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Article

New Specific and General Linearization Formulas of Some Classes of Jacobi Polynomials

Waleed Mohamed Abd-Elhameed [1,2,*] and Afnan Ali [2]

[1] Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt
[2] Department of Mathematics, College of Science, University of Jeddah, Jeddah 21589, Saudi Arabia; Ms.Afnan.Ali@outlook.com
* Correspondence: waleed@sci.cu.edu.eg

Abstract: The main purpose of the current article is to develop new specific and general linearization formulas of some classes of Jacobi polynomials. The basic idea behind the derivation of these formulas is based on reducing the linearization coefficients which are represented in terms of the Kampé de Fériet function for some particular choices of the involved parameters. In some cases, the required reduction is performed with the aid of some standard reduction formulas for certain hypergeometric functions of unit argument, while, in other cases, the reduction cannot be done via standard formulas, so we resort to certain symbolic algebraic computation, and specifically the algorithms of Zeilberger, Petkovsek, and van Hoeij. Some new linearization formulas of ultraspherical polynomials and third-and fourth-kinds Chebyshev polynomials are established.

Keywords: Jacobi polynomials; hypergeometric functions; linearization coefficients; recurrence relations; symbolic algorithms

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1. Introduction

The connection and linearization problems of polynomials in general and of orthogonal polynomials, in particular, are crucial in mathematical analysis and its applications. For example, the linearization coefficients are useful in the computation of physical and chemical properties of quantum-mechanical systems [1,2]. In addition, they serve in the numerical solution of some differential equations. As an example, a certain linearization formula of Chebyshev polynomials of the third- kind is employed to find a spectral solution for a nonlinear Riccati differential equation in Abd-Elhameed [3]. A variety of papers in the literature were interested in solving linearization and connection problems of different orthogonal polynomials via proposing several methods. For some articles in this direction, one can be referred to [4–10]. For the discussion of the linearization problem, and if we let $\{A_i(x)\}_{i\geq 0}$, $\{B_j(x)\}_{j\geq 0}$, and $\{C_k(x)\}_{k\geq 0}$ be three families of polynomials, then to solve the general linearization problem

$$A_i(x)\, B_j(x) = \sum_{m=0}^{i+j} L_{m,i,j}\, C_m(x),$$

it is required to find the linearization coefficients $L_{m,i,j}$.

Hypergeometric functions are very essential in the applications of mathematical analysis. Almost all elementary special functions can be represented by hypergeometric functions of certain arguments. There are theoretical studies about these kinds of functions, see, for example, [11,12]. Moreover, when solving the connection, duplication, and linearization problems, we have to find the connection, duplication, and linearization coefficients. These coefficients are often expressed in terms of hypergeometric functions of certain arguments; see, for example, [6,13].

It is well-known that the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, and their particular polynomials play distinguished parts in mathematical analysis theoretically and practically (see, for example, [14–17]). The class of Jacobi polynomials includes six celebrated classes of orthogonal polynomials. Four of them are symmetric, namely, ultraspherical, Legendre, and Chebyshev polynomials of the first- and second- kinds, while the other two classes, namely, Chebyshev polynomials of the third- and fourth- kinds are non-symmetric. In fact, the ultraspherical polynomials correspond to the case $\alpha = \beta$ and each is replaced by $(\alpha - \frac{1}{2})$; first- and second-kinds of Chebyshev polynomials correspond, respectively, to the cases $\left(\alpha = \beta = -\frac{1}{2}\right)$ and $\left(\alpha = \beta = \frac{1}{2}\right)$. The third- and fourth-kinds of Chebyshev polynomials correspond, respectively, to the cases $\left(\alpha = -\beta = -\frac{1}{2}\right)$ and $\left(\alpha = -\beta = \frac{1}{2}\right)$. For a survey on different kinds of Chebyshev polynomials, one can consult [18].

Due to the importance of the linearization formulas of Jacobi polynomials and their special polynomials, some contributions were devoted to establishing these formulas. For example, the authors in [19] established linearization formulas of the two non-symmetric classes, namely, Chebyshev polynomials of the third-and fourth-kinds, while the author in [20] established linearization formulas of other classes of non-symmetric Jacobi polynomials. The authors in [21] developed a linearization formula for the product $P_n^{(\alpha,\beta)}(x)\, P_n^{(\beta,\alpha)}(x)$ in terms of ultraspherical polynomials. The linearization coefficients are given in terms of a certain hypergeometric function of the type ${}_5F_4(1)$, which can be summed for particular choices of the involved parameters by making use of some well-known formulas in the literature or via the application of some suitable symbolic computation.

The current paper aims to establish new expressions for the linearization coefficients $A_{m,i,j}$ in the problem

$$\tilde{P}_i^{(\alpha,\beta)}(x)\, \tilde{P}_j^{(\lambda,\mu)}(x) = \sum_{m=0}^{i+j} A_{m,i,j}\, P_{i+j-m}^{(\alpha+\lambda,\beta+\mu)}(x), \tag{1}$$

for certain choices of the involved parameters in (1), where $\tilde{P}_i^{(\alpha,\beta)}(x)$ is the modified Jacobi polynomial of degree i which was defined in [10]. Chaggara and Koepf in [6] expressed the coefficients $A_{m,i,j}$ in terms of a product of two terminating hypergeometric functions of unit argument. We will show that these coefficients can be reduced in some cases to give simple forms via utilizing some standard formulas in the literature or with the aid of suitable symbolic computation, and, in particular, Zerilberger's, Petkovsek's algorithms [22], and van Hoeij's algorithm [23].

The rest of the paper is as follows: Section 2 displays some preliminaries concerned with the hypergeometric and the generalized hypergeometric functions and fundamental properties of Jacobi polynomials. In addition, we present a formula for the linearization coefficients of Jacobi polynomials. Section 3 is interested in establishing a linearization formula of two different ultraspherical polynomials. Linearization formulas of some other Jacobi polynomials are developed in Section 4. Finally, Section 5 presents some concluding remarks.

2. Some Preliminaries and Fundamental Properties of Jacobi Polynomials

This section is confined to presenting some essentials of the hypergeometric and the generalized hypergeometric functions and useful reduction formulas of some of these functions. In addition, some fundamental properties of the classical Jacobi polynomials are given. Furthermore, we give an account of the linearization coefficients of Jacobi polynomials.

2.1. Hypergeometric Functions, Generalized Hypergeometric Functions, and Some Useful Reduction Formulas

The ${}_2F_1(x)$ series is defined by Gauss as [24]

$${}_2F_1\left(\begin{matrix} b, c \\ d \end{matrix} \,\middle|\, x\right) = 1 + \frac{b\,c}{d}\,x + \frac{b(b+1)\,c(c+1)}{d\,(d+1)}\,\frac{x^2}{2!} + \cdots = \sum_{m=0}^{\infty} \frac{(b)_m\,(c)_m}{(d)_m}\,\frac{x^m}{m!}, \tag{2}$$

where d is neither zero nor a negative integer, and the symbol $(z)_m$ represents the well-known Pochhammer symbol that is $(z)_m = \dfrac{\Gamma(z+m)}{\Gamma(z)}$.

The series in (2) reduces to the elementary geometric series $\sum_{m=0}^{\infty} x^m$ for $b = 1$ and $c = d$.

The following two notes regarding the ${}_2F_1(x)$ in (2) are important:

- If none of b, c, or d is zero or a negative integer, then the series in (2) converges for all $|x| < 1$ and diverges for all $|x| > 1$.
- If either b or c (or both) are zero or a negative integer, then the series in (2) is finite and therefore converges for all x.

In general, we say that $\sum y_m(x)$ is a hypergeometric-type series if the ratio $\dfrac{y_{m+1}(x)}{y_m(x)}$ is a rational function of m. The generalized hypergeometric function is defined as ([25])

$$ {}_rF_s\left(\begin{matrix} c_1, c_2, \ldots, c_r \\ d_1, d_2, \ldots, d_s \end{matrix} \,\middle|\, x\right) = \sum_{m=0}^{\infty} \frac{(c_1)_m\,(c_2)_m\,\ldots(c_r)_m}{(d_1)_m\,(d_2)_m\,\ldots(d_s)_m}\,\frac{x^m}{m!}, \tag{3} $$

where r and s are non-negative integers, and no d_i, $1 \le i \le s$ is zero or a negative integer. It is to be noted here that, if one of the numerator parameters is a negative integer, then the series in (3) turns into a finite sum. For example, if $c_1 = -r$, for some positive integer r, then

$$ {}_rF_s\left(\begin{matrix} -r, c_2, \ldots, c_r \\ d_1, d_2, \ldots, d_s \end{matrix} \,\middle|\, x\right) = \sum_{m=0}^{r} \frac{(-r)_m\,(c_2)_m\,\ldots(c_r)_m}{(d_1)_m\,(d_2)_m\,\ldots(d_s)_m}\,\frac{x^m}{m!}. \tag{4} $$

There are several useful identities and transformations between hypergeometric functions and generalized hypergeometric functions. In this regard, one can consult [25,26].

The following two reduction formulas are useful in the sequel.

Theorem 1. *Watson's identity (see, [27]) For every non-negative integer j, one has*

$$ \begin{aligned} &{}_3F_2\left(\begin{matrix} -j, j+2c+2d-1, c \\ 2c, c+d \end{matrix} \,\middle|\, 1\right) \\ &= \begin{cases} \dfrac{j!\,\Gamma(c+\frac{j}{2})\,\Gamma(d+\frac{j}{2})\,\Gamma(2c)\,\Gamma(c+d)}{\left(\frac{j}{2}\right)!\,\Gamma(c+d+\frac{j}{2})\,\Gamma(2c+j)\,\Gamma(c)\,\Gamma(d)}, & j \text{ even}, \\ 0, & j \text{ odd}. \end{cases} \end{aligned} \tag{5} $$

Theorem 2. *Chu–Vandermonde identity ([24]) For every non-negative integer j, one has*

$$ {}_2F_1\left(\begin{matrix} -j, c \\ d \end{matrix} \,\middle|\, 1\right) = \frac{(d-c)_j}{(d)_j}. \tag{6} $$

2.2. An Overview on Jacobi Polynomials

The sequence of Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$, $x \in [-1,1]$, $k \ge 0$, and $\alpha > -1$, $\beta > -1$, (see, for example, [26,28]), can be generated with the aid of Rodrigues' formula:

$$ P_k^{(\alpha,\beta)}(x) = \frac{(-1)^k}{2^k\,k!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^k}{dx^k}\left[(1-x)^{\alpha+k}(1+x)^{\beta+k}\right]. $$

The hypergeometric representation of Jacobi polynomials is given by

$$ P_k^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_k}{k!}\,{}_2F_1\left(\begin{matrix} -k, k+\alpha+\beta+1 \\ \beta+1 \end{matrix} \,\middle|\, \frac{1-x}{2}\right). $$

Rahman in [10] defined the modified Jacobi polynomials as

$$\tilde{P}_k^{(\alpha,\beta)}(x) = {}_2F_1\left(\begin{array}{c} -k, k+\alpha+\beta+1 \\ \beta+1 \end{array} \middle| \frac{1-x}{2}\right). \tag{7}$$

The modified Jacobi polynomials are characterized by the property:

$$\tilde{P}_k^{(\alpha,\beta)}(1) = 1, \quad k = 0,1,2,\dots,$$

and, therefore, obtaining the six particular polynomials of Jacobi polynomial is easier. In fact, they can be given by the following formulas:

$$\begin{aligned} T_k(x) &= \tilde{P}_k^{(-\frac{1}{2},-\frac{1}{2})}(x), & U_k(x) &= (k+1)\,\tilde{P}_k^{(\frac{1}{2},\frac{1}{2})}(x), \\ V_k(x) &= \tilde{P}_k^{(-\frac{1}{2},\frac{1}{2})}(x), & W_k(x) &= (2k+1)\,\tilde{P}_k^{(\frac{1}{2},-\frac{1}{2})}(x), \\ C_k^{(\alpha)}(x) &= \tilde{P}_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x), & P_k(x) &= \tilde{P}_k^{(0,0)}(x), \end{aligned}$$

where $T_k(x)$, $U_k(x)$, $V_k(x)$ and $W_k(x)$ denote, respectively, the first-, second-, third-, and fourth- kinds of Chebyshev polynomials, while $C_k^{(\alpha)}(x)$ and $P_k(x)$ denote, respectively, the ultraspherical and Legendre polynomials.

Note that all four kinds of Chebyshev polynomials have trigonometric representations. They are given as follows:

$$\begin{aligned} T_k(x) &= \cos(k\,\theta), & U_k(x) &= \frac{\sin((k+1)\,\theta)}{\sin\theta}, \\ V_k(x) &= \frac{\cos\left(\left(k+\frac{1}{2}\right)\theta\right)}{\cos\left(\frac{\theta}{2}\right)}, & W_k(x) &= \frac{\sin\left(\left(k+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)}, \end{aligned}$$

with $\theta = \cos^{-1}(x)$.

In addition, note that the polynomials $V_k(x)$ and $W_k(x)$ satisfy the following relation:

$$W_k(x) = (-1)^k\,V_k(-x).$$

2.3. Linearization Coefficients of Jacobi Polynomials

There are many approaches followed to obtain new formulas of linearization coefficients of Jacobi polynomials and their special classes. The authors in [6] discussed the linearization problem:

$$P_i^{(\alpha,\beta)}(x)\,P_j^{(\eta,\theta)}(x) = \sum_{k=0}^{i+j} L_{k,i,j}\,P_{i+j-k}^{(\rho,\sigma)}(x), \tag{8}$$

and they established a formula for the linearization coefficients $L_{k,i,j}$, in which $L_{k,i,j}$ are expressed in terms of the Kampé de Fériet function $F^{2:2}_{2:1}$, which is represented as a double hypergeometric function. Regarding the linearization coefficients $L_{k,i,j}$, it is not easy to express them in reduced forms even for particular choices of the parameters of Jacobi polynomials in (8). The authors in [6] used the Gasper's reduction formula in [29] for developing a new formula for the coefficients $L_{k,i,j}$ corresponding to the particular choices: $\rho = \alpha+\eta$, $\sigma = \beta+\theta$, and they stated and proved in [6] an important theorem in this regard. The following theorem displays the corresponding result for the modified Jacobi polynomials, which are defined in (7).

Theorem 3. *For all non-negative integers i and j, the following linearization formula holds:*

$$\tilde{P}_i^{(\alpha,\beta)}(x)\,\tilde{P}_j^{(\eta,\theta)}(x) = \sum_{k=0}^{i+j} M_{k,i,j}\,\tilde{P}_{i+j-k}^{(\alpha+\eta,\beta+\theta)}(x), \tag{9}$$

where

$$\begin{aligned} M_{k,i,j} =& \frac{(1+\alpha+\beta)_{2i}\,(1+\alpha+\eta)_{i+j}\,(1+\eta+\theta)_{2j}\,(1+\alpha+\beta+\eta+\theta)_{i+j-k}}{(1+\alpha+\beta)_i\,(1+\alpha+\eta)_{i+j-k}\,(1+\eta+\theta)_j\,(1+\alpha+\beta+\eta+\theta)_{1+2(i+j)-k}} \\ &\times \frac{(i+j)!\,\Gamma(1+\alpha)\,\Gamma(1+\eta)\,\Gamma(1+i+j-k+\alpha+\eta)}{(i+j-k)!\,k!\,\Gamma(1+i+\alpha)\,\Gamma(1+j+\eta)\,\Gamma(1+\alpha+\eta)} \\ &\times \frac{(2i-k+\alpha+\beta+1)_k\,(\alpha+\beta+\eta+\theta+2i+2j-2k+1)}{(2j-k+\eta+\theta+1)_k} \\ &\times {}_3F_2\left(\left. \begin{array}{c} -k,-i,-\theta-\beta-2i-2j+k-\alpha-\eta-1 \\ -i-j,-\beta-2i-\alpha \end{array} \right| 1 \right) \\ &\times {}_3F_2\left(\left. \begin{array}{c} -k,-i-\alpha,-\theta-\beta-2i-2j+k-\alpha-\eta-1 \\ -\beta-2i-\alpha,-i-j-\alpha-\eta \end{array} \right| 1 \right). \end{aligned} \tag{10}$$

As mentioned in [6], we have to note that the two ${}_3F_2(1)$ which appear in (10) have no reduction formulas in general.

In the upcoming sections, and starting from Equation (9), we are going to establish some new linearization formulas of Jacobi polynomials for particular choices of their parameters. The basic idea behind the derivation of these formulas is based on reducing one or two of the ${}_3F_2(1)$ which appear in (10). We mention here that the desired reductions can be done through one of the following two approaches:

(a) Using some standard formulas such as Watson's and Chu–Vandermonde identities.

(b) Making use of some celebrated symbolic algebra. To be more precise, we make use of Zerilberger's algorithm [22] to obtain the recurrence relation satisfied by the hypergeometric function, and, after that, and thanks to any suitable symbolic algorithm such as Petkovsek's algorithm (Koepf [22]) or the improved version of van Hoeij ([23]), this recurrence relation can be exactly solved. This yields some linearization formulas in simple forms free of any hypergeometric functions.

3. Some Linearization Formulas of Ultraspherical Polynomials

Our goal in this section is to derive new linearization formulas of ultraspherical polynomials of different parameters. Some new specific and general linearization formulas are also deduced.

Theorem 4. *For all non-negative integers i and j, the following linearization formula is valid:*

$$\begin{aligned} C_i^{(\alpha)}(x)\,C_j^{(\eta)}(x) =& \frac{(i+j)!\,\Gamma(2i+2\alpha)\,\Gamma\left(\frac{1}{2}+\alpha\right)\,\Gamma\left(\frac{1}{2}+\eta\right)}{4^{i+\alpha}\,\Gamma(i+2\alpha)\,\Gamma(\alpha+\eta)\,\Gamma(j+2\eta)} \\ &\times \sum_{k=0}^{\left\lfloor \frac{i+j}{2} \right\rfloor} \frac{(-1)^k\,(-1+2i+2j-4k+2\alpha+2\eta)\,\Gamma(j-k+\eta)\,\Gamma(-1+i+j-2k+2\alpha+2\eta)}{k!\,(i+j-2k)!\,\Gamma\left(\frac{1}{2}+i-k+\alpha\right)\,\Gamma\left(\frac{1}{2}+i+j-k+\alpha+\eta\right)} \\ &\times {}_3F_2\left(\left. \begin{array}{c} -2k,-i,-2\alpha-2\eta-2i-2j+2k+1 \\ -i-j,-2\alpha-2i+1 \end{array} \right| 1 \right) C_{i+j-2k}^{(\alpha+\eta-\frac{1}{2})}(x). \end{aligned} \tag{11}$$

Proof. The substitution by $\beta = \alpha, \theta = \eta$ into the linearization coefficients (10) enables one to write the linearization coefficients $M_{k,i,j}$ in this case as

$$
\begin{aligned}
M_{k,i,j} =& \frac{4^{i+\alpha}\,(1+2i+2j-2k+2\alpha+2\eta)\,(i+j)!\,\Gamma(1+2j-k+2\eta)\,\Gamma(1+i+j-k+2\alpha+2\eta)}{\sqrt{\pi}\,k!\,(i+j-k)!\,\Gamma(1+i+2\alpha)\,\Gamma(1+2i-k+2\alpha)} \\
&\times \frac{\Gamma(1+\alpha)\,\Gamma\left(\frac{1}{2}+i+\alpha\right)\,\Gamma(1+2i+2\alpha)\,\Gamma(1+\eta)\,\Gamma(1+i+j+\alpha+\eta)}{\Gamma(1+j+\eta)\,\Gamma(1+\alpha+\eta)\,\Gamma(1+j+2\eta)\,\Gamma(2i+2j-k+2(1+\alpha+\eta))} \\
&\times {}_3F_2\left(\left.\begin{matrix} -k, -\alpha-i, -2\alpha-2\eta-2i-2j+k-1 \\ -2\alpha-2i, -\alpha-\eta-i-j \end{matrix}\right| 1\right) \\
&\times {}_3F_2\left(\left.\begin{matrix} -k, -i, -2\alpha-2\eta-2i-2j+k-1 \\ -i-j, -2\alpha-2i \end{matrix}\right| 1\right).
\end{aligned}
\tag{12}
$$

The first ${}_3F_2(1)$ which appears in (12) can be summed with the aid of Watson's identity (5). More definitely, setting $j = k,\ c = -i-\alpha$ and $d = -k-\eta$, in (5) immediately yields

$$
\begin{aligned}
&{}_3F_2\left(\left.\begin{matrix} -k, -\alpha-i, -2\alpha-2\eta-2i-2j+k-1 \\ -2\alpha-2i, -\alpha-\eta-i-j \end{matrix}\right| 1\right) = \\
&\begin{cases} \dfrac{(-1)^{\frac{k}{2}}\Gamma\left(\frac{k+1}{2}\right)\left(j-\frac{k}{2}+\eta+1\right)_{\frac{k}{2}}}{\sqrt{\pi}\left(i-\frac{k}{2}+\alpha+\frac{1}{2}\right)_{\frac{k}{2}}\left(i+j-\frac{k}{2}+\alpha+\eta+1\right)_{\frac{k}{2}}}, & k \text{ even}, \\ 0, & k \text{ odd}, \end{cases}
\end{aligned}
$$

and, therefore, the following linearization formula is obtained:

$$
\begin{aligned}
\tilde{P}_i^{(\alpha,\alpha)}(x)\,\tilde{P}_j^{(\eta,\eta)}(x) =& \frac{2^{-1-2i-2\alpha}\,(i+j)!\,\Gamma(\alpha+1)\,\Gamma(1+\eta)\,\Gamma(1+2i+2\alpha)}{\Gamma(1+i+2\alpha)\,\Gamma(1+\alpha+\eta)\,\Gamma(1+j+2\eta)} \\
&\times \sum_{k=0}^{\left\lfloor\frac{i+j}{2}\right\rfloor} \frac{(-1)^k\,(1+2i+2j-4k+2\alpha+2\eta)\,\Gamma\left(\frac{1}{2}+j-k+\eta\right)\,\Gamma(1+i+j-2k+2\alpha+2\eta)}{k!\,(i+j-2k)!\,\Gamma(1+i-k+\alpha)\,\Gamma\left(\frac{3}{2}+i+j-k+\alpha+\eta\right)} \\
&\times {}_3F_2\left(\left.\begin{matrix} -2k, -i, -1-2i-2j+2k-2\alpha-2\eta \\ -i-j, -2i-2\alpha \end{matrix}\right| 1\right) \tilde{P}_{i+j-2k}^{(\alpha+\eta,\alpha+\eta)}(x).
\end{aligned}
\tag{13}
$$

Noting the identity: $C_m^{(\alpha)}(x) = \tilde{P}_m^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x)$, it is easy to convert formula (13) into the desired linearization formula (11). $\square$

The result of Theorem 4 can be reduced to give some new linearization formulas free of any hypergeometric functions for specific choices of α and η. In the following, we state and prove some corollaries in this respect.

Corollary 1. *For all non-negative integers i and j, the following linearization formula is valid*

$$
\begin{aligned}
C_i^{(\frac{3}{2})}(x)\,P_j(x) =& \frac{1}{2\,(1+i)\,(2+i)\,\pi} \\
&\times \sum_{k=0}^{min(i+1,j)} \frac{(3+2i+2j-4k)(i+j-k+1)!\,\Gamma\left(\frac{1}{2}+k\right)\,\Gamma\left(\frac{3}{2}+i-k\right)\,\Gamma\left(\frac{1}{2}+j-k\right)}{k!\,(i-k+1)!\,(j-k)!\,\Gamma\left(\frac{5}{2}+i+j-k\right)} \\
&\times \left(2+j+i(3+i+j)-3k-2(i+j)k+2k^2\right) C_{i+j-2k}^{(\frac{3}{2})}(x).
\end{aligned}
\tag{14}
$$

Proof. If we substitute by $\alpha = \frac{3}{2}$, $\eta = \frac{1}{2}$ into (11), then we get

$$\begin{aligned} C_i^{(\frac{3}{2})}(x)\, P_j(x) = & \frac{(i+j)!\,(2i+2)!}{2^{3+2i}\, j!\,(i+2)!} \sum_{k=0}^{\lfloor \frac{i+j}{2} \rfloor} \frac{(-1)^k\,(3+2i+2j-4k)\,(i+j-2k+2)!\,\Gamma\left(\frac{1}{2}+j-k\right)}{(i+j-2k)!\,k!\,(i-k+1)!\,\Gamma\left(\frac{5}{2}+i+j-k\right)} \\ & \times {}_3F_2\left(\begin{matrix} -2k, -i, -3-2i-2j+2k \\ -2-2i, -i-j \end{matrix} \middle| 1 \right) C_{i+j-2k}^{(\frac{3}{2})}(x). \end{aligned} \tag{15}$$

To the best of our knowledge, no standard reduction formula exists in the literature to sum the ${}_3F_2(1)$ that appears in (15), so we employ symbolic computation for such purpose. First, set

$$R_{k,i,j} = {}_3F_2\left(\begin{matrix} -k, -i, -3-2i-2j+k \\ -2-2i, -i-j \end{matrix} \middle| 1 \right), \tag{16}$$

and utilize Zeilberger's algorithm (see [22]). The Maple software can be employed through "sumrecursion command". More precisely, to obtain the recurrence relation satisfied by the finite sum in (16), we use the command

$$\text{sumrecursion}\left(\frac{(-k)_m\,(-i)_m\,(-3-2i-2j+k)_m}{(-2-2i)_m\,(-i-j)_m\, m!}, m, R[k] \right),$$

to show that $R_{k,i,j}$ satisfies the following recurrence of order two:

$$\begin{aligned} & (k-1)(2+2j-k)(i+j+4-k)(-k+2+i+j)R_{k-2,i,j} \\ & -\left(2\,ij-2\,ik+2\,j^2-2\,jk+k^2+2\,i+6\,j-5\,k+4\right)(5+2\,i+2\,j-2\,k)R_{k-1,i,j} \\ & +(4+2\,i+2\,j-k)(-k+3+2\,i)(i+j+3-k)(-k+i+j+1)R_{k,i,j} = 0, \end{aligned} \tag{17}$$

with the initial values

$$R_{0,i,j} = 1, \qquad R_{1,i,j} = \frac{j}{(i+1)(i+j)}. \tag{18}$$

The recurrence relation (17) can be exactly solved with the aid of any suitable symbolic algorithm. The celebrated symbolic algorithm by Petkovsek [22], or the improved version of van Hoeij ([23]) may be employed for this purpose. The exact solution of (17) with the initial values in (18) is given explicitly as follows:

$$\begin{aligned} & {}_3F_2\left(\begin{matrix} -k, -i, -3-2i-2j+k \\ -2-2i, -i-j \end{matrix} \middle| 1 \right) = \\ & \frac{j!}{\sqrt{\pi}\,(1+i)(1+i+j-k)\,(2+i+j-k)\,\Gamma\left(\frac{3}{2}+i\right)\,(i+j)!} \\ & \times \begin{cases} \dfrac{(-1)^{\frac{k}{2}}\,\left(2(1+i)(2+i+j)-(3+2i+2j)\,k+k^2\right)\Gamma\left(\frac{3}{2}+i-\frac{k}{2}\right)\left(1+i+j-\frac{k}{2}\right)!\,\Gamma\left(\frac{1+k}{2}\right)}{2\left(j-\frac{k}{2}\right)!}, & k \text{ even}, \\ \dfrac{2(-1)^{\frac{k+3}{2}}\,\Gamma\left(2+i-\frac{k}{2}\right)\,\left(i+j-\frac{k-3}{2}\right)!\,\Gamma\left(1+\frac{k}{2}\right)}{\left(j-\frac{k+1}{2}\right)!}, & k \text{ odd}. \end{cases} \end{aligned} \tag{19}$$

Now, it is clear that from relation (19) that

$$\begin{aligned} & {}_3F_2\left(\begin{matrix} -2k, -i, -3-2i-2j+2k \\ -2-2i, -i-j \end{matrix} \middle| 1 \right) \\ & = \frac{(-1)^k\left(2+j+i(3+i+j)-3k-2(i+j)k+2k^2\right)\Gamma\left(\frac{1}{2}+k\right)(1+j-k)_k}{(1+i)\sqrt{\pi}(1+i+j-2k)(2+i+j-2k)\left(\frac{3}{2}+i-k\right)_k(2+i+j-k)_{k-1}}. \end{aligned} \tag{20}$$

The above reduction together with relation (15) yield the desired linearization Formula (14). □

Remark 1. *We mention here that all the recurrence relations in what follows can be treated similarly.*

Corollary 2. *For all non-negative integers i and j with $j \geq i$, one obtains*

$$P_i(x)\,T_j(x) = \frac{1}{\pi}\sum_{k=0}^{i}\frac{\Gamma\left(\frac{1}{2}+i-k\right)\Gamma\left(\frac{1}{2}+k\right)}{k!(i-k)!}\,T_{i+j-2k}(x). \tag{21}$$

Proof. If we substitute by $\alpha = \frac{1}{2}$ and $\eta = 0$ into (11), then we get

$$\begin{aligned} P_i(x)\,T_j(x) = & \frac{(i+j)!\,\Gamma\left(\frac{1}{2}+i\right)}{\sqrt{\pi}\,(j-1)!}\sum_{k=0}^{i}\frac{(-1)^k\,(j-k-1)!}{(i-k)!\,k!\,(j-k+i)!} \\ & \times\, {}_3F_2\left(\begin{matrix}-2k, -i, -2i-2j+2k\\ -2i, -i-j\end{matrix}\,\middle|\,1\right) T_{i+j-2k}(x). \end{aligned} \tag{22}$$

Now, in order to reduce the ${}_3F_2(1)$ in (22), first set

$$G_{k,i,j} = {}_3F_2\left(\begin{matrix}-k, -i, -2i-2j+k\\ -2i, -i-j\end{matrix}\,\middle|\,1\right).$$

In virtue of Zeilberger's algorithm (Koepf [22]), it can be shown that the following recurrence relation is satisfied by $G_{k,i,j}$:

$$\begin{aligned} &(k-1)(2j-k+1)(-2k+2i+2j+1)\,G_{k-2,i,j} - 2\left(2i^2+2ij-2ik-2jk+k^2+3i+2j-2k+1\right) \\ &\times G_{k-1,i,j} + (2i+2j-k+1)(2i-k+1)(2i-2k+3+2j)\,G_{k,i,j} = 0, \quad G_{0,i,j} = 1,\, G_{1,i,j} = \frac{1}{2(i+j)}. \end{aligned} \tag{23}$$

The above recurrence relation can be exactly solved to give

$$G_{k,i,j} = \frac{(j-1)!}{\sqrt{\pi}\,(i+j)!\,\Gamma\left(\frac{1}{2}+i\right)}\begin{cases} \dfrac{(-1)^{\frac{k}{2}}\left(i+j-\frac{k}{2}\right)!\,\Gamma\left(i-\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\left(j-\frac{k}{2}-1\right)!}, & k \text{ even},\\[2ex] \dfrac{(-1)^{\frac{k-1}{2}}\Gamma\left(i-\frac{k}{2}+1\right)\Gamma\left(i+j-\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(1+\frac{k}{2}\right)}{\Gamma\left(j-\frac{k}{2}+\frac{1}{2}\right)}, & k \text{ odd}. \end{cases} \tag{24}$$

Now, it is clear from (24) that

$${}_3F_2\left(\begin{matrix}-2k, -i, -2i-2j+2k\\ -2i, -i-j\end{matrix}\,\middle|\,1\right) = \frac{(-1)^k\,(j-1)!\,(i+j-k)!\,\Gamma\left(\frac{1}{2}+i-k\right)\Gamma\left(\frac{1}{2}+k\right)}{\sqrt{\pi}\,(i+j)!\,(j-k-1)!\,\Gamma\left(\frac{1}{2}+i\right)},$$

and hence the following linearization formula can be obtained:

$$P_i(x)\,T_j(x) = \frac{1}{\pi}\sum_{k=0}^{i}\frac{\Gamma\left(\frac{1}{2}+i-k\right)\Gamma\left(\frac{1}{2}+k\right)}{k!\,(i-k)!}\,T_{i+j-2k}(x).$$

□

Corollary 3. *For every non-negative integer i, one has*

$$\left(C_i^{(\alpha)}(x)\right)^2 = \frac{i!\,\Gamma\left(\frac{1}{2}+\alpha\right)}{2\sqrt{\pi}\,\Gamma(\alpha)\Gamma(i+2\alpha)} \times \sum_{k=0}^{i} \frac{(-1+4i-4k+4\alpha)\,\Gamma\left(\frac{1}{2}+k\right)\,\Gamma(i-k+\alpha)\,\Gamma\left(-\frac{1}{2}+i-k+2\alpha\right)\,\Gamma(2i-k+2\alpha)}{(i-k)!\,k!\,\Gamma\left(\frac{1}{2}+i-k+\alpha\right)\,\Gamma\left(\frac{1}{2}+2i-k+2\alpha\right)}\, C_{2i-2k}^{(2\alpha-\frac{1}{2})}(x), \tag{25}$$

and, in particular, one has

$$\left(P_i(x)\right)^2 = \frac{1}{2\pi} \sum_{k=0}^{i} \frac{(4i-4+1)\,(2i-k)!\,\Gamma\left(k+\frac{1}{2}\right)\left(\Gamma\left(i-k+\frac{1}{2}\right)\right)^2}{k!\,((i-k)!)^2\,\Gamma(2i-k+\frac{3}{2})}\, P_{2i-2k}(x),$$

$$\left(U_i(x)\right)^2 = \frac{i+1}{4} \sum_{k=0}^{i} \frac{(4i-4k+3)\,(2i-k+1)!\,\Gamma\left(k+\frac{1}{2}\right)}{k!\,\Gamma(2i-k+\frac{5}{2})}\, C_{2i-2k}^{(\frac{3}{2})}(x).$$

Proof. Setting $j = i$ and $\eta = \alpha$ in (11) yields

$$\begin{aligned}\left(C_i^{(\alpha)}(x)\right)^2 =& \frac{(2i)!\,\Gamma(2(i+\alpha))\left(\Gamma\left(\frac{1}{2}+\alpha\right)\right)^2}{4^{i+\alpha}\,\Gamma(2\alpha)\,(\Gamma(i+2\alpha))^2} \\ &\times \sum_{k=0}^{i} \frac{(-1)^k\,(-1+4i-4k+4\alpha)\,\Gamma(i-k+\alpha)\Gamma(-1+2i-2k+4\alpha)}{k!\,(2i-2k)!\,\Gamma\left(\frac{1}{2}+i-k+\alpha\right)\Gamma\left(\frac{1}{2}+2i-k+2\alpha\right)} \\ &\times {}_3F_2\left(\left.\begin{matrix}-i,-2k,1-4i+2k-4\alpha\\ -2i,1-2i-2\alpha\end{matrix}\right|1\right)\, C_{2i-2k}^{(2\alpha-\frac{1}{2})}(x).\end{aligned}$$

Making use of Watson's identity (5), it can be shown that

$$\begin{aligned}&{}_3F_2\left(\left.\begin{matrix}-k,-i,1-4i+k-4\alpha\\ -2i,1-2i-2\alpha\end{matrix}\right|1\right) = \\ &\begin{cases}\dfrac{(-1)^{\frac{k}{2}}\Gamma\left(\frac{1}{2}+i-\frac{k}{2}\right)\Gamma\left(\frac{1+k}{2}\right)\Gamma(i+2\alpha)\Gamma\left(2i-\frac{k}{2}+2\alpha\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+i\right)\Gamma(2(i+\alpha))\Gamma\left(i-\frac{k}{2}+2\alpha\right)}, & k \text{ even},\\ 0, & k \text{ odd}.\end{cases}\end{aligned} \tag{26}$$

Based on the above reduction formula, the linearization formula (25) can be obtained. □

Remark 2. *Relation* (25) *is in agreement with that obtained in Ref.* [30]*, but here it is derived differently.*

Corollary 4. *For every non-negative integer i, one has*

$$C_i^{(\alpha)}(x)\,C_{i+1}^{(\alpha)}(x) = \frac{(1+2i+2\alpha)\,i!\,\Gamma\left(\frac{1}{2}+\alpha\right)}{4\sqrt{\pi}\,\Gamma(\alpha)\,\Gamma(1+i+2\alpha)} \times \sum_{k=0}^{i} \frac{(1+4i-4k+4\alpha)\,\Gamma\left(\frac{1}{2}+k\right)\,\Gamma(i-k+\alpha)\,\Gamma\left(\frac{1}{2}+i-k+2\alpha\right)\,\Gamma(1+2i-k+2\alpha)}{(i-k)!\,k!\,\Gamma(\frac{3}{2}+i-k+\alpha)\,\Gamma(\frac{3}{2}+2i-k+2\alpha)}\, C_{2i-2k+1}^{(2\alpha-\frac{1}{2})}(x), \tag{27}$$

and, in particular, one has

$$U_i(x)\,U_{i+1}(x) = \frac{1}{8}(3+2i) \sum_{k=0}^{i} \frac{(5+4i-4k)\,(2i-k+2)!\,\Gamma\left(\frac{1}{2}+k\right)}{k!\,\Gamma(\frac{7}{2}+2i-k)}\, C_{2i-2k+1}^{(\frac{3}{2})}(x).$$

Proof. Setting $j = i + 1$ and $\eta = \alpha$ in (11) yields

$$\begin{aligned} C_i^{(\alpha)}(x)\, C_{i+1}^{(\alpha)}(x) = & \frac{(2i+1)!\,\Gamma(2(i+\alpha))\left(\Gamma\left(\frac{1}{2}+\alpha\right)\right)^2}{4^{i+\alpha}\,\Gamma(2\alpha)\,\Gamma(i+2\alpha)\,\Gamma(1+i+2\alpha)} \\ & \times \sum_{k=0}^{i} \frac{(-1)^k\,(1+4i-4k+4\alpha)\,\Gamma(1+i-k+\alpha)\,\Gamma(2i-2k+4\alpha)}{k!\,(2i-2k+1)!\,\Gamma\left(\frac{1}{2}+i-k+\alpha\right)\Gamma\left(\frac{3}{2}+2i-k+2\alpha\right)} \\ & \times {}_3F_2\left(\begin{matrix} -2k, -i, -1-4i+2k-4\alpha \\ -1-2i, 1-2i-2\alpha \end{matrix} \middle| 1 \right) C_{2i-2k+1}^{\left(2\alpha-\frac{1}{2}\right)}(x). \end{aligned} \tag{28}$$

In order to reduce the ${}_3F_2(1)$ that appears in (28), we perform similar procedures to those followed in the proof of Corollary 2 to obtain

$$\begin{aligned} & {}_3F_2\left(\begin{matrix} -2k, -i, -1-4i+2k-4\alpha \\ -1-2i, 1-2i-2\alpha \end{matrix} \middle| 1 \right) \\ & = \frac{(-1)^k\,(1+2i+2\alpha)\,\Gamma\left(k+\frac{1}{2}\right)(i-k+2\alpha)_k}{2\sqrt{\pi}\,(1+2i-2k+2\alpha)\,(i-k+\alpha)\left(i-k+\frac{3}{2}\right)_k(1-k+2(i+\alpha))_{k-1}}, \end{aligned}$$

and, therefore, the linearization formula (27) can be obtained. □

Corollary 5. *For every non-negative integer i, one has*

$$T_i(x)\,U_i(x) = \frac{i+1}{2\,i+1} + \frac{1}{4}\sum_{k=0}^{i-1} \frac{(1+4i-4k)\,(2i-k)!\,\Gamma\left(\frac{1}{2}+k\right)}{k!\,\Gamma\left(\frac{3}{2}+2i-k\right)}\, P_{2i-2k}(x). \tag{29}$$

Proof. Setting $\alpha = 0$, $\eta = 1$ and $j = i$ in (11) gives

$$\begin{aligned} T_i(x)\,U_i(x) = 4^{i-1}\left(\Gamma\left(\tfrac{1}{2}+i\right)\right)^2 & \sum_{k=0}^{i} \frac{(-1)^k\,(1+4i-4k)\,(i-k)!}{k!\,\Gamma\left(\frac{1}{2}+i-k\right)\Gamma\left(\frac{3}{2}+2i-k\right)} \\ & \times {}_3F_2\left(\begin{matrix} -2k, -i, -1-4i+2k \\ 1-2i, -2i \end{matrix} \middle| 1 \right) P_{2i-2k}(x). \end{aligned} \tag{30}$$

Now, in order to obtain a reduction formula for the ${}_3F_2(1)$ that appears in (30), we note that we have the following two cases:

(a) For $k = i$, the ${}_3F_2(1)$ in (30) reduces to ${}_2F_1\left(\begin{matrix} -i, -2i-1 \\ 1-2i \end{matrix} \middle| 1 \right)$, which can be summed with the aid of the Chu–Vandermonde identity, which stated in Theorem 2 to give

$${}_2F_1\left(\begin{matrix} -i, -2i-1 \\ 1-2i \end{matrix} \middle| 1 \right) = \frac{(-1)^i\,(i+1)!\,(i-1)!}{(2i-1)!}. \tag{31}$$

(b) For $0 \le k \le i-1$, let $H_{k,i} = {}_3F_2\left(\begin{matrix} -k, -i, -1-4i+k \\ 1-2i, -2i \end{matrix} \middle| 1 \right)$, and make use of Zeilberger's algorithm to obtain the following recurrence relation, which is satisfied by $H_{k,i}$:

$$\begin{aligned} & (k-1)(3+2\,i-k)H_{k-2,i} + (-2\,k+4\,i+3)H_{k-1,i} + (2+4\,i-k)(-k+2\,i)H_{k,i} = 0, \\ & \quad H_{0,i} = 1,\ H_{1,i} = \frac{1}{1-2i}. \end{aligned} \tag{32}$$

The last recurrence relation can be exactly solved to give

$$
{}_3F_2\left(\begin{matrix} -k, -i, -1-4i+k \\ 1-2i, -2i \end{matrix}\middle| 1\right) = \frac{1}{2^{2i}\left(\Gamma\left(\frac{1}{2}+i\right)\right)^2} \times \begin{cases} \dfrac{(-1)^{\frac{k}{2}}\left(2i-\frac{k}{2}\right)!\,\Gamma\left(i-\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\left(i-\frac{k}{2}\right)!}, & k \text{ even}, \\ \dfrac{(-1)^{\frac{k+1}{2}}\left(2i-\frac{k-1}{2}\right)!\Gamma\left(i-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}+1\right)}{\left(i-\frac{k-1}{2}\right)!}, & k \text{ odd}. \end{cases} \tag{33}
$$

Merging the two reduction formulas (31) and (33) leads to the reduction formula

$$
{}_3F_2\left(\begin{matrix} -2k, -i, -1-4i+2k \\ 1-2i, -2i \end{matrix}\middle| 1\right) = \begin{cases} \dfrac{(-1)^k\,\Gamma\left(\frac{1}{2}+k\right)\,(1+i-k)_k}{\sqrt{\pi}\,\left(\frac{1}{2}+i-k\right)_k\,(1+2i-k)_k}, & 0 \le k \le i-1, \\ \dfrac{(-1)^i\,(i+1)!\,(i-1)!}{(2i-1)!}, & k = i, \end{cases}
$$

and, therefore, some calculations lead to the following linearization formula:

$$
T_i(x)\,U_i(x) = \frac{i+1}{2\,i+1} + \frac{1}{4}\sum_{k=0}^{i-1}\frac{(1+4i-4k)\,(2i-k)!\,\Gamma\left(\frac{1}{2}+k\right)}{k!\,\Gamma\left(\frac{3}{2}+2i-k\right)}\,P_{2i-2k}(x).
$$

□

Theorem 5. *For all non-negative integers i and j, the following linearization formula is valid:*

$$
\begin{aligned}
\tilde{P}_i^{(\alpha,\alpha+1)}(x)\,C_j^{(\eta)}(x) = {} & \frac{(i+j)!\,\Gamma(1+\alpha)\,\Gamma(2(1+i+\alpha))\,\Gamma\left(\frac{1}{2}+\eta\right)}{2^{1+2i+2\alpha}\,\Gamma(2+i+2\alpha)\,\Gamma\left(\frac{1}{2}+\alpha+\eta\right)\,\Gamma(j+2\eta)} \\
& \times \left(\sum_{k=0}^{\left\lfloor\frac{i+j}{2}\right\rfloor} G_{k,i,j}\,\tilde{P}_{i+j-2k}^{(\alpha+\eta-\frac{1}{2},\alpha+\eta+\frac{1}{2})}(x) + \sum_{k=0}^{\left\lfloor\frac{i+j-1}{2}\right\rfloor}\bar{G}_{k,i,j}\,\tilde{P}_{i+j-2k-1}^{(\alpha+\eta-\frac{1}{2},\alpha+\eta+\frac{1}{2})}(x)\right),
\end{aligned} \tag{34}
$$

where

$$
\begin{aligned}
G_{k,i,j} = {} & \frac{(-1)^k\,\Gamma(j-k+\eta)\,\Gamma(1+i+j-2k+2\alpha+2\eta)}{k!\,(i+j-2k)!\,\Gamma(1+i-k+\alpha)\,\Gamma(1+i+j-k+\alpha+\eta)} \\
& \times {}_3F_2\left(\begin{matrix} -2k, -i, -1-2i-2j+2k-2\alpha-2\eta \\ -i-j, -1-2i-2\alpha \end{matrix}\middle| 1\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{G}_{k,i,j} = {} & \frac{(-1)^k\,\Gamma(j-k+\eta)\,\Gamma(i+j-2k+2\alpha+2\eta)}{k!\,(i+j-2k-1)!\,\Gamma(1+i-k+\alpha)\,\Gamma(1+i+j-k+\alpha+\eta)} \\
& \times {}_3F_2\left(\begin{matrix} -2k-1, -i, -2i-2j+2k-2\alpha-2\eta \\ -i-j, -1-2i-2\alpha \end{matrix}\middle| 1\right).
\end{aligned}
$$

Proof. If we substitute by $\beta = \alpha + 1$, and $\theta = \eta$ in (10), then the coefficients $M_{k,i,j}$ take the form

$$\begin{aligned} M_{k,i,j} =& \frac{2\,(\alpha+\eta+i+j-k+1)\,(i+j)!\,\Gamma(\alpha+1)\,\Gamma(\eta+1)\,(\Gamma(2(i+\alpha+1))^2}{k!\,(i+j-k)!\,\Gamma(\alpha+\eta+1)\,\Gamma(i+\alpha+1)\,\Gamma(i+2\alpha+2)\,\Gamma(j+\eta+1)} \\ &\times \frac{\Gamma(2j-k+2\eta+1)\,\Gamma(i+j+\alpha+\eta+1)\,\Gamma(i+j-k+2(\alpha+\eta+1))}{\Gamma(j+2\eta+1)\,\Gamma(2i-k+2\alpha+2)\,\Gamma(2i+2j-k+2\alpha+2\eta+3)} \\ &\times {}_3F_2\left(\left. \begin{array}{c} -k,-i,-2\alpha-2\eta-2i-2j+k-2 \\ -i-j,-2\alpha-2i-1 \end{array} \right| 1 \right) \\ &\times {}_3F_2\left(\left. \begin{array}{c} -k,-\alpha-i,-2\alpha-2\eta-2i-2j+k-2 \\ -2\alpha-2i-1,-\alpha-\eta-i-j \end{array} \right| 1 \right). \end{aligned}$$

Again, we can employ Zeilberger's and Petkovsek's algorithms to show that the second ${}_3F_2(1)$ in the above relation can be summed to give

$$\begin{aligned} &{}_3F_2\left(\left. \begin{array}{c} -k,-\alpha-i,-2\alpha-2\eta-2i-2j+k-2 \\ -2\alpha-2i-1,-\alpha-\eta-i-j \end{array} \right| 1 \right) = \\ &\begin{cases} \dfrac{(-1)^{\frac{k}{2}}\,\Gamma\left(\frac{k+1}{2}\right)\,\Gamma\left(i+j-\frac{k}{2}+\alpha+\eta+2\right)\left(j-\frac{k}{2}+\eta+1\right)_{\frac{k}{2}}}{\sqrt{\pi}\,(\alpha+\eta+i+j-k+1)\,\Gamma(i+j+\alpha+\eta+1)\left(i+\alpha+\frac{3-k}{2}\right)_{\frac{k}{2}}}, & k \text{ even}, \\ \dfrac{(-1)^{\frac{k+3}{2}}\,\Gamma\left(\frac{k}{2}+1\right)\Gamma(j+\eta+1)\Gamma\left(i-\frac{k}{2}+\alpha+1\right)\Gamma\left(i+j+\alpha+\eta+\frac{3-k}{2}\right)}{\sqrt{\pi}\,(\alpha+\eta+i+j-k+1)\,\Gamma\left(i+\alpha+\frac{3}{2}\right)\Gamma\left(j+\eta+\frac{1-k}{2}\right)\Gamma(i+j+\alpha+\eta+1)}, & k \text{ odd}. \end{cases} \end{aligned}$$

Some straightforward computations lead to the linearization formula (34). □

Corollary 6. *For all non-negative integers i and j with $i \geq j$, the following linearization formula holds:*

$$V_i(x)\,P_j(x) = \frac{1}{\pi}\sum_{k=0}^{j} \frac{\Gamma\left(\frac{1}{2}+j-k\right)\,\Gamma\left(\frac{1}{2}+k\right)}{k!\,(j-k)!}\,V_{i+j-2k}(x). \tag{35}$$

Proof. Substitution by $\alpha = -\frac{1}{2}$ and $\eta = \frac{1}{2}$ into (34) yields the following formula:

$$\begin{aligned} V_i(x)\,P_j(x) =& \frac{(i+j)!\,\Gamma\left(\frac{1}{2}+i\right)}{\sqrt{\pi}\,j!} \sum_{k=0}^{\lfloor\frac{i+j}{2}\rfloor} \frac{(-1)^k\,\Gamma\left(\frac{1}{2}+j-k\right)}{k!\,(i+j-k)!\,\Gamma\left(\frac{1}{2}+i-k\right)} \\ &\times {}_3F_2\left(\left. \begin{array}{c} -2k,-i,-1-2i-2j+2k \\ -2i,-i-j \end{array} \right| 1 \right) V_{i+j-2k}(x) \\ &+ \frac{(i+j)!\,\Gamma\left(\frac{1}{2}+i\right)}{\sqrt{\pi}\,j!} \sum_{k=0}^{\lfloor\frac{1}{2}(i+j-1)\rfloor} \frac{(-1)^k\,\Gamma\left(\frac{1}{2}+j-k\right)}{k!\,(i+j-k)!\,\Gamma\left(\frac{1}{2}+i-k\right)} \\ &\times {}_3F_2\left(\left. \begin{array}{c} -2k-1,-i,-2i-2j+2k \\ -2i,-i-j \end{array} \right| 1 \right) V_{i+j-2k-1}(x). \end{aligned} \tag{36}$$

Watson's identity again (5) leads to the following two identities:

$${}_3F_2\left(\left. \begin{array}{c} -2k,-i,-1-2i-2j+2k \\ -2i,-i-j \end{array} \right| 1 \right) = \frac{(-1)^k\,\Gamma\left(k+\frac{1}{2}\right)\,(j-k+1)_k}{\sqrt{\pi}\,\left(i-k+\frac{1}{2}\right)_k\,(i+j-k+1)_k},$$

and

$${}_3F_2\left(\left. \begin{array}{c} -2k-1,-i,-2i-2j+2k \\ -2i,-i-j \end{array} \right| 1 \right) = 0,$$

and, accordingly, the following linearization formula can be obtained:

$$V_i(x)\,P_j(x) = \frac{1}{\pi}\sum_{k=0}^{j}\frac{\Gamma\left(\frac{1}{2}+j-k\right)\Gamma\left(\frac{1}{2}+k\right)}{k!\,(j-k)!}\,V_{i+j-2k}(x).$$

□

Remark 3. *As a direct consequence of Corollary 6, and based on the two identities*

$$P_k(-x) = (-1)^k\,P_k(x), \quad V_k(-x) = (-1)^k\,W_k(x),$$

it is easy to see that, for all $i \geq j$, we have

$$W_i(x)\,P_j(x) = \frac{1}{\pi}\sum_{k=0}^{j}\frac{\Gamma\left(\frac{1}{2}+j-k\right)\Gamma\left(\frac{1}{2}+k\right)}{k!\,(j-k)!}\,W_{i+j-2k}(x). \tag{37}$$

Remark 4. *The two linearization formulas (35) and (37) can be translated into the following trigonometric identities:*

$$\frac{1}{\pi}\sum_{k=0}^{j}\frac{\Gamma\left(\frac{1}{2}+j-k\right)\Gamma\left(\frac{1}{2}+k\right)}{k!\,(j-k)!}\,\cos\left(\left(i+j-2k+\tfrac{1}{2}\right)\theta\right) = \cos\left(\left(i+\tfrac{1}{2}\right)\theta\right)P_j(\cos(\theta)),$$
$$\frac{1}{\pi}\sum_{k=0}^{j}\frac{\Gamma\left(\frac{1}{2}+j-k\right)\Gamma\left(\frac{1}{2}+k\right)}{k!\,(j-k)!}\,\sin\left(\left(i+j-2k+\tfrac{1}{2}\right)\theta\right) = \sin\left(\left(i+\tfrac{1}{2}\right)\theta\right)P_j(\cos(\theta)).$$

4. Some Other Linearization Formulas of Some Jacobi Polynomials

In this section, we are interested in deriving some other linearization formulas of certain Jacobi polynomials.

Theorem 6. *For every non-negative integer i, the following linearization formula is valid:*

$$\begin{aligned}\tilde{P}_i^{(\alpha,\beta)}(x)\,\tilde{P}_i^{(\eta,\alpha+\beta-\eta)}(x) =& \frac{i!\,\Gamma(\alpha+1)\,\Gamma(\eta+1)\,\Gamma(2i+\alpha+\beta+1)\,\Gamma(2i+\alpha+\eta+1)}{2^{2i+1}\,\Gamma(\alpha+\eta+1)\,\Gamma(i+\alpha+1)\,\Gamma(i+\eta+1)\,\Gamma(i+\alpha+\beta+1)}\\ &\times\sum_{k=0}^{i}\frac{(-1)^k\,(2\alpha+2\beta+4i-4k+1)\,\Gamma\left(i-k+\alpha+\beta+\frac{1}{2}\right)}{k!\,(i-k)!\,\Gamma\left(2i-k+\alpha+\beta+\frac{3}{2}\right)}\\ &\times {}_3F_2\left(\left.\begin{matrix}-2k,-\alpha-i,-2\alpha-2\beta-4i+2k-1\\ -\alpha-\beta-2i,-\alpha-\eta-2i\end{matrix}\right|1\right)\tilde{P}_{2i-2k}^{(\alpha+\eta,\alpha+2\beta-\eta)}(x).\end{aligned} \tag{38}$$

Proof. If we set $\theta = \alpha+\beta-\eta$, and $j = i$ in (9), then the following formula can be obtained:

$$\begin{aligned}\tilde{P}_i^{(\alpha,\beta)}(x)\,\tilde{P}_i^{(\eta,\alpha+\beta-\eta)}(x) =& \frac{(2i)!\,\Gamma(\alpha+1)\,\Gamma(\eta+1)\,\Gamma(2i+\alpha+\eta+1)(\Gamma(2i+\alpha+\beta+1))^2}{\Gamma(\alpha+\eta+1)\,\Gamma(i+\alpha+1)\,\Gamma(i+\eta+1)\,(\Gamma(i+\alpha+\beta+1))^2}\\ &\times\sum_{k=0}^{2i}\frac{(2\alpha+2\beta+4i-2k+1)\,\Gamma(2i-k+2\alpha+2\beta+1)}{k!\,(2i-k)!\,\Gamma(4i-k+2(\alpha+\beta+1))}\\ &\times {}_3F_2\left(\left.\begin{matrix}-k,-i,-4i+k-2\alpha-2\beta-1\\ -2i,-2i-\alpha-\beta\end{matrix}\right|1\right)\\ &\times {}_3F_2\left(\left.\begin{matrix}-k,-i-\alpha,-4i+k-2\alpha-2\beta-1\\ -2i-\alpha-\beta,-2i-\alpha-\eta\end{matrix}\right|1\right)\tilde{P}_{2i-k}^{(\alpha+\eta,\alpha+2\beta-\eta)}(x).\end{aligned} \tag{39}$$

The first ${}_3F_2(1)$ which appears in (39) can be summed with the aid of Watson's identity (5) to give

$$ {}_3F_2\left(\begin{matrix} -k,-i,-4i+k-2\alpha-2\beta-1 \\ -2i,-2i-\alpha-\beta \end{matrix}\,\middle|\,1\right) = \begin{cases} \dfrac{(-1)^{\frac{k}{2}}\Gamma\left(\frac{k+1}{2}\right)\left(i-\frac{k}{2}+\alpha+\beta+1\right)_{\frac{k}{2}}}{\sqrt{\pi}\left(i-\frac{k}{2}+\frac{1}{2}\right)_{\frac{k}{2}}\left(2i-\frac{k}{2}+\alpha+\beta+1\right)_{\frac{k}{2}}}, & k \text{ even}, \\ 0, & k \text{ odd}, \end{cases} $$

and hence, after performing some manipulations, the following linearization formula is obtained:

$$ \begin{aligned} \tilde{P}_i^{(\alpha,\beta)}(x)\,\tilde{P}_i^{(\eta,\alpha+\beta-\eta)}(x) = &\frac{i!\,\Gamma(\alpha+1)\,\Gamma(\eta+1)\,\Gamma(2i+\alpha+\beta+1)\,\Gamma(2i+\alpha+\eta+1)}{2^{2i+1}\,\Gamma(\alpha+\eta+1)\,\Gamma(i+\alpha+1)\,\Gamma(i+\eta+1)\,\Gamma(i+\alpha+\beta+1)} \\ &\times \sum_{k=0}^{i} \frac{(-1)^k\,(2\alpha+2\beta+4i-4k+1)\,\Gamma\left(i-k+\alpha+\beta+\frac{1}{2}\right)}{k!\,(i-k)!\,\Gamma\left(2i-k+\alpha+\beta+\frac{3}{2}\right)} \\ &\times {}_3F_2\left(\begin{matrix} -2k,-\alpha-i,-2\alpha-2\beta-4i+2k-1 \\ -\alpha-\beta-2i,-\alpha-\eta-2i \end{matrix}\,\middle|\,1\right) \tilde{P}_{2i-2k}^{(\alpha+\eta,\alpha+2\beta-\eta)}(x). \end{aligned} $$

The proof is now complete. □

Corollary 7. *For every non-negative integer i, the following linearization formula holds:*

$$ \begin{aligned} \tilde{P}_i^{(\alpha,\beta)}(x)\,\tilde{P}_i^{(\beta,\alpha)}(x) = &\frac{i!\,\Gamma(1+\alpha)\,\Gamma(1+\beta)\,\Gamma(1+2i+\alpha+\beta)^2}{2^{2i+1}\,\Gamma(1+i+\alpha)\,\Gamma(1+i+\beta)\,\Gamma(1+\alpha+\beta)\,\Gamma(1+i+\alpha+\beta)} \\ &\times \sum_{k=0}^{i} \frac{(-1)^k\,(1+4i-4k+2\alpha+2\beta)\,\Gamma\left(\frac{1}{2}+i-k+\alpha+\beta\right)}{k!\,(i-k)!\,\Gamma\left(\frac{3}{2}+2i-k+\alpha+\beta\right)} \\ &\times {}_3F_2\left(\begin{matrix} -2k,-i-\alpha,-1-4i+2k-2\alpha-2\beta \\ -2i-\alpha-\beta,-2i-\alpha-\beta \end{matrix}\,\middle|\,1\right) C_{2i-2k}^{(\alpha+\beta+\frac{1}{2})}(x). \end{aligned} \tag{40} $$

Proof. Corollary 7 is an immediate consequence of Theorem 6 for the case $\eta=\beta$. □

Remark 5. *The left-hand side of relation (40) was expressed before in [21], but in terms of a terminating hypergeometric function of the type ${}_4F_3(1)$. Therefore, a transformation formula between two different hypergeometric functions of unit argument can be deduced. This transformation is given in the following corollary.*

Corollary 8. *The following transformation formula holds:*

$$ \begin{aligned} &{}_4F_3\left(\begin{matrix} -k,\frac{1}{2}+i-k+\frac{\alpha}{2}+\frac{\beta}{2},1+i-k+\frac{\alpha}{2}+\frac{\beta}{2},1+2i-k+\alpha+\beta \\ 1+i-k+\alpha,1+i-k+\beta,\frac{3}{2}+2i-2k+\alpha+\beta \end{matrix}\,\middle|\,1\right) = \\ &\frac{(-1)^k\,2^{1+4i-4k+2\alpha+2\beta}\,\Gamma(1+i-k+\alpha)\,\Gamma(1+i-k+\beta)\,\Gamma(1+2i+\alpha+\beta)^2}{\sqrt{\pi}\,\Gamma(1+i+\alpha)\,\Gamma(1+i+\beta)\,\Gamma(1+2i-2k+\alpha+\beta)} \\ &\times \frac{\Gamma\left(\frac{3}{2}+2i-2k+\alpha+\beta\right)}{\Gamma(2(1+2i-k+\alpha+\beta))}\,{}_3F_2\left(\begin{matrix} -2k,-\alpha-i,-2\alpha-2\beta-4i+2k-1 \\ -\alpha-\beta-2i,-\alpha-\beta-2i \end{matrix}\,\middle|\,1\right). \end{aligned} \tag{41} $$

Proof. From formula (9), p. 159, which was introduced in [21], and if we set $\nu=\alpha+\beta+\frac{1}{2}$, then we get

$$ \tilde{P}_i^{(\alpha,\beta)}(x)\,\tilde{P}_i^{(\beta,\alpha)}(x) = \sum_{k=0}^{i} \xi_{k,i}\,C_{2i-2k}^{(\alpha+\beta+\frac{1}{2})}(x), \tag{42} $$

where

$$\xi_{k,i} = \frac{\binom{i}{i-k}\,(1+i+\alpha+\beta)_{i-k}\,\left(\frac{1}{2}(1+\alpha+\beta)\right)_{i-k}\left(\frac{1}{2}(2+\alpha+\beta)\right)_{i-k}}{(1+\alpha)_{i-k}(1+\beta)_{i-k}\left(\frac{1}{2}+i-k+\alpha+\beta\right)_{i-k}}$$
$$\times\ {}_4F_3\left(\begin{matrix} -k, \frac{1}{2}+i-k+\frac{\alpha}{2}+\frac{\beta}{2}, 1+i-k+\frac{\alpha}{2}+\frac{\beta}{2}, 1+2i-k+\alpha+\beta \\ 1+i-k+\alpha, 1+i-k+\beta, \frac{3}{2}+2i-2k+\alpha+\beta \end{matrix}\,\middle|\,1\right).$$

Comparing the two results in (40) and (42), the transformation formula (41) can be followed. □

Corollary 9. *For every non-negative integer i, the following linearization formula is valid:*

$$\begin{aligned}\left(\tilde{P}_i^{(\alpha,\beta)}(x)\right)^2 &= \frac{i!\,\Gamma(\alpha+1)\,\Gamma\left(i+\alpha+\frac{1}{2}\right)\Gamma(2i+\alpha+\beta+1)}{2\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(i+\alpha+1)\,\Gamma(i+\alpha+\beta+1)} \\ &\times \sum_{k=0}^{i} \frac{(2\alpha+2\beta+4i-4k+1)\,\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(i-k+\alpha+\beta+\frac{1}{2}\right)(i-k+\beta+1)_k}{k!\,(i-k)!\,\Gamma\left(2i-k+\alpha+\beta+\frac{3}{2}\right)\left(i-k+\alpha+\frac{1}{2}\right)_k(2i-k+\alpha+\beta+1)_k}\,\tilde{P}_{2i-2k}^{(2\alpha,2\beta)}(x).\end{aligned} \tag{43}$$

Proof. Setting $\eta = \alpha$ in (38) yields the relation

$$\begin{aligned}\left(\tilde{P}_i^{(\alpha,\beta)}(x)\right)^2 = &\frac{i!\,\Gamma(\alpha+1)\,\Gamma\left(i+\alpha+\frac{1}{2}\right)\Gamma(2i+\alpha+\beta+1)}{2\,\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(i+\alpha+1)\,\Gamma(i+\alpha+\beta+1)} \\ &\times \sum_{k=0}^{i} \frac{(-1)^k\,(2\alpha+2\beta+4i-4k+1)\,\Gamma\left(i-k+\alpha+\beta+\frac{1}{2}\right)}{k!\,(i-k)!\,\Gamma\left(2i-k+\alpha+\beta+\frac{3}{2}\right)} \\ &\times\ {}_3F_2\left(\begin{matrix} -2k, -\alpha-i, -2\alpha-2\beta-4i+2k-1 \\ -2\alpha-2i, -\alpha-\beta-2i \end{matrix}\,\middle|\,1\right)\tilde{P}_{2i-2k}^{(2\alpha,2\beta)}(x).\end{aligned}$$

Again, Watson's identity can be utilized to reduce the last ${}_3F_2(1)$. More precisely, we have

$${}_3F_2\left(\begin{matrix} -2k, -\alpha-i, -2\alpha-2\beta-4i+2k-1 \\ -2\alpha-2i, -\alpha-\beta-2i \end{matrix}\,\middle|\,1\right) = \frac{(-1)^k\,\Gamma\left(k+\frac{1}{2}\right)(i-k+\beta+1)_k}{\sqrt{\pi}\left(i-k+\alpha+\frac{1}{2}\right)_k(2i-k+\alpha+\beta+1)_k},$$

and, therefore, the linearization formula (43) can be obtained. □

Corollary 10. *For every non-negative integer i, the following linearization formulas are valid:*

$$\left(\tilde{P}_i^{(-\frac{1}{4},\frac{1}{4})}(x)\right)^2 = \frac{\sqrt{2}\left(\Gamma\left(\frac{3}{4}\right)\right)^2\Gamma\left(i+\frac{5}{4}\right)}{\pi^{3/2}\,\Gamma\left(i+\frac{3}{4}\right)}\sum_{k=0}^{i}\frac{(2i-k)!\,\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(i-k+\frac{1}{2}\right)}{k!\,(i-k)!\,\Gamma\left(2i-k+\frac{3}{2}\right)}\,V_{2i-2k}(x), \tag{44}$$

$$\left(\tilde{P}_i^{(\frac{1}{4},-\frac{1}{4})}(x)\right)^2 = \frac{\sqrt{2}\,(\Gamma(\frac{5}{4}))^2\,\Gamma(i+\frac{3}{4})}{\pi^{3/2}\,\Gamma(i+\frac{5}{4})}\sum_{k=0}^{i}\frac{(2i-k)!\,\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(i-k+\frac{1}{2}\right)}{k!\,(i-k)!\,\Gamma(2i-k+\frac{3}{2})}\,W_{2i-2k}(x). \tag{45}$$

Proof. Relations (44) and (45) can be immediately obtained as direct special cases of relation (43). □

Theorem 7. *For every non-negative integer i, the following linearization formula is valid:*

$$\tilde{P}_i^{(\alpha,\beta)}(x)\,\tilde{P}_{i+1}^{(\eta,\alpha+\beta-\eta)}(x)=\sum_{k=0}^{i} S_{k,i}\,\tilde{P}_{2i-2k}^{(\alpha+\eta,\alpha+2\beta-\eta)}(x)+\sum_{k=0}^{i}\bar{S}_{k,i}\,\tilde{P}_{2i-2k+1}^{(\alpha+\eta,\alpha+2\beta-\eta)}(x), \tag{46}$$

where

$$\begin{aligned} S_{k,i}=&\frac{(-1)^k\, i!\,(\alpha+\beta)\,(2\alpha+2\beta+4i-4k+1)\,\Gamma(\alpha+1)\,\Gamma(\eta+1)\,\Gamma(2i+\alpha+\eta+2)}{2^{2\,i+2}\,k!\,(i-k)!\,\Gamma(\alpha+\eta+1)\,\Gamma(i+\alpha+1)\,\Gamma(i+\eta+2)}\\ &\times\frac{\Gamma(2i+\alpha+\beta+1)\,\Gamma\left(i-k+\alpha+\beta+\frac{1}{2}\right)}{\Gamma(i+\alpha+\beta+2)\,\Gamma\left(2i-k+\alpha+\beta+\frac{3}{2}\right)}\\ &\times\,{}_3F_2\left(\left.\begin{matrix}-2k-1,-\alpha-i,-2\alpha-2\beta-4i+2k-2\\ -\alpha-\beta-2i,-\alpha-\eta-2i-1\end{matrix}\right|1\right), \end{aligned} \tag{47}$$

and

$$\begin{aligned} \bar{S}_{k,i}=&\frac{(-1)^k\, i!\,(\alpha+\beta+2i+2)\,(2\alpha+2\beta+4i-4k+3)\,\Gamma(\alpha+1)\,\Gamma(\eta+1)\,\Gamma(2\,i+\alpha+\beta+1)}{2^{2i+2}\,k!\,(i-k)!\,\Gamma(\alpha+\eta+1)\,\Gamma(i+\alpha+1)\,\Gamma(i+\eta+2)}\\ &\times\frac{\Gamma(2i+\alpha+\eta+2)\,\Gamma\left(i-k+\alpha+\beta+\frac{3}{2}\right)}{\Gamma(i+\alpha+\beta+2)\,\Gamma\left(2i-k+\alpha+\beta+\frac{5}{2}\right)}\\ &\times\,{}_3F_2\left(\left.\begin{matrix}-2k,-\alpha-i,-2\alpha-2\beta-4i+2k-3\\ -\alpha-\beta-2i,-\alpha-\eta-2i-1\end{matrix}\right|1\right). \end{aligned} \tag{48}$$

Proof. If we set $\theta=\alpha+\beta-\eta$, and $j=i+1$ in (9), then the following linearization formula is obtained:

$$\tilde{P}_i^{(\alpha,\beta)}(x)\,\tilde{P}_{i+1}^{(\eta,\alpha+\beta-\eta)}(x)=\sum_{k=0}^{2i+1} G_{k,i}\,\tilde{P}_{2i-k+1}^{(\alpha+\eta,\alpha+2\beta-\eta)}(x), \tag{49}$$

where

$$\begin{aligned} G_{k,i}=&\frac{(2i+1)!\,(\alpha+\beta+2i-k+1)\,(\alpha+\beta+2i-k+2)\,(2\alpha+2\beta+4i-2k+3)\,\Gamma(\alpha+1)\,\Gamma(\eta+1)}{k!\,(2i-k+1)!\,\Gamma(\alpha+\eta+1)\,\Gamma(i+\alpha+1)\,\Gamma(i+\eta+2)}\\ &\times\frac{(\Gamma(2i+\alpha+\beta+1))^2\,\Gamma(2i+\alpha+\eta+2)\,\Gamma(2i-k+2(\alpha+\beta+1))}{\Gamma(i+\alpha+\beta+1)\,\Gamma(i+\alpha+\beta+2)\,\Gamma(4i-k+2(\alpha+\beta+2))}\\ &\times\,{}_3F_2\left(\left.\begin{matrix}-k,-i,-2\alpha-2\beta-4i+k-3\\ -2i-1,-\alpha-\beta-2i\end{matrix}\right|1\right)\\ &\times\,{}_3F_2\left(\left.\begin{matrix}-k,-\alpha-i,-2\alpha-2\beta-4i+k-3\\ -\alpha-\beta-2i,-\alpha-\eta-2i-1\end{matrix}\right|1\right). \end{aligned}$$

Regarding the first ${}_3F_2(1)$ which appears in the last formula, it seems that it cannot be summed with any standard reduction formula, so we resort to the symbolic computation. More precisely, if we set

$$B_{k,i}=\,{}_3F_2\left(\left.\begin{matrix}-k,-i,-2\alpha-2\beta-4i+k-3\\ -2i-1,-\alpha-\beta-2i\end{matrix}\right|1\right),$$

then it can be shown with the aid of Zeilberger's algorithm that $B_{k,i}$ satisfies the following recurrence relation of order two:

$$\begin{aligned} &(k-1)(\alpha+\beta+2\,i-k+4)(-k+2+2\,i+\beta+\alpha)(2\,i+3+2\,\alpha+2\,\beta-k)\,B_{k-2,i}\\ &-(\alpha+\beta)(2+\alpha+\beta+2\,i)(2\,\alpha+2\,\beta+4\,i-2\,k+5)\,B_{k-1,i}\\ &+(2\,\alpha+2\,\beta+4\,i-k+4)(-k+2\,i+2)(\alpha+\beta+2\,i-k+3)(-k+\alpha+\beta+2\,i+1)\,B_{k,i}=0, \end{aligned}$$

with the initial values

$$B_{0,i} = 1, \quad B_{1,i} = \frac{\alpha+\beta}{(2i+1)(\alpha+\beta+2i)}.$$

The last recurrence relation has the following exact solution:

$$B_{k,i} = \frac{1}{\sqrt{\pi}(\alpha+\beta+2i-k+1)(\alpha+\beta+2i-k+2)} \times \begin{cases} \dfrac{(-1)^{\frac{k}{2}}\,(\alpha+\beta+2i+2)\,\Gamma\left(\frac{k+1}{2}\right)\left(i-\frac{k}{2}+\alpha+\beta+1\right)_{\frac{k}{2}}}{\left(i-\frac{k}{2}+\frac{3}{2}\right)_{\frac{k}{2}}\left(2i-\frac{k}{2}+\alpha+\beta+2\right)_{\frac{k}{2}-1}}, & k \text{ even}, \\ \dfrac{(-1)^{\frac{k-1}{2}}\,(\alpha+\beta)\,\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(i-\frac{k}{2}+1\right)\left(i-\frac{k}{2}+\alpha+\beta+\frac{3}{2}\right)_{\frac{k-1}{2}}}{\Gamma\left(i+\frac{3}{2}\right)\left(2i-\frac{k}{2}+\alpha+\beta+\frac{5}{2}\right)_{\frac{k-3}{2}}}, & k \text{ odd}. \end{cases}$$

Now, and based on the above reduction formula for $B_{k,i}$, the linearization coefficients $G_{k,i}$ can be reduced to give

$$\begin{aligned} G_{k,i} = {} & \frac{i!\,(2\alpha+2\beta+4i-2k+3)\,\Gamma(\alpha+1)\,\Gamma(\eta+1)\,\Gamma(2i+\alpha+\beta+1)\,\Gamma(2i+\alpha+\eta+2)}{2^{2i+2}\,\Gamma(\alpha+\eta+1)\,\Gamma(i+\alpha+1)\,\Gamma(i+\eta+2)\,\Gamma(i+\alpha+\beta+2)} \\ & \times {}_3F_2\left(\left.\begin{array}{c} -k, -\alpha-i, -2\alpha-2\beta-4i+k-3 \\ -\alpha-\beta-2i, -\alpha-\eta-2i-1 \end{array}\right| 1\right) \\ & \times \begin{cases} \dfrac{(-1)^{\frac{k}{2}}\,(\alpha+\beta+2i+2)\,\Gamma\left(i-\frac{k}{2}+\alpha+\beta+\frac{3}{2}\right)}{\left(\frac{k}{2}\right)!\left(i-\frac{k}{2}\right)!\,\Gamma\left(2i-\frac{k}{2}+\alpha+\beta+\frac{5}{2}\right)}, & k \text{ even}, \\ \dfrac{(-1)^{\frac{k-1}{2}}\,(\alpha+\beta)\,\Gamma\left(i-\frac{k}{2}+\alpha+\beta+1\right)}{\left(\frac{k-1}{2}\right)!\left(i-(\frac{k-1}{2})\right)!\,\Gamma\left(2i-\frac{k}{2}+\alpha+\beta+2\right)}, & k \text{ odd}. \end{cases} \end{aligned}$$

Now, it is not difficult to see that (49) is equivalent to

$$\tilde{P}_i^{(\alpha,\beta)}(x)\,\tilde{P}_{i+1}^{(\eta,\alpha+\beta-\eta)}(x) = \sum_{k=0}^{i} S_{k,i}\,\tilde{P}_{2i-2k}^{(\alpha+\eta,\alpha+2\beta-\eta)}(x) + \sum_{k=0}^{i} \bar{S}_{k,i}\,\tilde{P}_{2i-2k+1}^{(\alpha+\eta,\alpha+2\beta-\eta)}(x),$$

and $S_{k,i}$ and $\bar{S}_{k,i}$, are given respectively by (47) and (48). □

Now, the following corollary is a special case of Theorem 7.

Corollary 11. *If we set $\eta = \alpha$ in (46), then, for every non-negative integer i, the following linearization formula is valid:*

$$
\begin{aligned}
\tilde{P}_i^{(\alpha,\beta)}(x)\,\tilde{P}_{i+1}^{(\alpha,\beta)}(x) =& \frac{i!\,(\alpha+\beta+2i+2)^2\,\Gamma(\alpha+1)\,\Gamma(i+\beta+1)}{2\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(i+\alpha+2)\,\Gamma(i+\alpha+\beta+2)} \\
&\times \sum_{k=0}^{i} \frac{(2\alpha+2\beta+4i-4k+3)\,\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(i-k+\alpha+\frac{3}{2}\right)\Gamma\left(i-k+\alpha+\beta+\frac{3}{2}\right)}{k!\,(i-k)!\,(\alpha+\beta+2i-2k+1)\,(\alpha+\beta+2i-2k+2)\,\Gamma(i-k+\beta+1)} \\
&\times \frac{\Gamma(2i-k+\alpha+\beta+2)}{\Gamma\left(2i-k+\alpha+\beta+\frac{5}{2}\right)}\,\tilde{P}_{2i-2k+1}^{(2\alpha,2\beta)} + \frac{i!(\beta-\alpha)(\alpha+\beta)\Gamma(\alpha+1)\Gamma(i+\beta+1)}{2\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(i+\alpha+2)\Gamma(i+\alpha+\beta+2)} \\
&\times \sum_{k=0}^{i} \frac{(2\alpha+2\beta+4i-4k+1)\,\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(i-k+\alpha+\frac{1}{2}\right)\Gamma\left(i-k+\alpha+\beta+\frac{1}{2}\right)}{k!\,(i-k)!\,(\alpha+\beta+2i-2k)\,(\alpha+\beta+2i-2k+1)\,\Gamma(i-k+\beta+1)} \\
&\times \frac{\Gamma(2i-k+\alpha+\beta+2)}{\Gamma\left(2i-k+\alpha+\beta+\frac{3}{2}\right)}\,\tilde{P}_{2i-2k}^{(2\alpha,2\beta)}.
\end{aligned}
\tag{50}
$$

Proof. If we set $\alpha=\eta$ in relation (46), then the linearization coefficients $S_{k,i}, \bar{S}_{k,i}$ are given by

$$
\begin{aligned}
S_{k,i} =& \frac{(-1)^k\,i!\,\Gamma(\alpha+1)\,\Gamma\left(i+\alpha+\frac{3}{2}\right)\Gamma(2i+\alpha+\beta+1)}{2k!\,(i-k)!\,\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(i+\alpha+2)\,\Gamma(i+\alpha+\beta+2)} \\
&\times \frac{(\alpha+\beta+2i+2)\,(2\alpha+2\beta+4i-4k+3)\,\Gamma\left(i-k+\alpha+\beta+\frac{3}{2}\right)}{\Gamma\left(2i-k+\alpha+\beta+\frac{5}{2}\right)} \\
&\times {}_3F_2\left(\left.\begin{matrix}-2k,-\alpha-i,-2\alpha-2\beta-4i+2k-3\\ -2\alpha-2i-1,-\alpha-\beta-2i\end{matrix}\right|1\right), \\
\bar{S}_{k,i} =& \frac{(-1)^k\,i!\,\Gamma(\alpha+1)\,\Gamma\left(i+\alpha+\frac{3}{2}\right)\Gamma(2i+\alpha+\beta+1)}{2\,k!\,(i-k)!\,\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(i+\alpha+2)\,\Gamma(i+\alpha+\beta+2)} \\
&\times \frac{(\alpha+\beta)\,(2\alpha+2\beta+4i-4k+1)\,\Gamma\left(i-k+\alpha+\beta+\frac{1}{2}\right)}{\Gamma\left(2i-k+\alpha+\beta+\frac{3}{2}\right)} \\
&\times {}_3F_2\left(\left.\begin{matrix}-2k-1,-\alpha-i,-2\alpha-2\beta-4i+2k-2\\ -2\alpha-2i-1,-\alpha-\beta-2i\end{matrix}\right|1\right).
\end{aligned}
$$

Making use of Zeilberger's and Petkovsek's algorithms, it can be shown that the two hypergeometric functions appearing in the coefficients $S_{k,i}, \bar{S}_{k,i}$ can be reduced to give the following formulas:

$$
\begin{aligned}
&{}_3F_2\left(\left.\begin{matrix}-2k,-\alpha-i,-2\alpha-2\beta-4i+2k-3\\ -2\alpha-2i-1,-\alpha-\beta-2i\end{matrix}\right|1\right) = \\
&\frac{(-1)^k\,(\alpha+\beta+2i+2)\,\Gamma\left(k+\frac{1}{2}\right)(i-k+\beta+1)_k}{\sqrt{\pi}\,(\alpha+\beta+2i-2k+1)\,(\alpha+\beta+2i-2k+2)\left(i-k+\alpha+\frac{3}{2}\right)_k(2i-k+\alpha+\beta+2)_{k-1}},
\end{aligned}
$$

and

$$
\begin{aligned}
&{}_3F_2\left(\left.\begin{matrix}-2k-1,-\alpha-i,-2\alpha-2\beta-4i+2k-2\\ -2\alpha-2i-1,-\alpha-\beta-2i\end{matrix}\right|1\right) = \\
&\frac{(-1)^k\,(\beta-\alpha)\,(\beta+i)\,\Gamma\left(k+\frac{3}{2}\right)(i-k+\beta+1)_{k-1}}{\sqrt{\pi}\,(\alpha+\beta+2i-2k)\,(\alpha+\beta+2i-2k+1)\left(i-k+\alpha+\frac{1}{2}\right)_{k+1}(2i-k+\alpha+\beta+2)_{k-1}},
\end{aligned}
$$

and, therefore, the linearization formula (50) that is free of any hypergeometric function can be obtained. □

In the following, we are going to write some other linearization formulas of certain Jacobi polynomials. The proofs are omitted due to their similarity with the proofs of Theorems 6 and 7.

Theorem 8. *For every non-negative integer i, the following linearization formula is valid:*

$$\tilde{P}_i^{(\alpha,\beta)}(x)\,\tilde{P}_i^{(\eta,\alpha+\beta-\eta+1)}(x)=\sum_{k=0}^{i} Q_{k,i}\tilde{P}_{2i-2k}^{(\alpha+\eta,\alpha+2\beta-\eta+1)}(x)+\sum_{k=0}^{i-1}\bar{Q}_{k,i}\tilde{P}_{2i-2k-1}^{(\alpha+\eta,\alpha+2\beta-\eta+1)}(x), \tag{51}$$

where

$$Q_{k,i}=\frac{i!\,(-1)^k\,(1+2i-2k+\alpha+\beta)\,\Gamma(1+\alpha)\,\Gamma(1+\eta)\,\Gamma(1+2i+\alpha+\eta)\,\Gamma(1+2i+\alpha+\beta)}{2^{2i}\,k!\,(i-k)!\,\Gamma(1+i+\alpha)\,\Gamma(1+i+\eta)\,\Gamma(1+\alpha+\eta)\,\Gamma(2+i+\alpha+\beta)}$$
$$\times\frac{\Gamma\left(\frac{3}{2}+i-k+\alpha+\beta\right)}{\Gamma\left(\frac{3}{2}+2i-k+\alpha+\beta\right)}\,{}_3F_2\left(\begin{matrix}-2k,-\alpha-i,-2\alpha-2\beta-4i+2k-2\\-\alpha-\beta-2i,-\alpha-\eta-2i\end{matrix}\,\middle|\,1\right),$$

and

$$\bar{Q}_{k,i}=\frac{(-1)^k\,i!\,(2k-\alpha-\beta-2i)\,\Gamma(\alpha+1)\,\Gamma(\eta+1)\,\Gamma(2i+\alpha+\eta+1)\Gamma(2i+\alpha+\beta+1)}{2^{2i}\,k!\,(i-k-1)!\,\Gamma(\alpha+\eta+1)\Gamma(i+\alpha+1)\,\Gamma(i+\eta+1)\,\Gamma(i+\alpha+\beta+2)}$$
$$\times\frac{\Gamma\left(i-k+\alpha+\beta+\frac{1}{2}\right)}{\Gamma\left(2i-k+\alpha+\beta+\frac{3}{2}\right)}\,{}_3F_2\left(\begin{matrix}-2k-1,-\alpha-i,-2\alpha-2\beta-4i+2k-1\\-\alpha-\beta-2i,-\alpha-\eta-2i\end{matrix}\,\middle|\,1\right).$$

Theorem 9. *For every non-negative integer i, the following linearization formula is valid*

$$\tilde{P}_i^{(\alpha,\beta)}(x)\,\tilde{P}_i^{(\eta,\alpha+\beta-\eta-1)}(x)=\sum_{k=0}^{i} L_{k,i}\tilde{P}_{2i-2k}^{(\alpha+\eta,\alpha+2\beta-\eta-1)}(x)+\sum_{k=0}^{i-1}\bar{L}_{k,i}\tilde{P}_{2i-2k-1}^{(\alpha+\eta,\alpha+2\beta-\eta-1)}(x), \tag{52}$$

where

$$L_{k,i}=\frac{(-1)^k\,i!\,\Gamma(1+\alpha)\,\Gamma(1+2i+\alpha+\beta)\,\Gamma(1+\eta)\,\Gamma(1+2i+\alpha+\eta)}{k!\,2^{2i}\,(i-k)!\,\Gamma(1+i+\alpha)\Gamma(1+i+\alpha+\beta)\,\Gamma(1+i+\eta)\,\Gamma(1+\alpha+\eta)}$$
$$\times\frac{\Gamma\left(\frac{1}{2}+i-k+\alpha+\beta\right)}{\Gamma\left(\frac{1}{2}+2i-k+\alpha+\beta\right)}\,{}_3F_2\left(\begin{matrix}-2k,-i-\alpha,-4i+2k-2\alpha-2\beta\\-2i-\alpha-\beta,-2i-\alpha-\eta\end{matrix}\,\middle|\,1\right),$$

and

$$\bar{L}_{k,i}=\frac{(-1)^k\,i!\,\Gamma(1+\alpha)\,\Gamma(1+2i+\alpha+\beta)\,\Gamma(1+\eta)\,\Gamma(1+2i+\alpha+\eta)}{2^{2i}\,k!\,(i-k-1)!\,\Gamma(1+i+\alpha)\,\Gamma(1+i+\alpha+\beta)\,\Gamma(1+i+\eta)\Gamma(1+\alpha+\eta)}$$
$$\times\frac{\Gamma\left(\alpha+\beta+i-k-\frac{1}{2}\right)}{\Gamma\left(\alpha+\beta+2i-k+\frac{1}{2}\right)}\,{}_3F_2\left(\begin{matrix}-2k-1,-i-\alpha,1-4i+2k-2\alpha-2\beta\\-2i-\alpha-\beta,-2i-\alpha-\eta\end{matrix}\,\middle|\,1\right).$$

5. Conclusions

In this paper, we have considered the linearization problem of Jacobi polynomials. Various general and specific linearization formulas were derived. The derivation of these formulas depends on the reduction of the linearization coefficients which are expressed in terms of two terminating hypergeometric functions of unit argument. Some of these reductions can be obtained with the aid of some well-known standard formulas in the literature. Some other reductions can be obtained by employing some symbolic algorithms, and in particular via the algorithms of Zeilberger, Petkovsek, and van Hoeij. To the best of our knowledge, most of the results of this paper are new, and they are very useful.

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