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Concurrent Vector Fields on Lightlike Hypersurfaces

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Abstract: Concurrent vector fields lying on lightlike hypersurfaces of a Lorentzian manifold are investigated. Obtained results dealing with concurrent vector fields are discussed for totally umbilical lightlike hypersurfaces and totally geodesic lightlike hypersurfaces. Furthermore, Ricci soliton lightlike hypersurfaces admitting concurrent vector fields are studied and some characterizations for this frame of hypersurfaces are obtained.

Keywords: concurrent vector field; lightlike hypersurface; Lorentzian manifold; Ricci soliton

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1. Introduction

In 1943, K. Yano [1] proved that there exists a smooth vector field v, so-called concurrent on a Riemannian manifold (M, g) which satisfies the following condition: for every vector field X tangent to M;

$$\nabla_X v = X, \tag{1}$$

where ∇ is the Levi-Civita connection with respect to Riemannian metric g on M.

Applications of concurrent vector fields have been investigated and Riemannian and semi-Riemannian manifolds equipped with concurrent vector fields have been intensely studied by various authors (cf. [2–7]).

Beside these facts, the notion of a Ricci soliton is initially observed by Hamilton's Ricci flow and Ricci solitons drew attention after G. Perelman [8] applied Ricci solitons to solve the Poincaré conjecture.

A Riemannian manifold (M, g) with a metric tensor g is called a Ricci soliton if there exists a smooth vector field v tangent to M satisfying the following equation:

$$\frac{1}{2} L_v g + Ric = \lambda g, \tag{2}$$

where $L_v g$ is the Lie derivative of g with respect to v, Ric denotes the Ricci tensor and λ is a constant. A Ricci soliton is called shrinking if $\lambda > 0$, steady if $\lambda = 0$, expanding if $\lambda < 0$.

Some interesting applications and characterizations dealing the Ricci soliton equation given in (2) for Riemannian manifolds, semi-Riemannian manifolds and their submanifolds have been obtained in [9–16] recently.

The main purpose of this paper is to investigate concurrent vector fields on lightlike hypersurfaces and Ricci solitons lightlike hypersurfaces of a Lorentzian manifold. However, there are some difficulties to deal with while examining concurrent vector fields and Ricci solitons for these kinds of submanifolds. The first problem is that since the induced metric is degenerate and hence not invertible for a lightlike hypersurface, some significant differential operators such as the gradient, divergence, Laplacian operators with respect to the degenerate metric cannot be defined. To get rid of this problem, we consider the associated metric defined with the help of a rigging vector field. The second main problem is that the Ricci tensor of any lightlike hypersurface is not symmetric. In this case, the Ricci soliton equation loses its geometric and physical meanings. To get rid of this problem, we investigate this equation on lightlike hypersurfaces with the genus zero screen distribution whose Ricci tensor is symmetric.

2. Preliminaries

In this section, we shall recall some basic definitions and theorems related to lightlike hypersurfaces of a Lorentzian manifold by following [17–19].

Let $(\widetilde{M}, \widetilde{g})$ be a Lorentzian manifold with the Lorentzian metic $\widetilde{g}$ of constant index 1 and (M, g) be an $(n+1)-$dimensional lightlike hypersurface, $n \geq 2$, of $(\widetilde{M}, \widetilde{g})$, where g is the induced degenerate metric on M. Then the intersection of tangent bundle TM and normal bundle $TM^{\perp}$ is a one-dimensional subbundle such that this bundle is called the radical distribution of M and it is denoted by Rad TM. Therefore, we write the radical distribution at any point $p \in M$ by the following equation:

$$\text{Rad } T_pM = \{\xi \in T_pM : \ g_p(\xi, X) = 0, \ \forall X \in TM\}. \tag{3}$$

For any lightlike hypersurface (M, g), there exists the complementary non-degenerate (Riemannian) vector bundle of Rad TM in TM, called the screen distribution $\mathrm{S}(TM)$ of M such that we have

$$TM = \text{Rad } TM \oplus_{\text{orth}} \mathrm{S}(TM), \tag{4}$$

where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. For any ξ in Rad TM, there exists a unique section N of the lightlike transversal bundle $\mathrm{tr}(TM)$ such that we have

$$\widetilde{g}(N, X) = \widetilde{g}(N, N) = 0, \quad \widetilde{g}(N, \xi) = 1, \ \forall X \in \Gamma(S(TM)). \tag{5}$$

Therefore, the tangent bundle $T\widetilde{M}$ of $\widetilde{M}$ is decomposed as follows:

$$T\widetilde{M} = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus_{orth} S(TM), \tag{6}$$

where $\oplus$ denotes the direct sum which is not orthogonal.

From (5) and (6), one can consider a basis $\{e_1, \ldots\ldots, e_n, \xi, N\}$ on $T\widetilde{M}$ such that $\{e_1, \ldots\ldots, e_n\}$ is an orthonormal basis of $\Gamma(S(TM))$. The basis $\{e_1, \ldots\ldots, e_n, \xi, N\}$ is called a quasi-orthonomal basis on $T\widetilde{M}$.

Suppose that P to be the projection morphism of TM onto $\mathrm{S}(TM)$ and $\widetilde{\nabla}$ to be the Levi-Civita connection of $\widetilde{M}$. The Gauss and Weingarten formulas for the hypersurface are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{7}$$
$$\widetilde{\nabla}_X N = -A_N X + \tau(X)N, \tag{8}$$
$$\nabla_X Y = \nabla^*_X Y + C(X, Y)\xi, \tag{9}$$
$$\nabla_X \xi = -A^*_{\xi} X - \tau(X)\xi, \tag{10}$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^* are the induced linear connection on TM and $\mathrm{S}(TM)$, respectively [17,19]. It is well known that there exist the following equalities involving B and C and their shape operators A^*_{ξ} and A_N, respectively:

$$B(X, Y) = g(A^*_{\xi} X, Y), \tag{11}$$
$$C(X, PY) = g(A_N X, PY). \tag{12}$$

Note that $S(TM)$ is not unique [20] and the second fundamental form B is independent of the choice of a screen distribution and satisfies the condition $B(X, \xi) = 0$ for any

$X \in \Gamma(TM)$. It is known that the induced connection ∇ given in (7) is not metric connection and there exists the following relation for any $X, Y, Z \in \Gamma(TM)$:

$$\begin{aligned} (\nabla_Z g)(X,Y) &= Zg(X,Y) - g(\nabla_v X, Y) - g(\nabla_Z Y, X) \\ &= B(Z,X)\eta(Y) + B(Z,Y)\eta(X). \end{aligned} \tag{13}$$

The Lie derivative of $\widetilde{g}$ with respect to the its Levi-Civita connection $\widetilde{\nabla}$ is defined by

$$(L_Z\widetilde{g})(X,Y) = \widetilde{g}(\widetilde{\nabla}_X Z, Y) + \widetilde{g}(\widetilde{\nabla}_Y Z, X) \tag{14}$$

for any $X, Y, Z \in \Gamma(TM)$. For any lightlike hypersurface $(M, g, S(TM))$ of $(\widetilde{M}, \widetilde{g})$, we have from (13) and (14) that

$$\begin{aligned} (L_Z g)(X,Y) &= Zg(X,Y) - g([Z,X],Y) - g(X,[Z,Y]) \\ &= Zg(X,Y) - g(\nabla_Z X, Y) - g(\nabla_Z Y, X) + g(\nabla_X Z, Y) \\ &\quad + g(\nabla_Y v, X) \\ &= B(Z,X)\eta(Y) + B(Z,Y)\eta(X) + g(\nabla_X Z, Y) + g(\nabla_Y Z, X) \end{aligned} \tag{15}$$

or equivalently we can write the Equation (15) that

$$(L_Z g)(X,Y) = (\nabla_Z g)(X,Y) + g(\nabla_X Z, Y) + g(\nabla_Y Z, X) \tag{16}$$

for any $X, Y, Z \in \Gamma(TM)$.

If $B = 0$ on TM, then M is called totally geodesic in $\widetilde{M}$. A point $p \in M$ is called umbilical if

$$B(X,Y)_p = \lambda g_p(X,Y), \quad X, Y \in T_pM,$$

where λ is a constant. Furthermore, M is called totally umbilical in $\widetilde{M}$ if every points of M is umbilical [21].

A lightlike hypersurface $(M, g, S(TM))$ is called screen locally conformal if the shape operators A_N and A^*_ξ are related by

$$A_N = \varphi A^*_\xi. \tag{17}$$

Here, φ is a non-vanishing smooth function on a neighborhood U on M. We note that M is called screen homothetic if φ is a constant [22].

Let us denote the Riemann curvature tensors of $\widetilde{M}$ and M by $\widetilde{R}$ and R, respectively. The Gauss-Codazzi type equations are given as follows:

$$\begin{aligned} \widetilde{g}(\widetilde{R}(X,Y)Z, PU) &= g(R(X,Y)Z, PU) + B(X,Z)C(Y,PU) \\ &- B(Y,Z)C(X,PU), \end{aligned} \tag{18}$$

$$\begin{aligned} \widetilde{g}(\widetilde{R}(X,Y)Z, \xi) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &+ B(Y,Z)\tau(X) - B(X,Z)\tau(Y), \end{aligned} \tag{19}$$

$$\widetilde{g}(\widetilde{R}(X,Y)Z, N) = g(R(X,Y)Z, N), \tag{20}$$

$$\begin{aligned} \widetilde{g}(\widetilde{R}(X,Y)PZ, N) &= (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ &+ \tau(Y)C(X,PZ) - \tau(X)C(Y,PZ), \end{aligned} \tag{21}$$

for any $X, Y, Z, U \in \Gamma(TM)$.

Let $\Pi = \mathrm{Span}\{X, Y\}$ be a 2-dimensional non-degenerate plane in T_pM at a point $p \in M$. The sectional curvature of Π is given by

$$K(\Pi) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}. \tag{22}$$

We note that since C is not symmetric, it is clear from (18) and (22) that the sectional curvature map does not need to be symmetric on any lightlike hypersurface.

Theorem 1 ([17]). *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then the following assertions are equivalent:*

(i) $S(TM)$ is integrable.
(ii) C is symmetric on $\Gamma(S(TM))$.
(iii) A_N is self-adjoint on $\Gamma(S(TM))$ with respect to g.

As a result of Theorem 1, we see that the sectional curvature map is symmetric on every lightlike hypersurface whose screen distribution is integrable.

Let $\{e_1, \ldots\ldots, e_n, \xi, N\}$ be a quasi orthonormal basis on $T\widetilde{M}$, where $\{e_1, \ldots\ldots, e_n\}$ be an orthonormal basis of $\Gamma(\mathrm{S}(TM))$. The induced Ricci type tensor $R^{(0,2)}$ of M is defined by

$$R^{(0,2)}(X, Y) = \sum_{j=1}^{n} g(R(e_j, X)Y, e_j) + \widetilde{g}(R(\xi, X)Y, N) \tag{23}$$

for any $X, Y \in TM$. We note that the induced Ricci type tensor $R^{(0,2)}$ is not symmetric for any lightlike hypersurface.

Considering the Equation (20) and Theorem 1, we obtain the following corollary immediately:

Corollary 1. *The Ricci tensor $R^{(0,2)}$ is symmetric on lightlike hypersurface whose screen distribution is integrable.*

A lightlike hypersurface $(M, g, \mathrm{S}(TM))$ of a Lorentzian manifold is said to be of genus zero with screen $\mathrm{S}(TM)^0$ (cf. [23]) if

(a) M admits a canonical or unique screen distribution $\mathrm{S}(TM)^0$ that induces a canonical or unique lightlike transversal vector bundle N.
(b) M admits an induced symmetric Ricci tensor.

Let the Ricci tensor $R^{(0,2)}$ be symmetric on lightlike hypersurface $(M, g, \mathrm{S}(TM))$. The manifold M is called as an Einstein lightlike hypersurface [24] if, for any $X, Y \in \Gamma(TM)$, the following relation satisfies:

$$R^{(0,2)}(X, Y) = \gamma\, g(X, Y), \tag{24}$$

where γ is a constant.

3. Concurrent Vector Fields

For any lightlike hypersurface $(M, g, S(TM))$, some significant differential operators such as the gradient, divergence, Laplacian operators could be defined by the help of a rigging vector field and its associated metric (see [25–29]). Therefore, we shall initially recall some basic facts related to rigging vector fields and their some basic properties before studying concurrent vector fields on lightlike hypersurfaces.

Definition 1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold $(\widetilde{M}, \widetilde{g})$ and ζ be a vector field defined in some open set containing M. Suppose that $\zeta_p \notin T_pM$ for any $p \in M$. If there exists a 1-form η satisfying $\eta(X) = \widetilde{g}(X, \zeta)$ for any $X \in \Gamma(TM)$, then ζ is called a rigging vector field for M.*

Now, let $N \in \text{tr}(TM)$ be a rigging vector field for M and η be a 1-form defined by $\eta(X) = \widetilde{g}(X, N)$ for any $X \in \Gamma(TM)$. In this case, one can define a $(0,2)$ type tensor $\overline{g}$ as follows:

$$\overline{g}(X,Y) = g(X,Y) + \eta(X)\eta(Y) \tag{25}$$

for any $X, Y \in \Gamma(TM)$. We note that the associated metric $\overline{g}$ is non-degenerate. From (3), (5) and (25) we have

$$\overline{g}(\xi,\xi) = 1, \;\; \overline{g}(\xi, X) = \eta(X) \tag{26}$$

and

$$\overline{g}(X,Y) = g(X,Y), \;\; \eta(X) = 0 \quad \forall X, Y \in \Gamma(\mathrm{S}(TM)). \tag{27}$$

Let $f : U \subset M \to \mathbb{R}$ be a smooth function and $(x_1, x_2, \ldots, x_n)$ be a coordinate system on U. Then the gradient of f with respect to g is defined by

$$\text{grad } f = \sum_{i=1}^{n+1} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}. \tag{28}$$

Here, $[g^{ij}]$ denotes the inverse of $\overline{g}$ coincided with g and $[g^{ij}]$ is defined to be pseudo-inverse of g [25].

Now, let v be a concurrent vector field on $\Gamma(T\widetilde{M})$. Then, we can write v as the tangential and transversal components by

$$v = v^T + v^N, \tag{29}$$

where $v^T \in \Gamma(TM)$ and $v^N \in \text{tr}(TM)$. From (25) and (29), we have

$$\begin{aligned} \overline{g}(v^T, \xi) &= g(v^T, \xi) + \eta(v^T)\eta(\xi) \\ &= \eta(v^T) \\ &= \widetilde{g}(v^T, N). \end{aligned} \tag{30}$$

For any $X \in \Gamma(TM)$, we write

$$\begin{aligned} X\overline{g}(v^T, \xi) &= X\widetilde{g}(v^T, N) \\ &= \widetilde{g}(\widetilde{\nabla}_X v, N) + \widetilde{g}(v, \widetilde{\nabla}_X N) \\ &= \widetilde{g}(X, N) - \widetilde{g}(v, A_N X) + \tau(X)\widetilde{g}(v, N), \end{aligned}$$

which implies that

$$X\, \overline{g}(v^T, \xi) = \eta(X) + \tau(X)\, \eta(v) - \widetilde{g}(v, A_N X). \tag{31}$$

Now, we suppose that $v = v^T$, that is, v lies in $\Gamma(TM)$. In this case, we can write

$$v = v^s + a\xi, \tag{32}$$

where $v^s \in \Gamma(S(TM))$ and $\eta(v) = a$.

Taking into consideration the above facts, we get the following lemma:

Lemma 1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold $(\widetilde{M}, \widetilde{g})$. If v is a concurrent vector field on $\Gamma(TM)$, then we have*

$$\tau(v) = \frac{1}{a}(\widetilde{g}(v^s, A_N v)) - 1. \tag{33}$$

Proof. From (30)–(32), we have

$$(v^s + a\xi)\overline{g}(v^s + a\xi, \xi) = \widetilde{g}(v^s + a\xi, N) - \widetilde{g}(v^s + a\xi, A_N(v^s + a\xi)) + \tau(v^s + a\xi)\widetilde{g}(v^s + a\xi, N).$$

Therefore, we get

$$(v^s + a\xi)[\overline{g}(v^s, \xi)] + a(v^s + a\xi)[\overline{g}(\xi, \xi)] = a\widetilde{g}(\xi, N) - \widetilde{g}(v^s + a\xi, A_N v^s) - \widetilde{g}(v^s + a\xi, A_N a\xi) + \tau(v^s + a\xi)(\eta(v^s) + a\eta(\xi)).$$

By a straightforward computation, we have

$$0 = a - \widetilde{g}(v^s, A_N v) - a\widetilde{g}(v^s, A_N \xi) + a\tau(v),$$

which completes the proof of lemma. □

Lemma 2. *Let v be a concurrent vector field on $\Gamma(TM)$. Then we have*

$$\nabla_X v = X \ \text{and} \ B(X, v) = 0 \tag{34}$$

for any $X \in \Gamma(TM)$.

Proof. From (1) and (7), we get

$$\widetilde{\nabla}_X v = \nabla_X v + B(X, v)N = X. \tag{35}$$

Considering the tangential and transversal parts of (35), we get (34) immediately. □

Lemma 3. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface of a Lorentzian manifold $(\widetilde{M}, \widetilde{g})$. If v is a concurrent vector field on $\Gamma(TM)$, then we have*

$$\tau(v) = -1. \tag{36}$$

Proof. Under the assumption, we have from (9), (32) and Lemma 1 that

$$\tau(v) = \frac{1}{a}C(v, v^s) - 1,$$

which shows that

$$\tau(v) = \frac{1}{a}\frac{1}{\varphi}B(v, v^s) - 1. \tag{37}$$

From (34) and (37), we obtain (36). □

Lemma 4. *For any screen conformal lightlike hypersurface of a Lorentzian manifold, we have*

$$\nabla^*_\xi v = 0 \ \text{and} \ C(\xi, v) = 1, \tag{38}$$

where v is a concurrent vector field on $\Gamma(TM)$.

Proof. Using the fact the second fundamental form B vanishes on $\mathrm{Rad}(TM)$ and from (9), we write

$$\widetilde{\nabla}_\xi v = \nabla_\xi v = \nabla^*_\xi v + C(\xi, v)\xi = \xi. \tag{39}$$

From (4) and (39), the proof of lemma is completed. □

Theorem 2. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface and v be a concurrent vector field lying on $\Gamma(TM)$. Then at least one of the following statements occurs.*

(i) *v lies in $\Gamma(S(TM))$.*
(ii) *$\tau(\xi) = 0$.*

Proof. From the Gauss and Weingarten formulas and Lemma 3, we have

$$\nabla_v \xi = -A^*_\xi v + \xi \tag{40}$$

and

$$\widetilde{\nabla}_v N = -A_N v - \xi. \tag{41}$$

Since v is the concurrent vector field, we have from Lemma 2 that

$$\nabla_\xi v = \xi. \tag{42}$$

Using (40) in (42), we obtain

$$[\xi, v] = -A^*_\xi v. \tag{43}$$

If we write $v = v^s + a\xi$ in (43), we see that

$$[\xi, v^s + a\xi] = -A^*_\xi (v^s + a\xi).$$

Thus, we have

$$[\xi, v^s] = -A^*_\xi v^s - aA^*_\xi \xi. \tag{44}$$

From (44), we get

$$\nabla_\xi v^s - \nabla_{v^s} \xi = -A^*_\xi v^s - aA^*_\xi \xi$$

which implies that

$$\nabla^*_\xi v^s + C(\xi, v^s)\xi - A^*_\xi v^s + \tau(v^s)\xi = -A^*_\xi v^s - aA^*_\xi \xi. \tag{45}$$

Since A^*_ξ in $\Gamma(S(TM))$, we see from (45) that

$$\tau(v^s) + C(\xi, v^s) = 0. \tag{46}$$

Using Lemma 4 and (45), we get

$$\tau(v^s) = -1. \tag{47}$$

Using the fact that $\tau(v) = -1$, we have

$$\tau(v) = \tau(v^s + a\xi) = -1,$$

which shows that

$$a\tau(\xi) = 0. \tag{48}$$

Therefore, we get at least

$$a = 0 \quad \text{or} \quad \tau(\xi) = 0, \tag{49}$$

which implies the proof of theorem. □

From Theorem 2, we get the following corollary immediately:

Corollary 2. *Let $(M, g, S(TM))$ be a screen conformal lightlike hypersurface and v is a concurrent vector field such that $v \notin \Gamma(S(TM))$, that is $a \neq 0$. Then $\tau(\xi) = 0$.*

Now, we shall recall the following proposition of K. L. Duggal and B. Sahin [19]:

Proposition 1 (Proposition 2.5.4, Page 77). *Let $(M, g, S(TM))$ be a screen homothetic lightlike hypersurface of an indefinite space form $\widetilde{M}(c)$ with constant curvature c. Then the following relation holds:*

$$2\varphi\, \tau(\xi)\, B(X, PZ) = -c\, g(X, PZ). \tag{50}$$

Under the hypothesis of the above proposition, one can obtain the following corollary:

Corollary 3. *Let $(M, g, S(TM))$ be a screen homothetic lightlike hypersurface of a Lorentzain space form $\widetilde{M}(c)$. Then, the following assertions hold:*

(i) If $\tau = 0$ then $\widetilde{M}$ becomes the semi-Euclidean space.
(ii) If $\tau(\xi) \neq 0$, then M is totally umbilical.

In the light of Corollaries 2 and 3, we get the following corollary immediately:

Corollary 4. *Let $(M, g, S(TM))$ be a screen homothetic lightlike hypersurface of $(\widetilde{M}(c), \widetilde{g})$ and v be a concurrent vector field such that $v \notin \Gamma(S(TM))$. Then $\widetilde{M}$ is the semi-Euclidean space. If the concurrent vector field v lies on $\Gamma(S(TM))$ then M is totally umbilical.*

Theorem 3. *Every totally umbilical screen homothetic lightlike hypersurface $(M, g, S(TM))$ of a Lorenzian space form $\widetilde{M}(c)$ admitting a concurrent vector field v on $\Gamma(TM)$ is totally geodesic.*

Proof. Under the assumption, we have from Theorem 2 that $\tau(\xi) = 0$. Then we obviously have

$$\nabla_{\xi}\xi = -A^*_{\xi}\xi. \tag{51}$$

Furthermore, using the fact that $B(\xi, Y) = g(A^*_{\xi}\xi, Y)$ we obtain that $A^*_{\xi}\xi = 0$. From (51), we have

$$\nabla_{\xi}\xi = 0. \tag{52}$$

Therefore, we get

$$\begin{aligned} R(\xi, v)\xi &= \nabla_{\xi}\nabla_{v}\xi - \nabla_{v}\nabla_{\xi}\xi - \nabla_{[\xi,v]}\xi \\ &= \nabla_{\xi}\xi - \nabla_{[\xi,v]}\xi. \end{aligned} \tag{53}$$

Considering the Equation (43), we have

$$R(\xi, v)\xi = \nabla_{A^*_{\xi}v}\xi. \tag{54}$$

Using the fact that M is totally umbilical, we have

$$\begin{aligned} R(\xi, v)\xi &= \nabla_{\lambda v}\xi \\ &= \lambda\nabla_{v}\xi. \end{aligned} \tag{55}$$

Thus, we obtain from the Equation (40) that

$$R(\xi, v)\xi = \lambda(-A^*_{\xi}v + \xi). \tag{56}$$

From (20) and (56), we have

$$\widetilde{g}(\widetilde{R}(\xi, v)\xi, N) = \widetilde{g}(R(\xi, v)\xi, N) = \lambda. \tag{57}$$

Finally we see that $\lambda = 0$ by considering Proposition 1 and the equation (57). This fact shows us that M is totally geodesic. □

Remark 1. *From Theorem 2, we see that $\tau(\xi)$ does not have to be equal to zero. Therefore, Theorem 3 is not correct when the concurrent vector field v lies on $\Gamma(S(TM))$. In a similar manner, considering (53), we see that Theorem 3 is not correct when v lies on $Rad(TM)$.*

Now we shall investigate concurrent vector fields on the Levi-Civita connection with respect to the associated metric $\overline{g}$.

Let $\overline{\nabla}$ be the Riemannian connection of $\widetilde{M}$ with respect to the associated metric $\overline{g}$ given in the Equation (25) and ∇^1 be the induced Riemann connection from $\overline{\nabla}$ onto TM. Then we have the following:

Theorem 4. *Let v be a concurrent vector field with respect to $\widetilde{\nabla}$. Then v is also concurrent with respect to ∇^1.*

Proof. From (25), for any vector field $X \in \Gamma(TM)$, we write

$$\begin{aligned} X\overline{g}(v,v) &= Xg(v,v) + X\Big(\widetilde{g}(v,N)^2\Big) \\ &= 2\overline{g}(\nabla_X v, v) + 2\eta(v)\Big[\widetilde{g}(\widetilde{\nabla}_X v, N) + \widetilde{g}(v, \widetilde{\nabla}_X N)\Big] \\ &= 2\overline{g}(\nabla_X v, v) + 2\eta(v)[\widetilde{g}(X, N) + \widetilde{g}(X, -A_N X + \tau(X)N)] \\ &= 2\overline{g}(\nabla_X v, v) + 2\eta(v)[\eta(X) - \widetilde{g}(v, A_N X) + \tau(X)\eta(v)]. \end{aligned} \tag{58}$$

Beside this fact, we have

$$\begin{aligned} X\overline{g}(v,v) &= 2\overline{g}(\nabla^1_X v, v) \\ &= 2g(\nabla^1_X v, v) + 2\eta(\nabla^1_X v)\eta(v). \end{aligned} \tag{59}$$

From (58) and (59), we obtain

$$\nabla^1_X v + \eta(\nabla^1_X v)N = X + [\eta(X) - \widetilde{g}(v, A_N X) + \tau(X)\eta(v)]N. \tag{60}$$

Using (60), we get

$$\nabla^1_X v = X, \quad \eta(\nabla^1_X v) = \eta(X) - \widetilde{g}(v, A_N X) + \tau(X)\eta(v), \tag{61}$$

which show that v is also concurrent with respect to ∇^1. □

We note that the converge part of Theorem 4 is not correct.

Taking into consideration (61), we obtain the following corollaries.

Corollary 5. *Let v be a concurrent vector field with respect to ∇^1. Then v is also concurrent with respect to ∇ if and only if the following relation holds for all $X \in \Gamma(TM)$:*

$$\eta(\nabla^1_X v) = \eta(X) - \widetilde{g}(v, A_N X) + \tau(X)\eta(v). \tag{62}$$

Corollary 6. *Let $(M, g, S(TM))$ be screen conformal with respect to ∇ and v be a concurrent vector field with respect to ∇^1. Then v is also concurrent with respect to ∇ if and only if the following relation holds:*

$$\eta(\nabla^1_v v) = -\widetilde{g}(v, A_N v). \tag{63}$$

Corollary 7. *Let $(M, g, S(TM))$ be totally geodesic screen conformal with respect to ∇ and v be a concurrent vector field with respect to ∇^1. Then v is also concurrent with respect to ∇ if and only if the following relation holds:*

$$\eta(\nabla^1_v v) = 0. \tag{64}$$

Corollary 8. *Let $(M, g, S(TM))$ be totally umbilical screen conformal with respect to ∇. Suppose v is a concurrent vector field with respect to ∇^1. Then v is also concurrent with respect to ∇ if and only if the following relation holds:*

$$\eta(\nabla^1_v v) = -\lambda\, \widetilde{g}(v, v), \quad \lambda \in R. \tag{65}$$

4. Ricci Solitons on Lightlike Hypersurfaces

Let $(M, g, S(TM))$ be a lightlike hypersurface of a Lorenzian manifold $(\widetilde{M}, \widetilde{g})$. Suppose that v is a concurrent vector field on $\Gamma(T\widetilde{M})$. Then we can write the vector v as the tangential and transversal components by

$$v = v^T + fN, \tag{66}$$

where $v^T \in \Gamma(TM)$ and $f = \widetilde{g}(v, \xi)$. In this case we have the followings:

Lemma 5. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a Lorenzian manifold $(\widetilde{M}, \widetilde{g})$ admitting a concurrent vector field v on $\Gamma(T\widetilde{M})$. Then*

$$\nabla_X v^T = f A_N X + X \tag{67}$$

and

$$B(X, v^T) = f\tau(X) - X(f). \tag{68}$$

Proof. Since v is a concurrent vector field with respect to $\widetilde{\nabla}$, we have

$$X = \widetilde{\nabla}_X v = \widetilde{\nabla}_X v^T + \widetilde{\nabla}_X (fN) \tag{69}$$

for any $X \in \Gamma(TM)$. Therefore, we get

$$X = \nabla_X v^T + B(X, v^T) + X(f)N - f A_N X - f\tau(X)N. \tag{70}$$

From (69) and (70), we get the Equations (67) and (68) immediately. □

Now we recall the following proposition of K. L. Duggal and A. Bejancu [17]:

Proposition 2. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold. If the Ricci tensor with respect to ∇ is symmetric, then there exists a pair $\{\xi, N\}$ on U such that the 1-form τ vanishes.*

Theorem 5. *Let $(M, g, S(TM))$ be a totally geodesic lightlike hypersurface of a Lorentzian manifold $(\widetilde{M}, \widetilde{g})$ admitting a concurrent vector v defined as (66). Then the function f is constant.*

Proof. From (68), if $(M, g, S(TM))$ is totally geodesic, then we have

$$X(f) = f\tau(X), \quad \forall X \in \Gamma(TM). \tag{71}$$

Using the above equation, we get

$$[X, Y](f) = f(Y(\tau(X)) - X(\tau(Y))) \tag{72}$$

for any $X, Y \in \Gamma(TM)$. Considering Corollary 1 and Proposition 2, we get

$$[X,Y](f) = 0 \quad \forall X, Y \in \Gamma(TM),$$

which indicates that f is constant. □

Lemma 6. *For any lightlike hypersurface $(M, g, S(TM))$ of a Lorentzian manifold $(\widetilde{M}, \widetilde{g})$ we have*

$$\begin{aligned}(L_{v^T} g)(X,Y) &= B(v^T, X)\eta(Y) + B(v^T, Y)\eta(X) \\ &\quad + f g(A_N X, Y) + f g(A_N Y, X) + g(X,Y) \end{aligned} \tag{73}$$

or equivalent to (73) we have

$$\begin{aligned}(L_{v^T} g)(X,Y) &= (\nabla_{v^T} g)(X,Y) \\ &\quad + f[g(A_N X, Y) + g(A_N Y, X)] + 2g(X,Y) \end{aligned} \tag{74}$$

for any $X, Y \in \Gamma(TM)$.

Proof. From (15) and (16), we get

$$\begin{aligned}(L_{v^T} g)(X,Y) &= B(v^T, X)\eta(Y) + B(v^T, Y)\eta(X) \\ &\quad + g(\nabla_X v^T, Y) + g(\nabla_Y v^T, X) \end{aligned} \tag{75}$$

or equivalently

$$(L_{v^T} g)(X,Y) = (\nabla_{v^T} g)(X,Y) + g(\nabla_X v^T, Y) + g(\nabla_Y v^T, X). \tag{76}$$

Using (66) and (67) in (75) and (76), we get (73) and (74) respectively. □

If v lies on $\Gamma(TM)$, that is, $v = v^T$ (or $f = 0$), then we have the following special result:

Lemma 7. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold and v be a concurrent vector field on $\Gamma(TM)$. Then, for any $X, Y \in \Gamma(TM)$, we have*

$$(L_{v^T} g)(X,Y) = (\nabla_{v^T} g)(X,Y) + 2g(X,Y). \tag{77}$$

Definition 2. *A lightlike hypersurface $(M, g, S(TM))$ is called a Ricci soliton if the following relation satisfies for any $X, Y \in \Gamma(TM)$:*

$$L_v g(X,Y) + 2R^{(0,2)}(X,Y) = 2\lambda g(X,Y), \tag{78}$$

where λ is a constant. A Ricci soliton lightlike hypersurface is called shrinking if $\lambda > 0$, steady if $\lambda = 0$, expanding if $\lambda < 0$.

Proposition 3. *Let $(M, g, S(TM))$ be a Ricci soliton lightlike hypersurface with the potential vector field v^T. Then we have*

$$\begin{aligned}2R^{(0,2)}(X,Y) &= 2\lambda g(X,Y) - f[g(A_N X, Y) + g(A_N Y, X)] - 2g(X,Y) \\ &\quad - (\nabla_{v^T} g)(X,Y) \end{aligned} \tag{79}$$

or, equivalently we have

$$\begin{aligned}R^{(0,2)}(X,Y) &= (\lambda - 1)g(X,Y) - f[g(A_N X, Y) + g(A_N Y, X)] \\ &\quad - \frac{1}{2}(\nabla_{v^T} g)(X,Y). \end{aligned} \tag{80}$$

for any $X, Y \in \Gamma(TM)$.

Proof. From the Equation (2), we have

$$(L_{v^T}g) + 2R^{(0,2)} = 2\lambda g. \tag{81}$$

Considering Lemma 6 in (81), we get (79) and (80) respectively. □

Now, we shall give an example of Ricci solition lightlike hypersurface (see Examples 4, 9 and 17 in [18]):

Example 1 (Lightlike cone)**.** *Let $\mathbb{R}^4_1$ be the Minkowski space with signature $(-,+,+,+)$ of the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4\}$ and M be a submanifold of R^4_1 defined by*

$$\{(t, t\cos u\cos v, t\cos u\sin v, t\sin u) : \ t > 0, \ u \in (0, \frac{\pi}{2}), \ v \in [0, 2\pi]\}.$$

Then we have

$$\begin{aligned}
\partial u_0 &= \partial x_1 + \cos u\cos v\partial x_2 + \cos u\sin v\partial x_3 + \sin u\partial x_4, \\
N &= \frac{1}{2}(-\partial x_1 + \cos u\cos v\partial x_2 + \cos u\sin v\partial x_3 + \sin u\partial x_4) \\
\partial u_1 &= t(-\sin u\cos v\partial x_2 - \sin u\sin v\partial x_3 + \cos u\partial x_4) \\
\partial u_2 &= t(\frac{1}{\cos u} - \cos u\sin v\partial x_2 + \cos u\cos v\partial x_3),
\end{aligned}$$

where $\{u_1, u_2, u_3\}$ is the coordinate system on $U \subset M$ such that $Rad(TM) = Span\{\partial u_0\}$, $S(TM) = Span\{\partial u_1, \partial u_2\}$ and $tr(TM) = Span\{N\}$. In this case, the matrix of induced metric g on M with respect to the natural frame fields $\{\partial u_0, \partial u_1, \partial u_2\}$ is as follows:

$$[g] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & t^2 & 0 \\ 0 & 0 & t^2\cos^2 u \end{pmatrix}. \tag{82}$$

Let us consider the another coordinate system $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ on $\bar{U} \subset M$ satisfying

$$\partial\bar{u}_0, \qquad \partial\bar{u}_1 = t\sin u\,\partial u_0 + \cos u\,\partial u_1, \qquad \partial\bar{u}_2 = t\cos u\,\partial u_2.$$

Using the equation (7.1.2) in [18], we have the local field of frames $\{\partial u_0, N, \delta_{u_1}, \delta_{u_2}\}$ of R^4_1 such that the matrix of metric on R^4_1 satisfies

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & t^2 & 0 \\ 0 & 0 & 0 & t^2\cos^2 u \end{pmatrix}.$$

Then, the Ricci tensor is symmetric and

$$\begin{aligned}
&R^{(0,2)}(\delta_{u_1}, \delta_{u_1}) = -\frac{1}{2}, \quad R^{(0,2)}(\delta_{u_1}, \delta_{u_2}) = 0, \quad R^{(0,2)}(\delta_{u_2}, \delta_{u_2}) = -\frac{1}{2}\cos^2 u, \\
&R^{(0,2)}(\delta_{u_0}, \delta_{u_0}) = R^{(0,2)}(\delta_{u_0}, \delta_{u_1}) = R^{(0,2)}(\delta_{u_0}, \delta_{u_2}) = 0.
\end{aligned}$$

If we consider the position vector $v = t\xi$ of $(M, g, S(TM)$, we see that v is concurrent for M. From the above equations and (81), we get $(M, g, S(TM))$ is a expanding Ricci soliton with $\lambda = -\frac{1}{2}$, where v is the potential concurrent vector.

Now we shall give the following lemma for later use:

Lemma 8. *Let $(M, g, S(TM))$ be an $(n+1)$-dimensional lightlike hypersurface of the semi-Euclidean space $\mathbb{R}_q^{n+2}$. Then*

$$R^{(0,2)}(X, Y) = n\ traceA_N\ B(X, Y) - B(A_N X, Y) \tag{83}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Let $\{e_1, \ldots, e_n, \xi\}$ be a basis of M such that Span $= \{e_1, \ldots, e_n\}$ is an orthonormal basis on $\Gamma(\mathrm{S}(TM))$. Using the fact that $\widetilde{M} = IR_q^{n+2}$ is the semi-Euclidean space and from (18), we have

$$R(X, Y)Z = B(Y, Z)A_N X - B(X, Z)A_N Y$$

for any $X, Y, Z \in \Gamma(TM)$. Furthermore, we get

$$Ric(X, Y) = \sum_{i=1}^{m} \varepsilon_i g(R(e_i X)Y, e_i) + \widetilde{g}(R(\xi, X)Y, N).$$

Therefore, we have

$$R(e_i, X)Y = B(X, Y)A_N e_i - B(e_i, Y)A_N X$$

and putting $X = \xi$, we have

$$R(\xi, X)Y = B(X, Y)A_N \xi.$$

Thus, we get

$$\widetilde{g}(R(\xi, X)Y, N) = B(X, Y)g(A_N \xi, N) = 0.$$

So we obtain

$$\begin{aligned} R^{(0,2)}(X, Y) &= \sum_{i=1}^{n} \varepsilon_i B(X, Y) g(A_N e_i, e_i) - \sum_{i=1}^{m} \varepsilon_i B(g(A_N X, e_i)e_i, Y) \\ &= B(X, Y)n.\mathrm{trace}A_N - B\left(\sum_{i=1}^{n} \varepsilon_i g(A_N X, e_i)e_i, Y\right) \\ &= B(X, Y)n.\mathrm{trace}A_N - B(A_N X, Y), \end{aligned} \tag{84}$$

which completes the proof of the lemma. □

Proposition 4. *Let $(M, g, S(TM))$ be an $(n+1)-$dimensional Ricci soliton lightlike hypersurface of the semi-Euclidean space $\mathbb{R}_q^{n+2}$ admitting a potential vector field v. For any unit vector field $X \in \Gamma(S(TM))$, we have*

$$n\ B(X, X)\ traceA_N - B(A_N X, X) = \lambda - 1 - 2f[g(A_N X, X)]. \tag{85}$$

Proof. Suppose that $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\Gamma(\mathrm{S}(TM))$. From (83), we see that

$$R^{(0,2)}(e_i, e_i) = n.B(e_i, e_i).\mathrm{trace}A_N - B(A_N e_i, e_i). \tag{86}$$

Using (86) in (80) and putting $X = e_i$, the proof of proposition is straightforward. □

Corollary 9. *Let $(M, g, S(TM))$ be an $(n+1)-$dimensional Ricci soliton lightlike hypersurface of the semi-Euclidean space $\mathbb{R}_q^{n+2}$ admitting a potential vector field v. If $(M, g, S(TM))$ is totally umbilical and screen conformal with conformal factor φ then we have*

$$n^2[B(X,X)]^2\varphi - [B(X,X)]^2 = \lambda - 1 - 2f\, B(X,X) \tag{87}$$

for any unit vector field $X \in \Gamma(S(TM))$.

Remark 2. *In the Equation (87), we obtain a second order equation. Furthermore, there exists a unique solution of this equation. If we compute the discriminant $\Delta = 0$, we get*

$$f^2 = (1-\lambda)(\varphi n^2 - 1). \tag{88}$$

Corollary 10. *Let $(M, g, S(TM))$ be screen conformal totally umbilical lightlike hypersurface of the semi-Euclidean space $\mathbb{R}_q^{n+2}$. If M is a steady Ricci soliton, then the concurrent vector field v lies on $\Gamma(TM)$.*

Now, we shall investigate to the Ricci soliton equation for lightlike hypersurface $(M, g, S(TM))$ admitting a concurrent vector field which lies on $\Gamma(TM)$.

Proposition 5. *Let $(M, g, S(TM))$ be a Ricci soliton lightlike hypersurface of a Lorenzian manifold $(\widetilde{M}, \widetilde{g})$ admitting a concurrent vector field v on $\Gamma(TM)$. Then, for any $X, Y \in \Gamma(TM)$, we have*

$$R^{(0,2)}(X,Y) = (\lambda - 1)\, g(X,Y) - \frac{1}{2}[B(v,X)\eta(Y) + B(v,Y)\eta(X)]. \tag{89}$$

Proof. Since v lies on $\Gamma(TM)$ and $(M, g, S(TM))$ is a Ricci soliton, we write from (2) that

$$(L_v g) + 2R^{(0,2)} = 2\lambda g. \tag{90}$$

Using the assumption that v is a concurrent vector field on $\Gamma(TM)$, we have from Lemma 7 that

$$(L_v g)(X,Y) = (\nabla_v g)(X,Y) + 2g(X,Y) \tag{91}$$

for any $X, Y \in \Gamma(TM)$. Using (90) and (91), we obtain

$$R^{(0,2)}(X,Y) = (\lambda - 1)g(X,Y) - \frac{1}{2}(\nabla_v g)(X,Y). \tag{92}$$

From (15) and (92), the proof of proposition is straightforward. □

Now, suppose that $\{e_1, \ldots, e_n, \xi\}$ be a basis on $\Gamma(TM)$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$. Then we have

$$\begin{aligned} R^{(0,2)}(e_i, e_j) &= (\lambda - 1) - \frac{1}{2}[B(v,e_i)\eta(e_i) + B(v,e_i)\eta(e_i)] \\ &= (\lambda - 1)\, \delta_{ij}, \end{aligned} \tag{93}$$

where δ_{ij} is the Kronecker delta and $i, j \in \{1, \ldots, n\}$. Putting $X = Y = \xi$ in (89), then we get

$$\begin{aligned} R^{(0,2)}(\xi,\xi) &= (\lambda - 1) - \frac{1}{2}[B(v,\xi)\eta(\xi) + B(v,\xi)\eta(\xi)] \\ &= \lambda - 1. \end{aligned} \tag{94}$$

Considering (93) and (94), we obtain the following corollary:

Corollary 11. *Let $(M, g, S(TM))$ be a lightlike hypersurface and v be the concurrent vector field on $\Gamma(TM)$. The manifold $(M, g, S(TM))$ is a Ricci soliton if and only if it is an Einstein lightlike hypersurface.*

Now we recall the following theorem dealing screen homothetic lightlike hypersurfaces of C. Atindogbe, J.-P. Ezin, J. Tossa [24]:

Theorem 6. *Let $(M, g, S(TM))$ be a screen homothetic lightlike hypersurface of a Lorentzian space form $(\widetilde{M}, \widetilde{g})$, $c \geq 0$. If M is Einstein, that is $R^{(0,2)} = \gamma\, g$, (γ is constant), then γ is non-negative.*

From (93) and (94) and Theorem 6, we obtain the following corollary:

Corollary 12. *Let $(M, g, S(TM))$ be a screen homothetic lightlike hypersurface of a a Lorentzian space form. If M is a Ricci soliton, then M is shrinking with $\lambda \geq 1$.*

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