest graph with the above properties has 12 vertices and it is obtained using Proposition 1.

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